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How Does A Mathematician Do Research?

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Abstract

The aim of this paper is to unfold the thinking processes which lie at the heart of mathematics research. The understanding of these processes which seem to come naturally to mathematicians would certainly shed useful light on how to encourage, develop and foster these thinking skills in school children as well as in adults.

Introduction

Almost everyone agrees that mathematics is an important discipline -- it is needed in most branches of science and technology, and an ever greater number of professions cannot be exercised without some knowledge of it. Nevertheless, if one asks "What is a mathematician?" or "What does a mathematician do?" very often one will just get a preposterous answer.

We shall define a *mathematician* as a person who does research in mathematics, i.e., someone who creates new mathematics. When mathematicians have done some mathematics, they publish their results in journals. But what they write in their papers is mathematics, and is not about *how* they search, interpret and construct ideas. In other words, a mathematician's paper is not an exposition about the *thinking processes* involved in order to get the results. Consequently, outside of the closed circle of professional mathematicians, little is known of the true nature of mathematics research. With the use of several examples, in this paper I shall attempt to examine and illustrate the thinking that goes on in a mathematician's mind while he is engaging in research. It is hoped that the unfolding, which leads to the understanding, of the thinking processes that lie at the heart of mathematics research, would be able to shed useful light on how to encourage, develop and foster these thinking skills in school children as well as in adults.

How to Enter Mathematics Research?

To enter mathematics research, a mathematician needs input. He may start with reading what other mathematicians have done, which are usually documented in the form of journal papers and monographs. While immersing himself in others' discoveries, a *monitor* inside him is automatically switched on. What does this monitor do?

(i) It evaluates and checks others' ideas and arguments to see if they are correct in both logical as well as computational aspects.

(ii) It looks out for areas which have not been attended to. In other words, are there stones unturned?

(iii) It recognizes possibility of generalization / extension.

(iv) It identifies special cases in which further development is possible.

(v) It poses new questions which are motivated / stimulated by whatever is going on.

To cite a few examples, the oscillation of the differential equation

$$y'' + h(t)g(y) = 0 \tag{1}$$

has been studied by many mathematicians. A more general equation than (1), namely,

$$(a(t)y')' + Q(t,y) = P(t,y,y') \tag{2}$$

is being tackled later on. It is noted that equation (2) reduces to (1) when $a(t) = 1$, $Q(t,y) = h(t)g(y)$ and $P(t,y,y') = 0$.

Recently, Wong and Agarwal (1995) have discussed the differential equation

$$(a(t)(y')^{\sigma})' + Q(t,y) = P(t,y,y'), \tag{3}$$

where $0 < \sigma = p/q$ with p, q both odd integers, or p even and q odd integers. It is clear that when $\sigma = 1$, equation (3) becomes (2). Therefore, equation (3) is a generalization of both (1) and (2).

Further, motivated by equation (3), Wong and Agarwal (1996b) also consider the quasilinear differential equation

$$(a(t)|y'|^{\alpha-1}y')' + Q(t,y) = P(t,y,y') \tag{4}$$

where $\alpha > 0$. It is observed that equations (3) and (4) are different -- they are not particular cases of each other. Also, equation (4) reduces to (2) when $\alpha = 1$. Hence, once again equation (4) is *another* generalization of both (1) and (2).

A new dimension, namely the *discrete* analog of equations (1)--(4), has also been discussed in Agarwal and Wong (1997).

As a second example, some mathematicians have considered the oscillation of the difference equation

$$\Delta^{2m}y(k) + q(k)f(y(k - \sigma_k)) = 0 \tag{5}$$

Wong and Agarwal (1996a) have generalized equation (5) to

$$\begin{aligned} |\Delta^n y(k)|^{\alpha-1}\Delta^n y(k) &+ Q(k, y(k - \sigma_k), \Delta y(k - \sigma_k), \ldots, \Delta^{n-2}y(k - \sigma_k)) \\ &= P(k, y(k - \sigma_k), \Delta y(k - \sigma_k), \ldots, \Delta^{n-1}y(k - \sigma_k)) \end{aligned} \tag{6}$$

where $\alpha > 0$. It is noted that equation (5) is a particular case of (6) when $\alpha = 1$, $n = 2m$,

$$Q(k, y(k - \sigma_k), \Delta y(k - \sigma_k), \ldots, \Delta^{n-2}y(k - \sigma_k)) = q(k)f(y(k - \sigma_k))$$

and

$$P(k, y(k - \sigma_k), \Delta y(k - \sigma_k), \ldots, \Delta^{n-1}y(k - \sigma_k)) = 0.$$

In addition, motivated by the oscillation results on difference equations, Agarwal and Wong (1997) have also investigated the asymptotic behaviour of solutions of *partial* difference equations.

