PHYSICAL REVIEW A **71**, 022110 (2005)

Continuous multipartite entangled state in Wigner representation and violation of the Żukowski-Brukner inequality

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(Received 23 August 2004; published 25 February 2005)

We construct an explicit Wigner function for the N-mode squeezed state. Based on a previous observation that the Wigner function describes correlations in the joint measurement of the phase-space displaced parity operator, we investigate the nonlocality of the multipartite entangled state by the violation of the Żukowski-Brukner N-qubit Bell inequality. We find that quantum predictions for such a squeezed state violate these inequalities by an amount that grows with the number N.

DOI: 10.1103/PhysRevA.71.022110 PACS number(s): 03.65.Ud, 42.50.Dv

Einstein, Podolsky, and Rosen (EPR) questioned the completeness of quantum mechanics in a classic seminal paper in 1935 [1]. In their paper, they reasoned that the wave function of a two-particle system in which the particles are entangled in position and momentum (an EPR pair) and written explicitly as

$$\Psi(q_1,q_2) = \int_{-\infty}^{\infty} e^{(2\pi i/h)(q_1-q_2+q_0)p} dp \qquad (1)$$

is incompatible with the completeness postulate. It was subsequently argued that additional variables (local hidden variables) could be introduced to restore casuality and locality to quantum mechanics.

A scheme [2] for testing the compatibility of the theory of local hidden variables with quantum mechanics was subsequently proposed based on a different version of an EPR pair, namely, the entanglement of spin-1/2 particles first introduced by Bohm. In particular, the possibility of local realism implies logical constraints on the statistics of two or more physically separated systems. These constraints can be expressed in the form of Bell-type inequalities [2–8]. For quantum-mechanical systems, it was anticipated that these constraints could be violated with appropriate settings and measurements.

On the other hand, quantum correlation for position-momentum variables can be analyzed in position-momentum phase space using the Wigner function [9]. The Wigner function allows one to define a probability distribution in position-momentum phase space for a quantum-mechanical particle. This formalism led eventually to the formulation of a c-number approach to describe the quantum effects in phase space including the development of various other efficient tools in a number of fields in modern physics [10].

Indeed, a mixed state system can be represented by a density matrix ρ,

$$P_W(q,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dy \langle q-y|\rho|q+y\rangle e^{2ipy/\hbar}, \qquad (2)$$

where ρ is a density operator, $|q\rangle$ is the eigenvector of the coordinate operator. An intuitive physical meaning of the Wigner function is that its marginal distribution in one of the variables gives the probability distibution of the particle in that space. For momentum, we have $P_{mom}(p)=\int W(q,p)dq$ and with position $P_{pos}(q)=\int W(q,p)dp$.

Bell [11] had argued that the original EPR wave function does not violate local realism because its joint Wigner distribution function $W(q_1,p_1;q_2,p_2)$ is positive everywhere, and as such it will also admit a local hidden variable description of signed position correlations. However, the choice of appropriate observables is important for testing the existence of local realism for a given state. In a recent work, Banaszek and Wódkiewicz [12] considered parity measurement, a quantum observable which does not admit a local hidden variable description, and interpreted the Wigner function as a correlation function for these parity measurements. They then showed that the original EPR state and the two-mode squeezed vacuum state violate local realism since they violate generalized Bell inequalities such as the Clauser-Horne inequality [3] and the Clauser-Horne-Shimony-Holt (CHSH) [4] inequality. In particular, they considered the two-mode squeezed vacuum state produced through nondegenerate optical parametric amplification (NOPA) [13] in order to avoid problems related to the singularity of the unnormalizable EPR state. Moreover, in Ref. [12], it was shown that despite its positive definiteness, the Wigner function of the EPR state could provide direct evidence of the nonlocality.

