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SEEING THROUGH STUDENTS' EYES: THE BEST-HELP STRATEGY FOR PATTERN GENERALISATION

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This paper reports on the choices of 215 Singapore secondary school students' best-help strategy when presented in a questionnaire to one linear and one quadratic generalising task. The findings revealed different preferences of strategy between two groups of students–the more able and the less able–in that a figural strategy was favoured by the former while the latter tended to prefer a numerical strategy. There were no gender differences in student choices of strategy amongst the more able students and amongst the less able students. But a significant difference was present in both tasks between more able and less able girls. The findings are discussed and some implications for teaching drawn.

Keywords*: Pattern generalisation, generalising strategy, students' choice*

BACKGROUND AND THEORETICAL FRAMEWORK

Pattern generalising tasks are a common part of school mathematics in many countries. A typical generalising task involves, for instance, identifying a numerical pattern, extending the pattern to make a near and far generalisation, and articulating the functional relationship underpinning the pattern using words or symbols. Such tasks are a powerful vehicle not only for introducing the notion of variables (Mason, 1996) but also for developing two core aspects of algebraic thinking: the concept of relationships amongst quantities like between inputs and outputs (Radford, 2008) and the idea of expressing an explicit rule using letters to represent numerical values of the outputs (Kaput, 2008).

The wealth of research on students' generalising strategies and reasoning when they deal with pattern generalising tasks suggests that students use a variety of strategies to derive the rule between the term and its position in the pattern (Drury, 2007; Lannin, 2005; Rivera & Becker, 2008). For instance, Rivera and Becker (2008) identified three types of strategy that students might employ to derive the rule: (1) *numerical*, which uses only cues established from the pattern when listed as a sequence of numbers or tabulated in a table to derive the rule, (2) *figural,* which exploits visual cues established directly from the structure of configurations used to depict the pattern, and (3) a combination of both these approaches.

Different types of *numerical* strategy have been described. Bezuszka and Kenney (2008) identified three that involve recursion: (1) *comparison*, where the terms in a given number sequence are compared with corresponding terms of another sequence whose rule is already known, (2) *repeated substitution*, where each subsequent term in a number sequence is expressed in terms of the immediate term preceding it, and (3) *the method of differences*, also known in Mathematics as finite differences, which is an algorithm for finding explicit formulae when the pattern derived from a polynomial.

Likewise, different categories of *figural* strategy have also been identified. Rivera and Becker (2008) distinguished between (1) *constructive generalisation*, which occurs when the diagram given in a generalising task is viewed as a composite diagram made up of non-overlapping components and the rule is directly expressed as a sum of the various sub-components, and (2) *deconstructive generalisation*, which happens when the diagram is visualised as being made up of components that overlap, and the rule is expressed by separately counting each component of the diagram and then subtracting any overlapping parts. Chua and Hoyles (2010a) introduced two further strategies into the classification scheme developed by Rivera and Becker (2008). One involves rearranging one or more components of the original diagram to form something more familiar. The newly-rearranged configuration highlights the structure of the pattern which then facilitates the rule construction. The other entails viewing the original diagram as part of a larger composite configuration, from which the rule is generated by subtracting the sub-components from this composite configuration.

In some recent studies, students were given "different–looking" rules that described the same underpinning pattern and asked to justify how these rules could all be equivalent to one another (Drury, 2007; Rivera & Becker, 2008). However, none of these studies went further to ask students for the kind of strategy that they believed would best help them to construct those rules. Thus the present study sought to address this gap by examining secondary school student choices of generalising strategy to determine what they would judge as the most helpful strategy to establish the functional rule for deriving any term in the pattern. It seeks to answer some of the following questions: Which strategies did students believe would best help them to work out the rule? Would there be a difference between the best-help strategies chosen by more able students and those chosen by less able students? Would the best-help strategies chosen when the pattern was quadratic be the same as that chosen when the pattern was linear?

Data from the Third International Mathematics and Science Study (TIMSS) in 2003 and 2007 showed that, in Singapore secondary schools, girls had outperformed boys in pattern generalising tasks. In TIMSS-2003, 76% of the girls gave the correct number of matchsticks needed to make a certain figure, as compared to 70% of the boys. Similarly, when given a generic figure of a row of four squares formed using 13 matchsticks in TIMSS-2007, 42% of the girls and 41% of the boys correctly answered the number of squares in a row that could be made this way using 73 matchsticks. Why was the success rate for girls higher than that for boys? How did the girls figure out the answers? What were their generalising strategies? Was there a difference between the girls' generalising strategies and the boys'? These are also some questions that we seek to answer. So it is hoped that the findings of this present study could provide valuable insights for researchers, teachers, teacher educators and curriculum developers.

