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Violating Bell inequalities maximally for two $d$-dimensional systems

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We show the maximal violation of Bell inequalities for two $d$-dimensional systems by using the method of the Bell operator. The maximal violation corresponds to the maximal eigenvalue of the Bell operator matrix. The eigenvectors corresponding to these eigenvalues are described by asymmetric entangled states. We estimate the maximum value of the eigenvalue for large dimension. A family of elegant entangled states $|\Psi\rangle_{\text{app}}$ that violate Bell inequality more strongly than the maximally entangled state but are somewhat close to these eigenvectors is presented. These approximate states can potentially be useful for quantum cryptography as well as many other important fields of quantum information.

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I. INTRODUCTION

The famous Clauser-Horne-Shimony-Holt (CHSH) inequality [1,2] for two entangled spin-1/2 particles has always provided an excellent test-bed for experimental verification of quantum mechanics against the predictions of local realism [3]. It is well known that all pure entangled states in two dimensions violate the CHSH inequality: the maximum quantum violation of $2\sqrt{2}$ being often called the Tsirelson’s bound [4]. In 2000, contrary to previous study, Kaszlikowski et al. showed numerically, based on linear optimization techniques, that the violations of local realism increase with dimensions for two maximally entangled $d$-dimensional systems (qudits) ($3 \leq d \leq 9$) [5]. A year later, Durt et al. extended the analysis to $d=16$ under special experimental settings [6]. In the same year, an analytical proof was constructed for two maximally entangled three-dimensional systems (qutrits) [7]. In 2002, two research teams independently developed Bell inequalities for high-dimensional systems: the first one is a Clauser-Horne type (probability) inequality for two qutrits [8]; and the second one is a CHSH type (correlation) inequality to two arbitrary $d$-dimensional systems [9], now known as the Collins-Gisin-Linden-Masser-Popescu (CGLMP) inequalities. The tightness of the CGLMP inequality was demonstrated in Ref. [10].

It was generally felt that maximally entangled states $|\Psi\rangle_{\text{mes}}$ would maximally violate the Bell inequality, just as the CHSH inequality has worked for two qubit. Moreover, the results of Ref. [9] was numerically consistent with Ref. [5]. However, contrary to prevalent belief, Acín et al. studied the quantum nonlocality of two qutrit as well as two $d$-dimensional systems up to $d=8$ and discovered another unexpected result: there existed nonmaximally entangled states that lead to greater violation of the CGLMP inequalities compared with maximally entangled states [11]. Moreover, it was shown in this study that the maximal violation increases with dimension $d$, and it reaches $3.1013$ for $d=8$.

This gives rise to a natural question: “What are the maximal violations of the CGLMP inequalities if one increases the dimension $d$, especially when $d$ goes to infinity?” The purpose of this paper is to investigate the maximal violations of the CGLMP inequalities for higher dimensional systems. We also present a family of elegant entangled states $|\Psi\rangle_{\text{app}}$ whose corresponding maximal violations are approximate to the real ones.

The article is organized as follows. In Sec. II, we discuss briefly the higher dimensional generalization of Bell inequality and considered the quantum violations of the CGLMP inequality. In Sec. III, we explain the method for extracting Bell operator and we proceed to compute the quantum violations in Sec. IV. Finally, we concluded the article with some final remarks.

II. BELL INEQUALITIES, ITS GENERALIZATION, AND UPPER BOUND

In 1964, John Bell published his seminal paper [1] in which he formulated an inequality which must be obeyed by all local realistic theory. As it turns out, local measurements on quantum entangled states produce correlations that can violate the Bell or CHSH inequality. Bell original inequalities were idealized in some sense and therefore not suited for experimental verification. Other inequalities better for experimental work were then proposed, including the CHSH inequality [2]. Indeed, it was subsequently shown that there exist Bell inequalities that are violated by all pure entangled states [4].

Both Bell and CHSH inequalities apply for a system of two two-dimensional particles (for instance, two spin-1/2 particles). The inequalities for $N$ two-dimensional particles were subsequently proposed [12–14]. In 2002, Bell-type inequalities were proposed for higher dimensional systems, including the CGLMP inequality [8,9]. Extension to three three-dimensional systems were subsequently proposed in Ref. [15].

