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PRE-SERVICE TEACHERS' USE OF SYMMETRY OF QUADRATIC GRAPHS IN PROBLEM SOLVING

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This paper reports on the performance of pre-service mathematics teachers with regards to the use of symmetry in problem solving. The current study reveals that pre-service teachers do not make use of symmetry as their main problem-solving tool, even in situations where symmetry is the obvious notion to consider. In addition, a quantitative comparison of the effectiveness of problem solving among approaches (those relying on symmetry versus those relying on other conventional methods) is reported herein. This formal comparison validates the opinion that active usage of symmetry in problem solving significantly enhances the chance of solving non-routine problems.

Keywords : Problem solving, Symmetry, Quadratic graphs, Pre-service teachers

INTRODUCTION

In Singapore schools, the notion of symmetry is first introduced in the lower primary syllabus where children of age 9 years old are expected to locate and draw lines of symmetry of a given figure, such as a square. In the upper primary, students built upon this knowledge in their study of angles, for instance, it is expected of a Primary 4 student to obtain the size of the acute angle made by the diagonal of a square with its side as half that of the right angle, i.e., 45°. Moving up to the secondary level, symmetry is invoked primarily in connection with geometrical properties of congruent figures as well as the graphs of quadratic functions. It is clear that symmetry in itself *is not a fundamental concept* in the Singapore Mathematics Syllabus; at least not as important as mensuration concepts, such as area and volume, just to name one example. However, symmetry has long been hailed as one of the most powerful and commonly-used problem-solving tools by mathematicians (Weyl, 1952; Pólya, 1981; Schoenfeld, 1985; Hilton & Pedersen; 1986; Dreyfus & Eisenberg, 1990).The aforementioned discussion about symmetry compels us to raise the following questions:

(1) Does the lack of emphasis on symmetry in the current Singapore Mathematics Syllabus significantly handicap the students' and teachers' problem solving ability in mathematics?

(2) To what extent is symmetry perceived as one of the heuristics or cognitive resources in problem solving, and how often is it invoked?

Incidentally, the lack of emphasis of symmetry in the Israeli national mathematics curriculum was reported, and its effects studied in detail, by Leikin, Berman, and Zaslavsky (2000) as well as Leikin (2003). These studies reported that the concept of symmetry is taught only in connection with the graphs of quadratic functions in the Israeli mathematics syllabus, and this lack of emphasis of symmetry has resulted in the situation that symmetry is now rarely used in problem solving at the secondary level. Even worse was the revelation that Israeli mathematics teachers not only viewed problem solving via symmetry as non-rigorous (i.e., as compared to conventional methods), but also were increasingly ignorant of the importance of symmetry as an elegant and convenient problem-solving tool in mathematics.

This paper takes a first step towards investigating whether Singapore school teachers are sharing the same fate as their Israeli counterpart, e.g., How often do teachers employ symmetry in problem solving? Are the chances of successful problem solving dampened if symmetry is not one of the many problem solving heuristics considered or used in the problem solving process?

RATIONALE AND METHODOLOGY

The National Institute of Education (NIE) is the sole teacher-training institute in Singapore, where courses such as Postgraduate Diploma in Education and Undergraduate Degree (in Education) are conducted for pre-service teachers. These programs aim to equip the trainee teachers with both the content and pedagogical knowledge so that they can function effectively as classroom teachers. It is mandatory for all teachers in Singapore government-based schools to be trained in NIE. Problem solving, being the central theme of the Singapore Mathematic Curriculum, is a key component in the curricula study of these courses. The Pólya's problem-solving model, comprising of the four strategies (1) Understand the problem; (2) Devise a plan; (3) Implement the plan; (4) Check and extend, is the main framework taught. Additionally, students are introduced to 13 different heuristics which are rules of the thumb for discovering the solution to a problem (Toh et al, 2008, p.13).