How to Pose a Problem?

Now that the mathematician has got an idea of what he is going to do, he will pose his problem. He has to clarify what he *KNOWS* and what he *WANTS*. To ascertain what he *KNOWS*, the mathematician may have to take a particular case to

discover what is really involved. Further, he may have to reflect on his experience to see if he has ever come across anything similar, or of an analogous structure. To have a better idea of what he *WANTS*, the mathematician may have to classify and sort information, and to examine a specific scenario to discover what the real question is. After the mathematician is clear of what he *KNOWS* and what he *WANTS*, he will need to *INTRODUCE* notations, representations and assumptions.

As an example, *cubic* spline interpolation is well known in the literature. Here, given a partition $\rho : x_0 < x_1 < \ldots < x_N$, the cubic spline interpolate s(x) of a function f(x) satisfies

$$\begin{aligned} &s(x_i) = f(x_i),\ 0 \le i \le N \\ &s'(x_0) = f'(x_0), \\ &s'(x_N) = f'(x_N). \end{aligned} \tag{7}$$

Suppose we want to consider *quintic* spline interpolation. Then, analogous to the interpolating conditions (7), we may have the following two types of conditions

$$\begin{aligned} &s(x_i) = f(x_i),\ 0 \le i \le N \\ &s'(x_i) = f'(x_i),\ 0 \le i \le N \\ &s''(x_0) = f''(x_0), \\ &s''(x_N) = f''(x_N) \end{aligned} \tag{8}$$

or

$$\begin{aligned} &s(x_i) = f(x_i),\ 0 \le i \le N \\ &s'(x_0) = f'(x_0), \\ &s'(x_N) = f'(x_N), \\ &s''(x_0) = f''(x_0), \\ &s''(x_N) = f''(x_N). \end{aligned} \tag{9}$$

To unify (7)--(9), the following notations are introduced (Agarwal & Wong, 1993). Let $m \geq 2$ be an integer and $m \leq \tau \leq 2m - 2$. Given a partition ρ, the (2m-1)th degree spline interpolate $s_{m,\tau}(x)$ of a function $f(x)$ satisfies

$$s^{(k)}_{m,\tau}(x_i) = f^{(k)}(x_I); \quad 1 \leq i \leq N-1, 0 \leq k \leq 2m-2-\tau$$
$$i = 0, N, 0 \leq k \leq m-1. \tag{10}$$

We observe that the sets of conditions (7), (8) and (9) are actually (10) when $(m,\tau) = (2,2)$, (3,3) and (3,4) respectively.

How to Tackle a Problem?

(a) Specializing

Very often it is difficult to tackle the problem as it is. The mathematician will have to specialize, i.e., take a particular case and focus his attention on it. The problem is now simplified (to a large extent, sometimes) and it makes investigation easier.

By considering the particular case, the mathematician finds out more about what he *KNOWS*, what he *WANTS* and what he might sensibly *INTRODUCE*. Further, by examining the techniques used and the resolution of the particular case, the mathematician may uncover patterns that lead to a generalization.

(b) Conjecturing

By articulating the generalization, the mathematician comes up with *CONJECTURES*. A *conjecture* is something that appears reasonable, but whose truth has not yet been established.

So, at least the conjectures that the mathematician writes down must not contradict the particular case. To check that the conjectures are reasonable, the mathematician may generalize the particular case a little or look at another particular case, ~~and~~ see if the conjectures still hold. It is at this point that sometimes the conjectures are found to be false and need modification.

The processes of making conjectures and checking conjectures go back and forth. From these processes, the mathematician gets a sense of why the conjectures are right, or how to modify them on new particular cases.