The CHSH inequality is defined by the inequality

$$I_d = Q_{11} + Q_{12} - Q_{21} + Q_{22} \leq 2,$$

where $Q_{ij}$ are the correlation functions. It has been shown that quantum mechanics yields a factor $\sqrt{2}$ so that maximal
value of the correlations in Eq. (1) is $2\sqrt{2}$. For estimating the upper bound of CGLMP inequality under quantum mechanics, we need to recast the CGLMP inequality into a form similar to the CHSH inequality [17] in Eq. (1) with the correlation functions $Q_{ij}$ defined by probabilities in the following way:

$$Q_{ij} = \frac{1}{S} \sum_{m=0}^{d-1} \sum_{n=0}^{d-1} f^i(m,n) P(A_i = m, B_j = n),$$

(2)

where $S=(d-1)/2$ is the spin of the particle for the $d$-dimensional system, $f^i(m,n)=S-M(e(i-j)(m+n),d)$, $e(x)=1$ and $-1$ for $x \geq 0$ and $x < 0$, respectively; $M(x,d)=x \mod d$ and $0 \leq M(x,d) \leq d-1$. References [9,17] have proved that $I_d \leq 2$ for hidden variable theory. From Eq. (2), one notes that the extreme values of $Q_{ij}$ are $\pm 1$ for both local realistic description and quantum mechanics, therefore it is impossible that the maximum quantum violation of $I_d$ is larger than 4. Furthermore $I_d$ cannot reach 4, because $Q_{ij}$ are constrained to each other, if three of them are set to be 1, the fourth must also be 1. Consequently, one can conclude easily from above analysis that the maximal quantum violation of $I_d$ is a number between 2 and 4.

Indeed, for a maximally entangled state of two-qubit given by $|\Psi\rangle_{\text{mes}} = \frac{1}{\sqrt{2}} \sum_{j=0}^{d-1} |jj\rangle$, where $|j\rangle$ is the orthonormal base in each subsystem, the Tsirelson’s bound or Bell expression $I_2$ is given by [9]

$$I_2(|\Psi\rangle_{\text{mes}}) = 4d \sum_{l=0}^{l=d} \left(1 - \frac{2k}{d-1}\right) \frac{1}{2d^3 \sin^2 \left(\pi \frac{k + 1}{4} / d\right)} - \frac{1}{2d^3 \sin^2 \left(\pi \frac{k - 1 + 1}{4} / d\right)},$$

(3)

showing that the maximum quantum violation increases with the dimension $d$ (here $l = [d/2 - 1]$ represents the integer part of $(d/2 - 1)$). When $d \to \infty$, it is interesting to note that

$$\lim_{d \to \infty} I_2(|\Psi\rangle_{\text{mes}}) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{(k+1/4)^2} - \frac{1}{(k+3/4)^2} = 2.96981,$$

(4)

a number related to the Catalan’s constant.

The restriction of quantum violation to $|\Psi\rangle_{\text{mes}}$ is due to two simple reasons: on one hand, $|\Psi\rangle_{\text{mes}}$ is a simple state, on the other hand, there is a very natural view point that maximally entangled states would maximally violate the Bell inequality, just as the CHSH inequality has worked for two qubits. The results of Ref. [9] was numerically consistent with Ref. [5]. By considering an approach similar to Sec. III, Acín et al. showed that there existed nonmaximally entangled states that can lead to greater violation of the CGLMP inequalities compared with maximally entangled states [11]. This surprising result still lacks of a satisfactory explanation. From Table I of Ref. [11], one observes that the maximal violation increases with dimension $d$, and it reaches 3.1013 for $d=8$. In this article, we extend this computation to very high dimensions and show that it may approach some asymptotic limit.