The two-mode squeezed vacuum state generated in a nondegenerate optical parametric amplifier (NOPA) [13] is given by

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$$|\text{NOPA}\rangle = e^{r(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)}|00\rangle = \sum_{n=0}^{\infty} \frac{(\tanh r)^n}{\cosh r}|nn\rangle, \qquad (3)$$

where r is known as the squeezing parameter and $|nn\rangle = |n\rangle_1 \otimes |n\rangle_2 = (1/n!)(\hat{a}_1^\dagger)^n(\hat{a}_2^\dagger)^n|00\rangle$. The NOPA states $|\text{NOPA}\rangle$ can also be written as [12]

$$|\text{NOPA}\rangle = \sqrt{1-\tanh^2 r}\int dq \int dq' g(q,q';\tanh r)|qq'\rangle, \qquad (4)$$

where $g(q,q';x) \equiv 1/\sqrt{\pi(1-x^2)}\exp[-(q^2+q'^2-2qq'x)/2(1-x^2)]$ and $|qq'\rangle \equiv |q\rangle_1 \otimes |q'\rangle_2$, with $|q\rangle$ being the eigenstates of the position operator. Since $\lim_{x\to 1} g(q,q';x) = \delta(q-q')$, one has $\lim_{r\to\infty}\int dq \int dq' g(q,q';\tanh r)|qq'\rangle = \int dq|qq\rangle = |\text{EPR}\rangle$, which is just the original EPR states. Thus, in the infinite squeezing limit, $|\text{NOPA}\rangle|_{r\to\infty}$ becomes the original, normalized EPR states. Since the original EPR state is an unnormalized δ function, a normalizable state generated in a NOPA can avoid problem arising from this singularity.

The Wigner function can be associated directly with the parity operator. The connection between the parity operator $(-1)^{\hat{n}}$ and the Wigner function provides an equivalent definition of the latter [14]. The Wigner representation of the parity operator is not a bounded reality corresponding to the dichotomic result of the measurement. This enables violation of Bell inequality for quantum states described by positive-definite Wigner function.

The relation between the nonlocality of arbitrary multipartite entangled states and the Wigner function remains an open question. Recently, tripartite entangled state representation of the Wigner operator and the corresponding Wigner function have been found by Fan and Jiang [15]. They focused principally on a generalization of the Wigner function and its marginal distributions, without invoking the nonlocality issue. A general Bell inequality which is a sufficient and necessary condition for the correlation function for N particles has been described in Ref. [8]. In this work, measurements on each particle were chosen from two arbitrary dichotomic observables. Namely, the Zukowski-Brukner inequality was derived for observables with eigenvalues +1 or −1, which are also the spectrum of the displaced parity operator. Thus this general Bell theorem for general N-qubit states provides a useful tool to test the violation of local realism of multipartite quantum states described by Wigner function. With this motivation, we derive an expression for the Wigner function of a N-mode squeezed state in this paper. By expressing the correlation function using the Wigner function, we show that the multipartite entangled state violates local realism, and this violation is enhanced with increasing dimension, N.

To this end, we first choose parity operators as the observables for testing violation of local realism for a squeezed state. The Wigner function can be expressed as the expectation value of a product of displaced parity operators as follows:

$$W(\alpha_1,\alpha_2,\ldots,\alpha_N) \propto \Pi(\alpha_1,\alpha_2,\ldots,\alpha_N), \qquad (5)$$

where $\Pi(\alpha_1,\alpha_2,\ldots,\alpha_N)$ is the expectation value of the joint displaced parity operator (i.e., the measured observable)

$$\hat{\Pi}(\alpha_1,\alpha_2,\ldots,\alpha_N) = \hat{D}_1(\alpha_1)\ldots\hat{D}_N(\alpha_N)(-1)^{\hat{n}_1+\cdots+\hat{n}_N} \times \hat{D}_N^{-1}(\alpha_N)\cdots\hat{D}_1^{-1}(\alpha_1). \qquad (6)$$