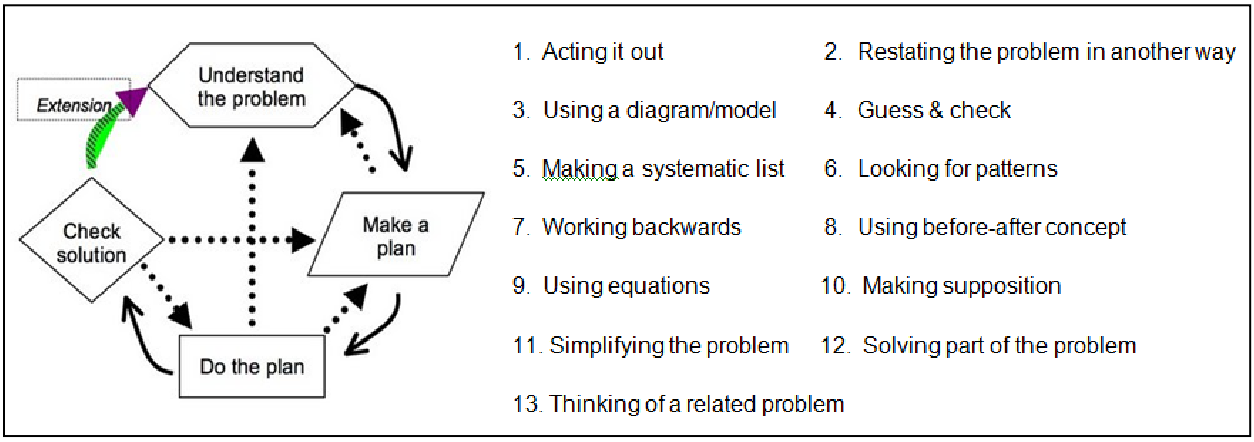

Figure 1 : G. Polya's problem solving model and the 13 problem solving heuristics

Figure 1 above depicts the Polya's problem solving model (left), and the 13 problem solving heuristics (right). Since problem-solving expertise is strongly connected to teachers' content knowledge (e.g., Polya, 1963; Silver & Marshall, 1990; Yerushalmy, Chazan & Gordon, 1990), it is best to minimize the potential interference of (the lack of) content knowledge on the outcome of the study by targeting the experiment to a specific group of teachers: *the pre-service teachers*. As such, pre-service teachers who participated in the study, had just completed or were in the process of completing their undergraduate studies in mathematics (or a mathematics-related discipline), and so the content knowledge was still fresh in their minds by virtue of the principle of recency. Thus, any significant deficiency in the use of symmetry during problem solving observed for this group of subjects could not have been due to a lack of content knowledge.

A sample of 96 pre-service teachers (they had been told in advance that the activity would not count towards their grades for the course they were taking), who had already received prior training in problem solving using the Pólya's model and the 13 heuristics, were given 15 minutes to solve the following problem:

Item 1
A quadratic curve $y = ax^2 + bx + c$ passes through two points (–2,4) and (4,4). Find the range of values of *a* such that the curve has a minimum point above the *x*-axis.

The design of the problem was intentional. In order to be able to draw valid parallel conclusions in relation to the Israeli context (as reported in Leikin, 2003), the mathematical topic of choice *had to be* the graphs of quadratic functions and their symmetrical properties. The mathematical concepts of "quadratic curve", "Cartesian coordinates-system", "range of values" and "minimum points" are familiar to all Secondary 3 students in Singapore. Moreover, the problem had been phrased in a way that was very similar to frequent examination items for this topic, such as the one shown below:

Find the range of values of a for which the quadratic expression $(a-2)x^2 - ax + (2a+3)$ is positive for all real values of *x*.

Examining both the explicit and implicit requirements of the item immediately reveals its non-routine nature. Since only two points are supplied for the interpolating quadratic curve, the Lagrange Interpolating Polynomial Theorem dictates the degree of freedom in this problem to be 1, i.e., one of the quantities (amongst *a*, *b* and *c*) is free to vary. The constraint that the minimum point of the curve must lie above the *x*-axis further imposes on the freedom of that varying quantity.

$\boldsymbol{y = ax^2 + bx + c}$ and Symmetry

The curve $y = ax^2 + bx + c$ is a parabola that belongs to the general family of conics and is geometrically defined as the locus of points equidistant from a fixed line (directrix) and a fixed point (focus). Other than the vertex (V) which lies on the axis of the parabola, there are always two points symmetrically placed with respect to the axis for which the above locus conditions apply. From Figure 2, we can note that $\frac{\mathrm{P1F}}{\mathrm{P1D1}} = \frac{\mathrm{P2F}}{\mathrm{P2D2}}$, where F is the focus, P1 and P2 are points symmetrically placed on the parabola, and D1 and D2 are on the directrix.