### III. GENERALIZED APPROACH

To generalize Acín’s approach, we first note that it has been shown that unbiased multiport beam splitter [16] can be used to test violation of local realism of two maximally entangled qubits. Unbiased $d$-port beam splitter is a device with the following property: if a photon enters any of the $d$ single input ports, its chances of exit are equally split among the $d$ output ports. In fact one can always build the device with the distinguishing trait that the elements of its unitary transition matrix $T$ are solely powers of the root of unity $\gamma = \exp(i2\pi/d)$, namely, $T_{kl} = \gamma^{|k-l|}$. In front of $i$th input port of the device a phase shifter is placed to change the phase of the incoming photon by $\phi(i)$. These $d$ phase shifts, denoted for convenience as a “vector” of phase shifts $\vec{\phi}=(\phi(0), \phi(1), \ldots, \phi(d-1))$, are macroscopic local parameters that can be changed by the observer. Therefore, unbiased $d$-port beam splitter together with the $d$ phase shifters perform the unitary transformation $U(\vec{\phi})$ with the entries $U_{kl} = T_{kl} \exp[i\phi(l)]$. Devices (Bell multiports) endowed with such a matrix were proposed, and readers who are interested in it can refer to Refs. [5,16]. The approach developed in Ref. [11] is related to Bell operator. An arbitrary entangled state of two qubits reads

$$|\Phi\rangle = \sum_{j,j'=0}^{d-1} \alpha_{jj'} |jj'\rangle,$$

(5)

The quantum prediction of the joint probability $P(A_a = k, B_b = l)$ when $A_a$ and $B_b$ are measured in the initial state $|\Phi\rangle$ is given by

$$P(A_a = k, B_b = l) = \text{Tr}[[(U(\hat{\phi}_a)^\dagger \otimes U(\hat{\phi}_b)^\dagger)] \hat{P}_k \otimes \hat{P}_{l}] U(\hat{\phi}_a) \otimes U(\hat{\phi}_b) |\Phi\rangle \langle \Phi|],$$

(6)

where $\hat{P}_k = |k\rangle \langle k|$ and $\hat{P}_l = |l\rangle \langle l|$ are the projectors for systems $A$ and $B$, respectively. Substituting Eq. (6) into the CGLMP inequality, one gets the Bell expression for the state $|\Phi\rangle$:
The first subspace spanned by the vectors $H_{2083}$ reduces to the determination of the maximal eigenvalue $B^*$ of matrix $\hat{B}$ decomposed into the sum of $d$ states that maximally violate the Bell inequality. Thus, the highest quantum prediction of the Bell expression $\text{VIOLATING BELL INEQUALITIES MAXIMALLY FOR}$ can be further simplified, i.e., it can be expressed as

$$I_d(\Phi) = \text{Tr}(\hat{B}|\Phi\rangle\langle\Phi|) = \langle\Phi|\hat{B}|\Phi\rangle,$$

where $\hat{B}$ is the so-called Bell operator. Starting with the CGLMP inequality and choosing the suitable experimental settings [6,11],

$$\phi_1 = 0, \quad \phi_2 = \frac{j\pi}{d}, \quad \varphi_1 = \frac{j\pi}{2d}, \quad \varphi_2 = -\frac{j\pi}{2d},$$

optimal for maximal violations, we can derive the element of the Bell operator matrix as

$$B_{mm',jj'} = \frac{1}{d^2} \sum_{l=0}^{d-1} e^{i(2\pi d)[(j-m)-(j'-m')]} \left\{ e^{i(\pi/2d)(j'-m') \sum_{k=0}^l \left( 1 - \frac{2k}{d-1} \right) (e^{i(2\pi d)(k+j-m)} - e^{i(2\pi d)(k+j'-m')}) + e^{i\pi d(j-m)} \right\} \times \left\{ e^{i(\pi/2d)(j'-m') \sum_{k=0}^l \left( 1 - \frac{2k}{d-1} \right) (e^{i(2\pi d)(k+j'-m')} - e^{i(2\pi d)(k+j-m)}) + e^{i\pi d(j-m)} \right\},$$