In the above expression, $\hat{D}_i(\alpha_i) = \exp(\alpha_i \hat{a}^\dagger - \alpha_i^* \hat{a})$ denotes the displacement operators for the subsystem i, where $\hat{a}$ $(\hat{a}^\dagger)$ is annihilation (creation) operator. We also equate the correlation functions given by the displaced parity operator $(-1)^{\hat{n}_1+\cdots+\hat{n}_N}$ as the equivalent Wigner function [16]. In this way, we see that the nonlocal realistic description is embedded in the dichotomic correlation measurements given by the phase-space Wigner function for the entangled state,

$$E(\alpha_1,\alpha_2,\ldots,\alpha_N) \equiv \Pi(\alpha_1,\alpha_2,\ldots,\alpha_N). \qquad (7)$$

The Wigner function, or equivalently the correlation function for multipartite system, can be calculated using the expectation value of the operator under the N-mode squeezed state. This new squeezed state, a SU(1,1) coherent state, is given as

$$|r\rangle = V|\mathbf{0}\rangle = \exp[r(W_+ - W_-)]|\mathbf{0}\rangle, \qquad (8)$$

where $|\mathbf{0}\rangle = |00\cdots 0\rangle$ is a N-mode vacuum state, and

$$W_+ = x\sum_{i=1}^{N}\hat{a}_i^{\dagger 2} + y\sum_{i<j=1}^{N}\hat{a}_i^\dagger\hat{a}_j^\dagger,$$

$$W_- = x\sum_{i=1}^{N}\hat{a}_i^{2} + y\sum_{i<j=1}^{N}\hat{a}_i\hat{a}_j,$$

$$B = \frac{1}{2}\sum_{i=1}^{N}\hat{a}_i^\dagger\hat{a}_i + \frac{N}{4}. \qquad (9)$$

W_+ is a N-mode squeezing operator and x and y are the coefficients which can be determined by the fact that the above formula satisfies the closed SU(1,1) Lie algebra: $[W_+,W_-] = -2B$, $[W_+,B] = -W_+$, $[W_-,B] = W_-$. The final result is

$$W_+ = \frac{2-N}{2N}\sum_{i=1}^{N}\hat{a}_i^{\dagger 2} + \frac{2}{N}\sum_{i<j=1}^{N}\hat{a}_i^\dagger\hat{a}_j^\dagger, \qquad (10)$$

$$W_- = \frac{2-N}{2N}\sum_{i=1}^{N}\hat{a}_i^{2} + \frac{2}{N}\sum_{i<j=1}^{N}\hat{a}_i\hat{a}_j. \qquad (11)$$

The N-mode squeezed state is characterized by the squeezing parameter r. The Wigner function of the squeezed state is calculated in the following way. When r is zero, namely when no squeezing occurs, the Wigner function is given by

$$E(\alpha_1,\alpha_2,\ldots,\alpha_N) = \langle \mathbf{0}|\hat{\Pi}(\alpha_1,\alpha_2,\ldots,\alpha_N)|\mathbf{0}\rangle = \exp\left[-2\sum_{i=1}^{N}|\alpha_i|^2\right]. \quad (12)$$

When $r\neq 0$, the new Wigner function can be constructed from

$$E'(\alpha_1,\alpha_2,\ldots,\alpha_N) = \langle r|\hat{\Pi}(\alpha_1,\alpha_2,\ldots,\alpha_N)|r\rangle = \langle \mathbf{0}|\hat{\Pi}(\alpha_1',\alpha_2',\ldots,\alpha_N')|\mathbf{0}\rangle = \exp\left[-2\sum_{i=1}^{N}|\alpha_i'|^2\right], \quad (13)$$

where $\hat{\Pi}(\alpha_1',\alpha_2',\ldots,\alpha_N')$ is the squeezed displaced parity operator given in Eq. (6). After some lengthy calculation, we arrive at the following relations:

$$V^{-1}\hat{a}_iV = \cosh r\hat{a}_i + \sinh r\left(\frac{2-N}{N}\hat{a}_i^\dagger + \frac{2}{N}\sum_{j\neq i}^{N}\hat{a}_j^\dagger\right), \quad (14)$$

$$V^{-1}\hat{a}_i^\dagger V = \cosh r\hat{a}_i^\dagger + \sinh r\left(\frac{2-N}{N}\hat{a}_i + \frac{2}{N}\sum_{j\neq i}^{N}\hat{a}_j\right). \quad (15)$$