METHODS

Student data were collected through a questionnaire administered to 215 Secondary Two students (aged 14 years) from a secondary school in Singapore. 108 of the students (56 girls, 52 boys) came from the Express course and 107 from the Normal (Academic) course (52 girls, 55 boys). The students were placed in these courses based on their performance in a national examination taken at the end of their primary education when they were 12 years old. The Express students were considered academically more able than the Normal (Academic) students. The participating students in each course were from three intact classes, selected by the school.

Having learnt the topic of number patterns in the Singapore mathematics curriculum before participating in this study, these students should be able to continue any pattern when presented with a few numerical terms or diagrams, make a near and far generalisation and establish the functional rule in the form of an algebraic expression for predicting any term. Further, they should also be more familiar in dealing with linear than non-linear patterns since the latter are less commonly featured in mathematics textbooks.

Before administering the questionnaire, a worksheet comprising the two generalising tasks used in the questionnaire was distributed to every student. The two tasks, *High Chair* and *Christmas Party Decoration*, are presented in Figures 1 and 2 respectively below. The first task involves a linear rule whereas the latter involves a quadratic rule. These two tasks differ from the typical textbook tasks in that they are less structured, thus allowing more room for the students to explore the pattern structure. The students were asked individually to work out the functional rules in terms of the size number using any strategy with which they were familiar. The purpose was to familiarise them with these tasks so that they could better understand the questionnaire tasks that they had to do later.

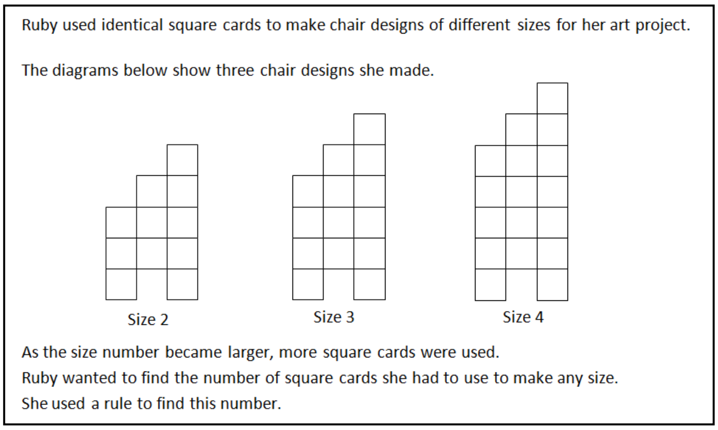

Figure 1. High Chair

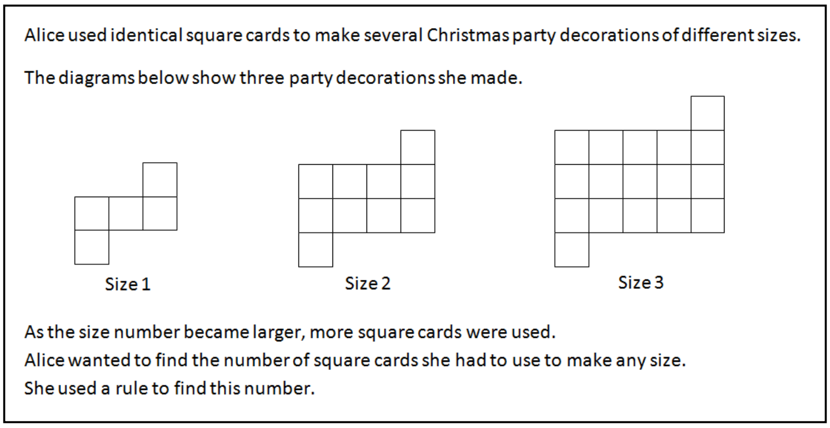

Figure 2. Christmas Party Decoration

The questionnaire containing these two generalising tasks, each accompanied by four possible student solutions, was subsequently distributed to each student. Figures 3 and 4 below show the four distinct student solutions for the two respective tasks.