In short, a conjecture is an informal guess about a possible pattern or regularity which might explain *WHAT* is puzzling in a problem. Once formulated, the conjecture is investigated to see whether it must be modified. Articulating, testing and modifying conjectures actually form the backbone of a resolution.

(c) **Justifying**

Whatever the number of particular cases that the conjectures satisfy, they remain as conjectures until a justification, or in the mathematicians' language, a *proof* is found. So, the process of seeking *WHAT* is true and articulating it as a conjecture is swiftly followed by the process of seeking *WHY* it is true.

The answer to *WHY* is a structure which *links* what the mathematician knows to what he has conjectured. His argument will be an exposition of that link. Once he has found that link, it is a matter of stating it carefully and clearly.

As with conjuctures about *WHAT*, the mathematician's conjectures about *WHY* may need several modifications. It is frequent that the first version of proof is tried out on paper and is discussed with a colleague. After various weaknesses have been probed and repaired, by which time the proof has already been reviewed many times, a paper is written and is then submitted for publication.

The paper is read critically by at least one expert in that field. Only upon the recommendation of the referee will the paper be published. Moreover, it is frequent that the referee requires the paper to be revised before it is finally accepted for publication. As long as the published proof convinces the mathematical community, it is deemed to constitute a justification. Even so, it sometimes happens that a result is published and accepted, yet years later a fallacy is found.

The final version of the paper tends to be abstract and formal as it tries to be precise and to avoid hidden assumptions. As a consequence, the reader will have to work harder at decoding in order to understand what is going on. For example, denoting $[1,N] = \{1,2,\ldots,N\}$ where $N \geq 1$ is an integer, the set Ω given by

$$\Omega = [1,N_1] \times [1,N_2] \times \ldots \times [1,N_n]$$

has boundary $\partial\Omega$ defined by

$$\partial\Omega = \bigcup_{\ell=1}^{n}\{(i_1,\ldots,i_{\ell-1},0,i_{\ell+1},\ldots,i_n),(i_1,\ldots,i_{\ell-1},N_\ell+1,i_{\ell+1},\ldots,i_n);$$
$$i_k \in [1,N_k], 1 \le k \le n\}.$$

Upon decoding the set $\partial\Omega$ actually consists of the following n-tuples

$(0,i_2,\ldots,i_n)$, $(N_1+1,i_2,\ldots,i_n)$; $\quad i_k \in [1,N_k]$, $k = 2,3,\ldots,n$

$(i_1,0,i_3,\ldots i_n)$, $(i_1,N_2+1,i_3,\ldots,i_n)$; $\quad i_k \in [1,N_k]$, $k = 1,3,4,\ldots,n$

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$(i_1,\ldots,i_{n-1},0)$, $(i_1,\ldots,i_{n-1},N_n+1)$; $\quad i_k \in [1,N_k]$, $k = 1,2,\ldots,n-1$.

Further, in the journal papers we normally see that a theorem is stated first and is followed by its proof, and one often wonders how the conditions in the theorem come about. Those conditions actually arise from the proof itself. It is usual that in the mathematician's first draft of working, the proof actually comes before the statement of the theorem.

What to do if Problem remains Unsolved?

Suppose that after specializing and examining the arguments many times, the problem still remains unsolved. What will the mathematician do then? Well, he can

(i) give up the problem;
(ii) put it aside for a while, with intention of entering the problem again at a later date; or
(iii) continue working on it.

If the mathematician chooses option (ii) or (iii), then he will probably distill the problem to a sharp question that he will always hold in his mind and mull over. Another thing that the mathematician can do is to alter the original problem, for example, to impose more conditions, or to change some of the assumptions, until some progress is possible. It is the mathematician's previlege that if he cannot solve his current problem, he may alter it until it is solvable. Needless to say, the alteration must not reduce the problem to one that is trivial.

Conclusion

In summary, the thinking processes that a mathematician used are specializing, generalizing, conjecturing and justifying. These thinking processes cannot be rushed and they require ample time. Moreover, it is possible to improve these thinking skills through practice. It is clear that thinking involves both pain and pleasure : the pain of incomprehension and struggle to understand, and the pleasure of insight and convincing arguments. At the end of the day, it is the one who perseveres that will succeed.

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