The latter relation can be employed to yield the Wigner function of N-mode squeezing state,

$$E'(\alpha_1,\alpha_2,\ldots,\alpha_N) = \exp\left\{-2\cosh 2r\sum_{i=1}^{N}|\alpha_i|^2 + \frac{4}{N}\sinh 2r\sum_{i<j}^{N}(\alpha_i\alpha_j+\alpha_i^*\alpha_j^*) - \frac{N-2}{N}\sinh 2r\sum_{i=1}^{N}(\alpha_i^2+\alpha_i^{*2})\right\}. \quad (16)$$

The Wigner function of the original EPR state is recovered in the limit of $r\to\infty$ for $N=2$.

The N-NOPA field modes are equivalent to an entangled state of N oscillators. When $N=3$, the Wigner function is

$$E'(\alpha_1,\alpha_2,\alpha_3) = \exp\left\{-2\cosh 2r\sum_{i=1}^{3}|\alpha_i|^2 + \frac{4}{3}\sinh 2r \times\sum_{i<j}^{3}(\alpha_i\alpha_j+\alpha_i^*\alpha_j^*) - \frac{1}{3}\sinh 2r\sum_{i=1}^{3}(\alpha_i^2+\alpha_i^{*2})\right\} \quad (17)$$

and this is the same as the result given in Ref. [15]. The correlation function is determined by considering measurements corresponding to the settings $\alpha_1^1=0$, $\alpha_1^2=a$, $\alpha_2^1=0$, $\alpha_2^2=a$, and $\alpha_3^1=-a$, $\alpha_3^2=0$, where α_i^j ($i=1,2,3$ and $j=1,2$) is the jth measurement setting for ith particle, and a is a positive constant associated with the displacement magnitude. From these combinations, the following quantity can be constructed:

$$\begin{aligned}\mathcal{B}(3) &= E'(\alpha_1^1,\alpha_2^1,\alpha_3^2)+E'(\alpha_1^1,\alpha_2^2,\alpha_3^1)+E'(\alpha_1^2,\alpha_2^1,\alpha_3^1) - E'(\alpha_1^2,\alpha_2^2,\alpha_3^2)\\ &= E'(0,0,0)+E'(0,a,-a)+E'(a,0,-a)-E'(a,a,0)\\ &= 1+2\exp\left\{\left(-4\cosh 2r-\frac{8}{3}\sinh 2r-\frac{4}{3}\sinh 2r\right)a^2\right\} - \exp\left\{\left(-4\cosh 2r+\frac{8}{3}\sinh 2r-\frac{4}{3}\sinh 2r\right)a^2\right\}.\end{aligned} \quad (18)$$

For local hidden variables theories, we have the inequality [8] $-2\leqslant\mathcal{B}(3)\leqslant 2$. If we perform an asymptotic analysis for large $|r|$ with $r<0$, $\cosh 2r$ and $\sinh 2r$ can be replaced by $e^{-2r}/2$ and $-e^{-2r}/2$, respectively, and Eq. (18) becomes $\mathcal{B}(3)=3-\exp\{-\frac{8}{3}e^{-2r}a^2\}$. We see that when a^2/e^{2r} is large enough, the Bell inequality for three qubits is violated when $\mathcal{B}(3)$ approaches the value $\mathcal{B}_{\rm opt}=3$.