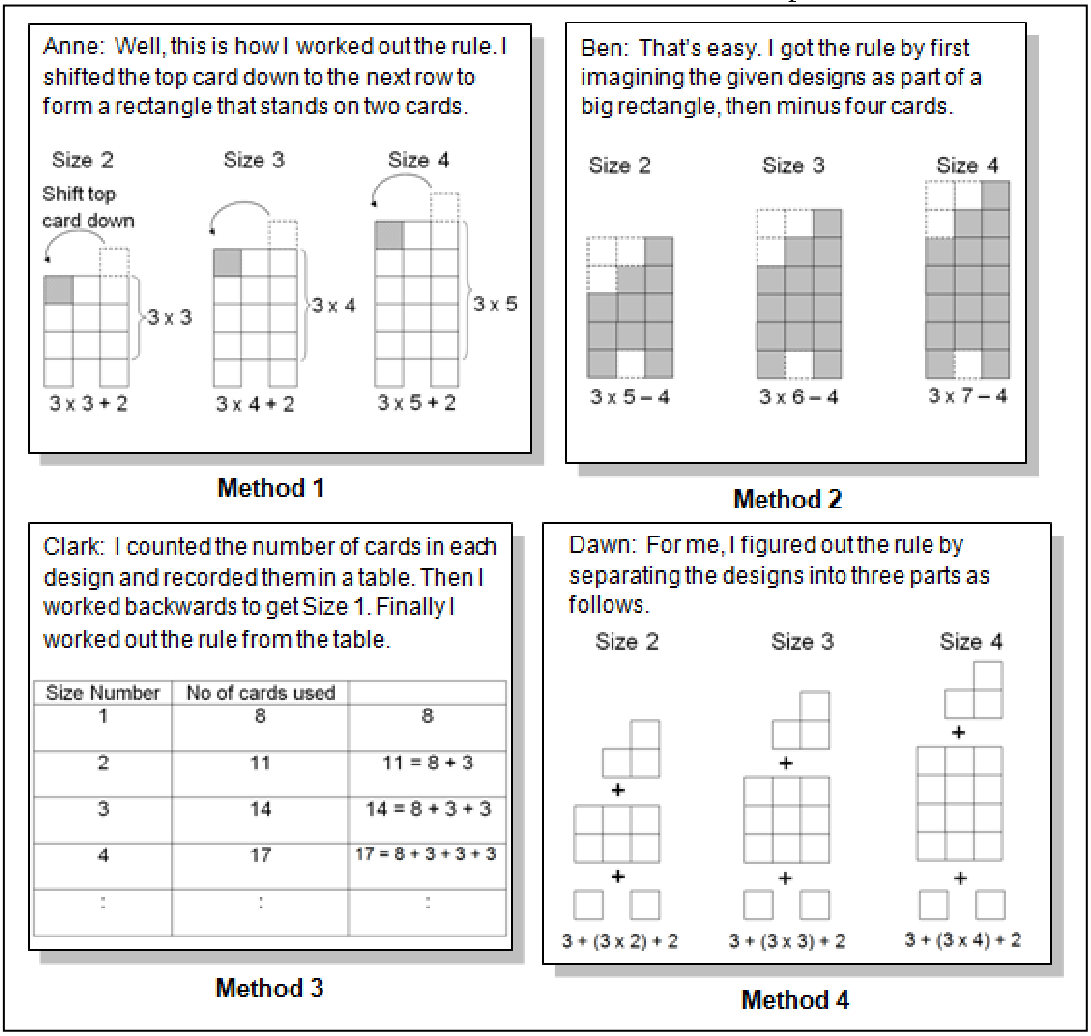

Figure 3. Four different methods for High Chair

Set in a context of a discussion amongst four students, each student solution represented a different way of deriving the rule based on the classification scheme described above. To illustrate using High Chair: Method 1 involves rearranging the original configurations into something more familiar (S3); Method 2 involves viewing the original configurations as part of a larger rectangle with four missing cards (S4); Method 3 uses a numerical strategy (S1) known as the repeated substitution strategy (Bezuszka & Kenney, 2008); and Method 4 employs a constructive strategy (S2). For Christmas Party Decoration, Methods 1, 2, 3 and 4 correspond to S4, S2, S3 and S1 respectively. The students were asked to choose the method that they believed would best help them to construct the functional rule, as well as to provide justifications for their choices of the best-help method.