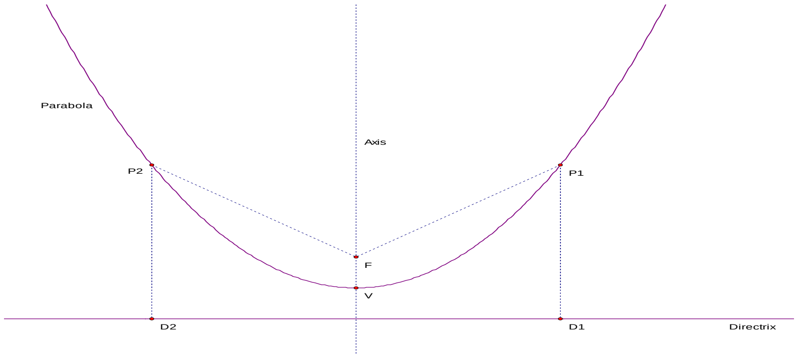

Figure 2. Symmetry in the parabola

Thus, apprehending the bilateral symmetry in a parabola involves identifying the axis and the points symmetrically placed on each side of the axis. Accordingly, knowing points symmetrically placed on the parabola, the use of symmetry involves identifying the axis by finding the perpendicular bisector of the line joining any two pairs of symmetrically placed points on the parabola. At the school level in Singapore, students are not expected to know the terms "conics", "directrix" or "focus" of a parabola. However, the bilateral symmetry about the axis or line of symmetry is taught to students.

On the other hand, the general equation $y = ax^2 + bx + c$ can also be written in the form $y = a(x + \frac{b}{2a})^2 + \frac{4ac-b^2}{4a}$. Simple observation of this equation highlights the algebraic symmetry that replacing $x + \frac{b}{2a}$ by $-x - \frac{b}{2a}$ will not change the equation. What does this mean? Since $x = -\frac{b}{2a}$ is the equation of axis of the parabola or more specifically the equation of the line of symmetry, for any point $P(x, y)$ on the parabola, $x + \frac{b}{2a}$ represents the distance from the axis of the parabola. Thus, the point Q symmetrically placed on the parabola will be twice this distance from P. Accordingly, the x-coordinate of Q is $x - 2(x + \frac{b}{2a})$ or $-x - \frac{b}{a}$, which satisfies the equation of the parabola $y = a(x + \frac{b}{2a})^2 + \frac{4ac-b^2}{4a}$, as it yields $-x - \frac{b}{2a}$ inside the brackets. Although this algebraic discussion may seem hard for the average student, we anticipate that replacing the parameters a, b and c by specific numbers can be an easier option.

INSTRUMENT

Our discussion of the solutions S1 and S2 (stated below) to the problem stated as Item 1 follows the ideas of symmetry of the parabola discussed above.

The item design anticipates the use of a diagram (Heuristic 3 of Figure 1) to clarify problem at hand. This, together with the cognitive resource that the graphs of quadratic functions exhibit symmetry, would lead the problem solver to exploit the geometrical symmetry (i.e., notice that the y-coordinates of the given points are equal) to locate the axis of symmetry at $x = \frac{(-2)+4}{2} = 1$. The problem solver then draws on his/her cognitive resource that the line of symmetry is $x = -\frac{b}{2a}$ and so, $b = -2a$. Again using the line of symmetry $x = 1$ enables us to write $Q(x) = a(x-1)^2 + (c-a)$. A direct substitution of $x = -2$ (or $x = 4$) then results in $c = 4 - 8a$. Since that the y-value at $x = 1$ is positive, $Q(1) = a + b + c = a + (-2a) + (4 - 8a) > 0 \Leftrightarrow a < \frac{4}{9}$. Since a necessary condition for the minimum point to lie above the x-axis is that the curve is concave upwards, it follows that $a > 0$. Combining these inequalities yields the desired range of values of a, i.e., $\{a \in \mathbb{R} | 0 < a < \frac{4}{9}\}$. We label this track of solutions as (S1).