where $\hat{B}$, in general, is a $d^2 \times d^2$ matrix. Reference [11] found that the maximal eigenvalue of matrix $\hat{B}$ is nothing but the highest quantum prediction of the Bell expression $I_d(\Phi_{\text{max}})$ and the corresponding eigenvector $|\Phi_{\text{max}}\rangle$ is the state that maximally violates the Bell inequality. Thus, the problem of computing the maximal violation of $I_d(\Phi)$ reduces to the determination of the maximal eigenvalue of matrix $\hat{B}$. Due to $\sum_{l=0}^{d-1} e^{i(2\pi d)pq l} = 0$, where $\delta_{pq}=1$ when $p=q$ modulo $d$ and 0 otherwise, the matrix $\hat{B}$ can be further simplified, i.e., it can be decomposed into the sum of $d$ decoupled operators that act individually within the subspaces spanned by the vectors $|\{00\},\{11\},\ldots,\{|d-1\rangle\rangle\}$, $|\{01\},\{12\},\ldots,\{|d-1\rangle\rangle\}$, $|\{0(d-1)\},\{10\},\ldots,\{|(d-1)(d-2)\}\rangle\rangle$, respectively. For the first subspace spanned by the vectors $|\{00\},\{11\},\ldots,\{|d-1\rangle\rangle\}$, and with the constraint $j-m=j'-m'$, one can reduce $B_{mm',jj'}$ to

$$B_{mj}^{\text{red}} = \frac{8}{d} \sin \left[ \frac{\pi}{2d} (j-m) \right] \times \sum_{k=0}^{l} \left( 1 - \frac{2k}{d-1} \right) \sin \left[ \frac{2\pi}{d} \left( k + \frac{1}{2} \right) (j-m) \right] = \frac{2}{d-1} \cos \left[ \frac{\pi}{2d} (j-m) \right] (1 - \delta_{mj}),$$

where $B_{mj}^{\text{red}}$ is the matrix element located at the $m$th row and the $j$th column of the reduced Bell operator matrix. 

032106-3
For instance, for the maximally entangled state $|\Psi\rangle_{\text{mes}}$ with $a_j = 1/\sqrt{d}$, the summation $(\Sigma_{m=0}^{d-1} B_{m,j})/d = (\Sigma_{m=0}^{d-1} (d-r) B_{m,j})/d$ recovers the results of Eq. (3).

IV. NUMERICAL COMPUTATION

As the analysis shown in Ref. [11], the maximal violation of the CGLMP inequality with the experimental settings (9) corresponds to the maximal eigenvalue of $\hat{B}_{\text{red}}^1$, and indeed its corresponding eigenvector is a nonmaximally entangled state of two qudits. For instance, for $d=3$, one has

$B_0=0$, $B_1=2\sqrt{3}/3$, $B_2=2$, the eigenvector $|\Psi\rangle_{\text{eig}}^3 = \sqrt{\frac{2}{11}} |00\rangle + \frac{\sqrt{3}}{11} |11\rangle + |22\rangle$ corresponds to a maximal violation $I_{d=3}(|\Psi\rangle_{\text{eig}}^3) = 1 + \sqrt{11}/3 \approx 2.9149$, which is larger than $I_{d=3}(|\Psi\rangle_{\text{mes}}^3) \approx 2.8729$; for $d=4$, one has $B_0=0$, $B_1=2\sqrt{4-2\sqrt{2}}/3$, $B_2=2\sqrt{2}/3$, $B_3=2\sqrt{4+2\sqrt{2}}/3$, the eigenvector $|\Psi\rangle_{\text{eig}}^4 = \frac{1}{2\sqrt{2}} |00\rangle + (1/2 |11\rangle + a |22\rangle + |33\rangle)$, with $a = (\sqrt{2} + \sqrt{2} + \sqrt{8-3\sqrt{2}} + 4\sqrt{2-2\sqrt{2}} + 2\sqrt{2} - \sqrt{4+2\sqrt{2}})^{-1} = 0.73937$, corresponds to a maximal violation $I_{d=4}(|\Psi\rangle_{\text{eig}}^4) \approx 2\sqrt{2} + \sqrt{2} + \sqrt{8-3\sqrt{2}} + 4\sqrt{2-2\sqrt{2}} \approx 2.9727$, which is larger than $I_{d=4}(|\Psi\rangle_{\text{mes}}^4) \approx 2.8962$.