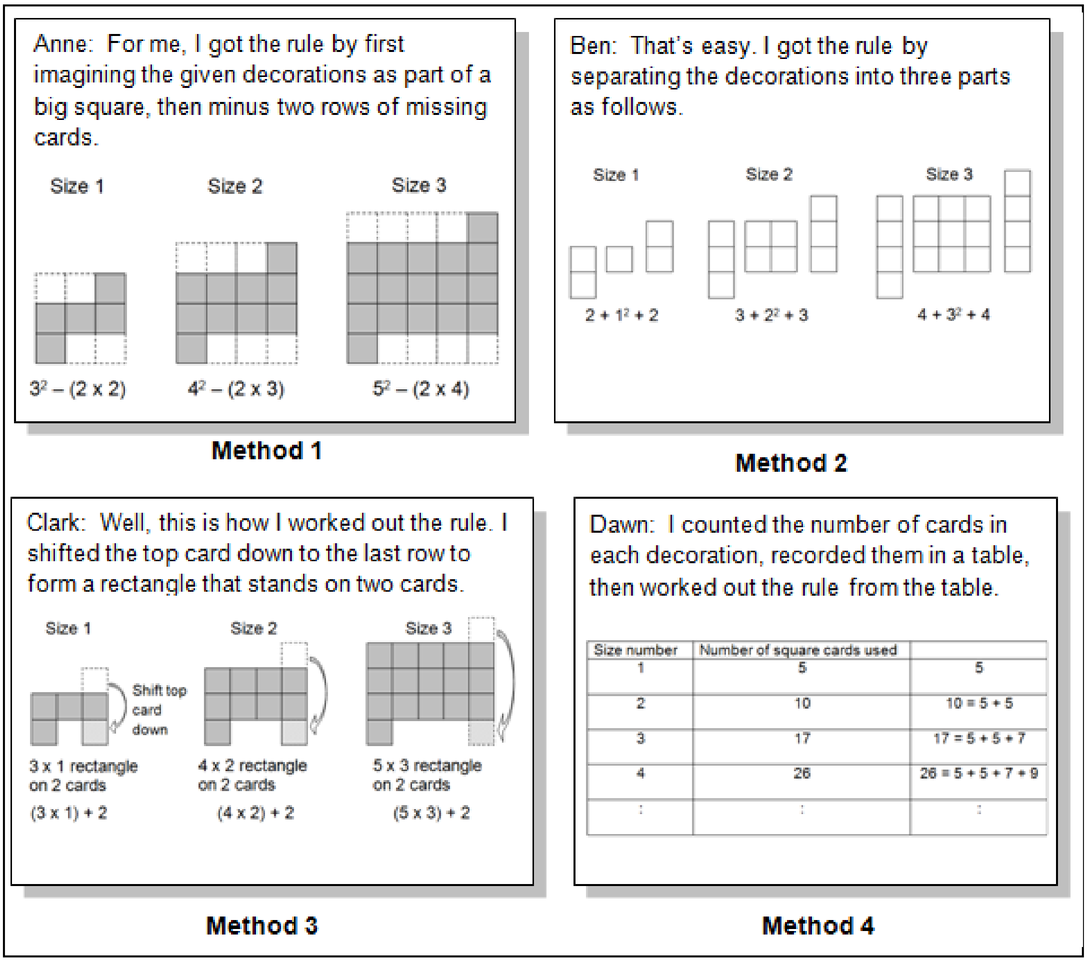

Figure 4. Four different methods for Christmas Party Decoration

All 215 questionnaires were collected and responses analysed. The frequencies of the four student methods by gender and course for each generalising task were then counted. χ^2-tests were conducted to determine whether there were any significant differences in students' choices of best-help generalising strategies for each generalising task across (a) gender, and (b) course. The student justifications were also analysed to gain a better understanding of the reasons behind the student choices of best-help strategies.

RESULTS

This section presents the findings to the three questions that guided this study.

1. *Which strategies did students believe would best help them to work out the linear rule for High Chair?*

Table 1: Student Choices of Best-help Method for High Chair

		Best-help Method				
High Chair		S1	S2	S3	S4	Total
Express Students (n = 108)						
	Girls	8	26	17	5	56
	Boys	12	14	20	6	52
	Total	20	40	37	11	108
Normal (Academic) students (n = 107)						
	Girls	18	13	11	10	52
	Boys	16	12	16	11	55
	Total	34	25	27	21	107

S1: numerical; S2: constructive; S3: rearranging the original configurations; S4: viewing the original configurations as part of a larger rectangle

Table 1 shows the distribution of student choices of best-help strategy for the linear generalising task (*High Chair*) for this sample. It indicates that by far the most popular choices of best-help strategies amongst the Express students were the figural strategies S2 (37%) and S3 (34%). S2 was favoured by the girls whereas S3 by the boys. The numerical strategy S1 was selected by 19% of the Express students, which is half the number who chose S2. The rest of about 10% of the Express students selected S4. On the other hand, the percentages of Normal (Academic) students across the four methods were fairly close. S1 was the most popular method amongst the girls and boys, with 32% believing that it would best help them to derive the rule. This was then followed in descending order by S3 (25%), S2 (23%) and S4 (20%).