Another variant of symmetry that can be exploited is algebraic symmetry. An instance of algebraic symmetry typically involves 'completing the squares', i.e., by writing the quadratic expression $Q(x) = ax^2 + bx + c \equiv a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$ and observing that $Q(-2) = Q(4)$, one has $\left(-2 + \frac{b}{2a}\right) = -\left(4 + \frac{b}{2a}\right) \Leftrightarrow \frac{b}{a} = -2 \Leftrightarrow b = -2a$. Because the perfect square creates an algebraic symmetry about zero (i.e., $\alpha^2 = (-\alpha)^2$), we still regard this method as one that invokes symmetry. The problem is subsequently solved, following the remaining part of the argument in (S1). We label this track of solutions as (S2). However, we expect few solutions belonging to (S2) since the argument involved seems less natural.

Understandably, we anticipate other solutions which, in general, do not appeal to symmetry at all. We label such a track of solutions as (C), called *conventional solutions*. Solutions in (C) typically begin with the routine substitutions of $x = -2$ and $x = 4$ to yield a pair of simultaneous linear equations:

$$4a - 2b + c = 4 \ldots (1)$$
$$16a + 4b + c = 4 \ldots (2)$$

Subtracting (1) from (2) yields $12a + 6b = 0 \Leftrightarrow b = -2a$. Thus, $c = 4 - (4a - 2b) = 4 - (4a + 4a) = 4 - 8a$. At this point, we expect a bifurcation of methods. (CS1) consists of solutions which exploit differential calculus. Typically, this involves differentiating the quadratic function $Q(x) = ax^2 + bx + c$ to yield $Q'(x) = 2ax + b$. Equating this to zero gives $x = -\frac{b}{2a} = 1$. Substituting this into Q to calculate the corresponding y-value, one has $Q(1) = a + b + c = a + (-2a) + (4 - 8a) = 4 - 9a$. Setting this to be positive yields $a < \frac{4}{9}$. Applying the second derivative test, one requires further that $\frac{dQ}{dx} = 2a > 0$ so that $a > 0$. The desired range of values of a is obtained by combining the two inequalities. (CS2) consists of solutions that make use of the discriminant condition. Relying on the background knowledge that $Q(x) > 0$ for all real x is possible only when $a > 0$, and additionally, the discriminant of Q must be negative, i.e., $D = b^2 - 4ac < 0$. We expect many attempts to fall within this track.

In summary, Tracks (S1) and (S2) make use of symmetry, and Track (CS) makes use of conventional methods such as calculus but not symmetry. Thus, our classification dictates that any solution must fall within one of these tracks, regardless of whether that solution eventually leads to the complete solution of the problem.

Measuring the degree of success of an attempted solution is a separate business. For comparison purposes, such a measure should be, at worst, unbiased towards any particular track of solutions and at the same time give credit according to how much progress is made in the solution for a given solution track. To do this, we devise a 'marking scheme' within each track, which is given in Table 1. Each marking scheme awards constructive progress towards the solution of the problem, and maintains consistency with the other parallel marking schemes. For instance, 2 marks are awarded for a solution that reaches the deduction of $b = -2a$ or any of its logical equivalent, e.g., $x = 1$ is the line of symmetry.

The instruments are designed to meet two purposes: (1) compare the degree of success in problem among the different solution tracks, and (2) obtain the relative frequency of the usage of symmetry in problem solving.

DATA ANALYSIS

We begin with a quantitative data analysis. Owing to the fact that the observed frequencies in symmetry track (S2) falls below 5, we choose to amalgamate (S1) and (S2) into a combined track (S). It was observed that the conventional tracks (CS1) and (CS2) are the main solution tracks used, and each of these is of sufficient weightage that they can be considered as two distinct tracks. Similarly, owing to small class sizes, in terms of the marks scored, we have collapsed the data into the various progress classes (0-1: Poor; 2: Satisfactory; Good: 3; Excellent: 4-5). The observed absolute and relative frequencies (expressed in percentages) for the various progress classes within the tracks (S), (CS1) and (CS2) are tabulated in Table 2 below. Depicting relative frequencies as histograms, those associated to each of the performance classes are depicted in Figure 3(a), while those associated to each solution track (S), (CS1) or (CS2) are depicted in Figure 3(b).

A majority (about 2.5 times) of the subjects used the conventional methods as compared to those who used symmetry. This confirms our expectations that in Singapore schools, symmetry is not the mainstream method to be considered or used for problem solving. At a 5% level of significance, a Pearson's χ^2-test indicates that there is a dependency between the measure of success of problem solving (marks scored) and the solution tracks used, with a p-value of 0.04556 (5 d.p.).