By diagonalizing exactly the matrix $\hat{B}_{\text{red}}^1$, we have obtained the real maximal violations $I_d(|\Psi\rangle_{\text{eig}})$ for two entangled qudits. The highest dimension that we have calculated is $d=8000$, the corresponding $I_{d=8000}(|\Psi\rangle_{\text{eig}}) \approx 3.7362$. In Fig. 1, one may observe that $I_d(|\Psi\rangle_{\text{eig}})$ increases with dimension $d$. Based on the data of $I_d(|\Psi\rangle_{\text{eig}})$ from $d=2$ to $d=8000$, one has an empirical formula fitting $I_d(|\Psi\rangle_{\text{eig}})$ numerically to the dimension $d$:

$$I_d^\text{rough}(|\Psi\rangle_{\text{eig}}) = 3.9132 - 1.2891x^{-0.2226}$$

(14)

from which one can see that $I_d^\text{rough}(|\Psi\rangle_{\text{eig}}) \approx 3.9132$ is a coarse-grained limit of the maximal violation for the CGLMP inequality when $d$ tends to infinity.

Analysis of the eigenvectors $|\Psi\rangle_{\text{eig}}$ shows that these eigenvectors numerically satisfy some general properties: for instance, $|\Psi\rangle_{\text{eig}} = \Sigma_{j=0}^{d-1} a_{j\sigma} (|j\rangle |j\sigma\rangle)$ with maximal eigenvalue has the following symmetric properties for the coefficients: $a_j = a_{d-j}$ and $a_0; a_1; a_2; a_3; \cdots = 1; 1/\sqrt{2}; 1/\sqrt{3}; \cdots$ for large $d$. Thus, we may approximate a family of elegant entangled states

$$|\Psi\rangle_{\text{app}} = \Sigma_{j=0}^{d-1} a_{j\sigma} (|j\rangle |j\sigma\rangle), \quad a_{j\sigma} = \frac{1}{\sqrt{N}} \frac{1}{\sqrt{(j+1)(d-j)}}$$

(15)

whose corresponding Bell expressions $I_d(|\Psi\rangle_{\text{app}})$ can be calculated as follows;

\[ \text{FIG. 1. (Color online) Variations of } I_d(|\Psi\rangle) \text{ with increasing dimension } d (2 \leq d \leq 8000). \]
It is found that $I_d(|\Psi\rangle_{\text{app}})$ are close to the actual ones $I_d(|\Psi\rangle_{\text{eig}})$. For example, for $d=3$, $I_{d=3}(|\Psi\rangle_{\text{app}}) = 2.90909$ and $I_{d=3}(|\Psi\rangle_{\text{eig}}) = 2.9149$; for $d=4$, $I_{d=4}(|\Psi\rangle_{\text{app}}) = 2.96466$ and $I_{d=4}(|\Psi\rangle_{\text{eig}}) = 2.9727$. For $d=8000$, the error rate between $I_d(|\Psi\rangle_{\text{eig}}) = 3.7362$ and $I_d(|\Psi\rangle_{\text{app}}) = 3.70829$ is only about 0.745%. We have also drawn the corresponding curves of $I_d(|\Psi\rangle_{\text{app}})$ and $I_d(|\Psi\rangle_{\text{max}})$ in Fig. 1. One may observe that for $d=50000$, $I_d(|\Psi\rangle_{\text{max}}) = 3.96981$ has almost reached the limit as shown in Eq. (4), and for $d=60000$, $I_d(|\Psi\rangle_{\text{app}}) = 3.8008$ has exceeded 3.8. Although we do not have the analytical form of the states $|\Psi\rangle_{\text{eig}}$ which maximally violate the CGLMP inequalities, we construct the entangled states $|\Psi\rangle_{\text{app}}$. The states $|\Psi\rangle_{\text{app}}$ may be of importance in many fields of quantum information theory since they are highly resistant to noise.

V. CONCLUSION

In summary, we have investigated the maximal violation of the CGLMP inequalities for two entangled qudits. The maximal violation occurs at the nonmaximally entangled state, which is the eigenvector of Bell operator with the maximal eigenvalue. The maximal violations increase with growing dimension $d$, coming close to 4 when $d$ approaches infinity. Experimental test has been performed to verify the CGLMP inequalities for the first few high-dimensional systems, and indeed there exist nonmaximally entangled states that violate these inequalities more strongly than the maximal entangled ones [18]. Bell inequalities are applicable to quantum cryptography and quantum communication complexity [19], previous researches are mostly based on maximally entangled states. For stronger violation, it may be useful to generate the approximate state in this paper. Nevertheless, it may be significant and interesting to apply the elegant symmetric entangled states $|\Psi\rangle_{\text{app}}$ to quantum cryptography as well as other important fields of quantum information.

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