2. *Which strategies did students believe would best help them to work out the quadratic rule for Christmas Party Decoration?*

Table 2: Student Choices of Best-help Method for Christmas Party Decoration

		Best-help Method				
Christmas Party Decoration		S1	S2	S3	S4	Total
Express Students (n = 108)						
	Girls	6	28	13	9	56
	Boys	8	21	15	8	52
	Total	14	49	28	17	108
Normal (Academic) students (n = 107)						
	Girls	16	12	10	14	52
	Boys	11	22	9	13	55
	Total	27	34	19	27	107

S1: numerical; S2: constructive; S3: rearranging the original configurations; S4: viewing the original configurations as part of a larger rectangle

Table 2 shows the distribution of student choices of best-help strategy for the quadratic generalising task (Christmas Party Decoration). Again, S2 and S3 remained the clear favourites amongst the Express students, in that 49 (45%) and 28 (26%) students chose them respectively. This time, S2 was the top choice for both girls and boys. Of the remaining students, 17 (16%) picked S4 and 14 (13%) selected S1. Amongst the Normal (Academic) students, S2 was also a clear favourite, selected by 34 (32%) of them. However, it was preferred more by the boys. The girls, on the other hand, seemed to prefer S1, which was equally popular as S4 for the second choice, each with 25% of students. S3 (18%) was the least popular choice of best-help strategy amongst the four methods.

3. *Is there any difference between students' choices of best-help generalising strategies and student characteristics such as gender and course?*

Table 3: Student Choices of Best-help Strategy Across Gender and Course

Tasks		χ^2	df	p-value	phi (ϕ)
High Chair	between girls and boys in Exp course	4.592	3	.204	.204
	between girls and boys in N(A) course	1.048	3	.790	.790
	between Exp girls and N(A) girls	10.999	3	.012*	.319
	between Exp boys and N(A) boys	2.558	3	.465	.155
Christmas Party Decorations	between girls and boys in Exp course	1.341	3	.719	.111
	between girls and boys in N(A) course	3.876	3	.275	.275
	between Exp girls and N(A) girls	12.292	3	.006*	.337
	between Exp boys and N(A) boys	3.106	3	.376	.170

*Significant at $p < .05$; Exp for Express and N(A) for Normal (Academic)

Table 3 provides the descriptive statistics for the eight χ^2-tests conducted. As the table indicates, the χ^2-test demonstrated that the differences between girls and boys in both Express and Normal (Academic) courses were not statistically significant for both tasks. Further, there were also no significant differences between Express boys and Normal (Academic) boys for High Chair ($\chi^2 = 2.558$, df = 3, $p = .465$) and for Christmas Party Decorations ($\chi^2 = 3.106$, df = 3, $p = .376$). However, significant differences were present between Express girls and Normal (Academic) girls for High Chair ($\chi^2 = 10.999$, df = 3, $p < .05$) and for Christmas Party Decorations ($\chi^2 = 12.292$, df = 3, $p < .05$). The effect sizes were .319 for High Chair and .337 for Christmas Party Decorations, both indicating medium effect according to Cohen's (1998) conventions.

DISCUSSION AND CONCLUSION

The findings revealed different preferences of best-help strategy between the more able Express and the less able Normal (Academic) students. A substantial majority of the Express students seemed to favour using a figural strategy to work out the functional rule for both the linear and quadratic generalising tasks. The two most popular choices of best-help strategy for each respective task were rather consistent. In particular, they preferred either to view a composite diagram as being made up of non-overlapping components (S2) or to rearrange one or more components of the original diagram to form a familiar shape (S3). According to the students' explanations, one reason for favouring a figural strategy is that the link between the size number and the number of cards used was more explicit, thus providing a means to easily recognise the structure of the pattern. To illustrate what the students meant, we consider Method 4 in High Chair, which employs the strategy S2. The configuration in Size 2 can be perceived as three columns of two cards with three cards above and another two cards below, Size 3 as three columns of three cards with three above and two below, and Size 4 as three columns of four cards with three above and two below. So what remains the same across all three configurations is the number of columns of cards in the middle block, the three cards above and the two cards below this block. What changes across the three configurations is then the number of cards in each column, which corresponds to the size number in this case. This way of viewing the configurations makes it easy for students to abstract from all three of them that Size *n* has three columns of *n* cards plus three above and two below, or $3n+5$ when expressed in symbols.