Table 1. Marking scheme deriving the degree of success of an attempted solution within each track

Marks	S1	S2	CS1	CS2
0	Nothing is correct.	Nothing is correct.	Nothing is correct.	Nothing is correct
1	Symmetry at $x = 1$.	Completing squares correctly.	Solving simultaneous equations to get: $b = -2a$.	Solving simultaneous equations to get: $b = -2a$
2	$b = -2a$.	$b = -2a$ and obtain the line of symmetry at $x = 1$.	Solving $Q'(x)= 0$ to yield $x = 1$.	$b = -2a$ and either $a > 0$ or $D < 0$.
3	$b = -2a$ and $a > 0$. Rewrite $Q(x)$ as $a(x-1)^2+(c-a)$ or set $Q(1) > 0$. Did not get c in terms of a.	$b = -2a$ and $a > 0$. Did not get c in terms of a.	$b = -2a$ and $a > 0$. Using $Q''(x) > 0$ to get $a > 0$. Did not get $Q(1)$ or c in terms of a.	$b = -2a$ and $a > 0$. $D < 0$ or completing the square correctly. Did not get c in terms of a.
4	Achieving 3 marks and $Q(1) = 4-9a$ or $c = 4-8a$. Did not obtain or did not explain the answer $0<a<9/4$.	Achieving 3 marks and $Q(1) = 4-9a$ or $c = 4-8a$. Did not obtain or did not explain the answer $0<a<9/4$.	Achieving 3 marks and $Q(1) = 4-9a$ or $c = 4-8a$. Did not obtain or did not explain the answer $0<a<9/4$.	Achieving 3 marks and $Q(1) = 4-9a$ or $c = 4-8a$. Did not obtain or did not explain the answer $0<a<9/4$.
5	Achieving 4 marks and obtain $0<a<9/4$ with correct explanations.	Achieving 4 marks and obtain $0<a<9/4$ with correct explanations.	Achieving 4 marks and obtain $0<a<9/4$ with correct explanations.	Achieving 4 marks and obtain $0<a<9/4$ with correct explanations.

Table 2. Observed absolute (resp. relative) frequencies of marks scored within each solution track

Score	S = S1 & S2	CS1	CS2	Total
0-1	7 (25.00%)	7 (31.82%)	18 (39.13%)	32 (33%)
2	4 (14.29%)	8 (36.36%)	9 (19.57%)	21 (22%)
3	12 (42.86%)	6 (27.27%)	7 (15.22%)	25 (26%)
4-5	5 (17.86%)	1 (4.55%)	12 (26.09%)	18 (19%)
Total	28 (100%)	22 (100%)	46 (100%)	96 (100%)