The most popular choice of best-help strategy for the Normal (Academic) students, on the other hand, varied between the two generalising tasks. For the linear task, the students preferred a *numerical* strategy (S1) whereas the *constructive* strategy (S2) was their top preference for the quadratic task although S1 was also fairly popular amongst a sizeable number of students. How might such a finding be explained? According to student justifications, the popularity of the numerical strategy lies in its simplicity for representing the changes across the different cases without having to draw any diagrams, thus making the workings easier to understand. In addition, some students commented that this strategy is well-organised and systematic for them to detect the pattern and then derive the rule by inductive reasoning. Furthermore, students might have also learnt from their teachers such a strategy, which is a clear favourite amongst the teachers in another of our studies (Chua & Hoyles, 2010b). In that study, 16 in-service secondary school teachers were asked to establish the functional rule for generalising tasks using the strategy that they would employ in their classroom teaching. From the analysis of the teachers' responses, 10 were classified as *numerical*, with over half of them showing workings similar to *Method 3* in High Chair. So it is reasonable to presume students to be familiar with the *numerical* strategy, which they take as their first choice.

These justifications for choosing the numerical strategy, however, seemed to hold for linear and *not* for quadratic tasks. This is because the numerical strategy, despite showing clearly how the number of cards used to create each configuration changes with the size number in an orderly tabular format, is not as straightforward as it appears to be for deriving the quadratic rule. Not surprisingly, it is evident that some students had underestimated the difficulty of deriving the quadratic rule using the numerical strategy and continued to opt for it. But some students might have noticed such difficulty and then decided to stay clear of it by picking a figural strategy instead. Thus, as Table 2 indicates, there was a greater preference for a figural strategy in the case of the quadratic task. So, students' choice of best-help strategy might depend on the type of rule underpinning the pattern. A linear rule tends to evoke the numerical strategy. But when the rule changes to a quadratic relationship, some students will switch to use a figural strategy when they realise that the numerical strategy is not helpful in deriving the quadratic rule.

The findings also showed that there were no gender differences in student choice of best-help strategy in both Express and Normal (Academic) courses. Additionally, there were also no differences in the choice of best-help strategy between the boys in the two courses. However, there was a significant difference in both tasks between the girls in the two courses. At the time of preparing this paper, the research is still on-going and so the reasons for the difference are not clear yet. But they would certainly seem to be worth investigating further to find out how far they might be teacher or curriculum-related.

Despite the limitations of the study as derived from one secondary school, we argue that the findings provide a window for teachers on the types of generalising strategies that might better facilitate student visualisation of the pattern structure. Such research-based knowledge has the potential for significant impact on the teaching and learning of number patterns. For instance, teachers, seeking an idea of what might be an appropriate generalising strategy to employ in class when demonstrating examples, could use the findings to help them make informed decisions, rather than relying on their beliefs about what students are capable of understanding and how best they will learn (Chua & Hoyles, 2010b). In addition, not only can teachers guide students to realise that there are *multiple* ways of seeing the structure of the pattern and that the various ways can lead to constructing "different-looking" but equivalent rules, they can also encourage students to justify how these rules can all be equivalent to one another – a crucial goal of generalisation advocated by many researchers (see Drury, 2007; Noss et al., 2009; Rivera & Becker, 2008). By aligning their choices of generalising strategies with students' preferences and promoting classroom discussion about the various ways of constructing the equivalent rules, teachers can improve the efficacy of teaching and learning outcomes.

Finally, how can one tell if the chosen method really helps the students? So what is left after students had made their choices of best-help generalising methods in the survey is to investigate the efficacy of the chosen method on their construction of the functional rule. The next step of the present study would, therefore, involve students developing the functional rule using the best-help method they have chosen. As described in the *Methods* section, the students had to attempt the generalising tasks first before taking the survey. One difficulty students face when making generalisations is partly due to an ignorance of appropriate generalising strategies (Moss & Beatty, 2006). So an objective here is to find out whether students who were unsuccessful in deriving the rule prior to the survey are *now* able to do so using their choice of best-help method. Another objective is to probe *why* some students, in the survey, opted for another method different from the one used earlier in their rule construction. By seeing through the students' eyes, valuable insights about their learning experience may be gleaned. In particular, it would illuminate how students develop awareness of the pattern structure, think and reason in the abstraction process, as well as make generalisations when establishing the functional rule.

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