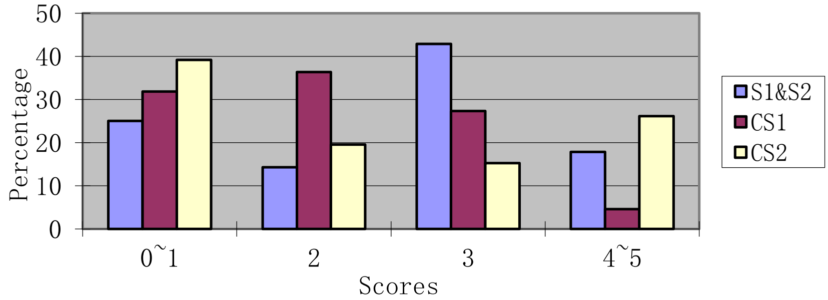

Figure 3(a): Histogram of observed relative frequencies within progress class

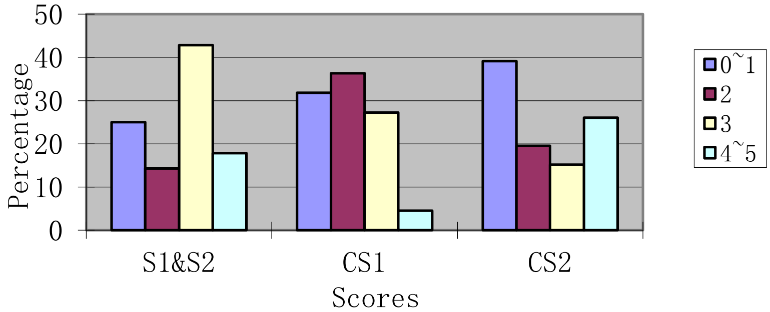

Figure 3(b): Histogram of observed relative frequencies within solution tracks

Figure 3(a) suggests that people using conventional methods are more prone to mistakes, leading to 'Poor' progress in problem solving. Interestingly, the highest relative frequency for 'Good' performance is achieved by the symmetry track. This somewhat indicates that symmetry as a problem solving tool can be effective in *putting people on the right track* in problem solving, i.e., there is a high chance of achieving good progress if symmetry has been considered or invoked. Also, subjects who employed symmetry had a better chance (more precisely, 4 times as likely) of achieving 'Excellent' progress than those who employed the differential calculus. To get a sense of the overall effective of each solution track in terms of achieving progress in problem solving, we compute the weighted average of the score (based on the statistics corresponding to Figure 3(b)) as follows:

S : 25(0.5) + 14.29(2) + 42.86(3) + 17.86(4.5) = 250.03
CS1 : 31.82(0.5) + 36.36(2) + 27.27(3) + 4.55(4.5) = 190.915
CS2 : 39.13(0.5) + 19.57(2) + 15.22(3) + 26.09(4.5) = 221.77

Based on this rubric, there is a suggestion that symmetry, when employed in problem solving, ensures better progress.

Next, we make some qualitative remarks concerning the use of symmetry in this problem. A total of 69 responses included a sketch of the quadratic graph, and yet only 5 students made use of the symmetry present in the graph to solve the problem. What this means is that even when a pictorial representation (such as a graph) is given, the trigger to invoke symmetry is rarely pulled. Also, there was a sizable portion (about 25% of those who used symmetry) calculated the position of the line of symmetry wrongly, given that they knew it was located mid-way between $x = -2$ and $x = 4$.

DISCUSSION AND IMPLICATIONS

The written responses of the subjects provide sufficient evidence for us to raise the following alarm: Singapore teachers *do not employ the concept of symmetry* in problem solving as frequently as they should, even in cases where symmetry seemed to be the natural track to take (e.g., problems related to the bilateral symmetry arising from quadratic graphs). One of the most likely causes for this, we reckon, is the lack of emphasis of symmetry as an important mathematical concept in the Singapore mathematics syllabus. More specifically, the relationship between zeros (i.e., the *x*-intercepts) and the line of symmetry is not stressed in most textbooks. Regarding curve sketching, for instance, there is only one example found on p. 53 of the textbook (Fan, 2007). As for use of symmetry in sketching the quadratic curve, there is only one example found on p. 20-21 of the textbook (Teh et al, 2007). Such a deficit can be easily remedied by supplying more examples which make salient use of symmetry in the solution of problems related to quadratic functions and their graphs. Classroom lessons can be enriched with activities that are centred about the theme of symmetry in several contexts, such as those suggested in Weyl (1952) and Voloshinov (1996).

CONCLUDING REMARKS

Using and teaching symmetry should never be seen as an ad-hoc approach or “one-off” business. In order that symmetry be exploited in problem solving, there must be a concerted effort to raise the awareness that symmetry is a powerful and convenient problem-solving tool. This can be made very explicitly by including it in the list of problem solving heuristics, or by mentioning it when examples involve the 6^{th} heuristics of “looking for patterns”. Additionally, the concept of symmetry can be intentionally grafted at every possible location, especially when geometry is concerned. Take for instance, an alternative definition of a rhombus (as opposed to the traditional one that says it is a quadrilateral with all sides equal) is a quadrilateral with two lines of symmetry each passing through a pair of opposite vertices.

As mathematics educators, we have a dual responsibility of equipping pre-service and in-service teachers with sound mathematical problem solving skills. The problem used in the present study demonstrated that solutions which exploit symmetry tend to be more effective and elegant. More importantly, our study suggests that by considering symmetry as one of the problem solving heuristics or tool one heightens the chance of success in the problem solving endeavour. Symmetry seems to tease out the essence of the entire problem. The present study revealed that our pre-service teachers lack a *habit of mind* of using symmetric property of a mathematical entity, such as a graph, in problem solving. We attribute this deficiency to the fact that textbook problems do not give ample exposure on the application of symmetry throughout the mathematics syllabus.

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