For $N=4$, and choosing all α_i to be real, the Wigner function can be written as

$$E'(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \exp\left\{(-2\cosh 2r-\sinh 2r)\times\sum_{i=1}^{4}\alpha_i^2 + 2\sinh 2r\sum_{i<j}^{4}\alpha_i\alpha_j\right\}. \quad (19)$$

Evaluating the quantity $\mathcal{B}(4)$, and applying the N-qubit Bell inequality, we have

$$\begin{aligned}\mathcal{B}(4) = &-E'(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^1)+E'(\alpha_1^1,\alpha_2^1,\alpha_3^1,\alpha_4^2)\\ &+E'(\alpha_1^1,\alpha_2^1,\alpha_3^2,\alpha_4^1)+E'(\alpha_1^1,\alpha_2^1,\alpha_3^2,\alpha_4^2)\\ &+E'(\alpha_1^1,\alpha_2^2,\alpha_3^1,\alpha_4^1)+E'(\alpha_1^1,\alpha_2^2,\alpha_3^1,\alpha_4^2)\end{aligned}$$

TABLE I. Threshold visibility for $2\leqslant N\leqslant 7$.

$V=2/\mathcal{B}_{\rm opt}(N)$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$
ME states	0.707	0.5	0.354	0.25	0.177	0.125
Oscillator	0.913	0.667	0.544	0.4	0.318	0.229

$$+E'(\alpha_1^1,\alpha_2^2,\alpha_3^2,\alpha_4^1)-E'(\alpha_1^1,\alpha_2^2,\alpha_3^2,\alpha_4^2)$$
$$+E'(\alpha_1^2,\alpha_2^1,\alpha_3^1,\alpha_4^1)+E'(\alpha_1^2,\alpha_2^1,\alpha_3^1,\alpha_4^2)$$
$$+E'(\alpha_1^2,\alpha_2^1,\alpha_3^2,\alpha_4^1)-E'(\alpha_1^2,\alpha_2^1,\alpha_3^2,\alpha_4^2)$$
$$+E'(\alpha_1^2,\alpha_2^2,\alpha_3^1,\alpha_4^1)-E'(\alpha_1^2,\alpha_2^2,\alpha_3^1,\alpha_4^2)$$
$$-E'(\alpha_1^2,\alpha_2^2,\alpha_3^2,\alpha_4^1)-E'(\alpha_1^2,\alpha_2^2,\alpha_3^2,\alpha_4^2). \quad (20)$$

Under a local realistic description, $\mathcal{B}(4)\leqslant 4$. By choosing appropriate measurements, we have $\mathcal{B}_{\rm opt}(4)=7.357$. That is to say that the four-mode NOPA state shows strong nonlocality, which is stronger compared with three-mode or two-mode NOPA states.

We also consider the strength of violation or visibility (V) as the minimal amount V of the given entangled state $|\psi\rangle$ that one has to add to pure noise, $\rho_{\rm noise}$, so that the resulting state violates local realism. The quantity V is thus the threshold visibility above which the state cannot be described by local realism, and it is sometimes called the critical visibility. More specifically, we consider Werner state of the form $\rho_w = V|\psi\rangle\langle\psi| + (1-V)\rho_{\rm noise}$ where $\rho_{\rm noise}=I/2^N$ is the completely mixed state. As shown in Ref. [8], for the maximally entangled state $|\psi\rangle_{\rm GHZ}=(1/\sqrt{2})(|0\rangle_1\cdots|0\rangle_N+|1\rangle_1\cdots|1\rangle_N)$, the Werner state cannot be described by local realism if and only if $V>(1/\sqrt{2^{N-1}})$.

We repeat the calculation for entangled states for N oscillators ($N=2,3,4,5,6,7$) and their results are succinctly summarized in Table I and compared to the values for maximally entangled states. To see the variation of V with N, we also plot V versus the number of particles N both for maximally entangled (ME) states and entangled states of oscillators (Fig. 1. Naturally it is not surprising to see that the two systems show similar variations of V with increasing dimension N. Alternatively, if one considers the optimal value of the violation for the Żokowski-Brukner inequalities, the optimal value for this violation grows with N. Increasing the number of qubits, in this case, will not bring us any closer to the classical regime, but rather it appears to discriminate better the quantum and the classical boundary. We also see that the entangled states of the oscillator do not violate the N-qubit Bell inequality as much as the maximally entangled states do since the NOPA state is not maximally entangled. However, from the experimental perspective, the NOPA state is easier to generate than $|\psi\rangle_{\rm GHZ}$.

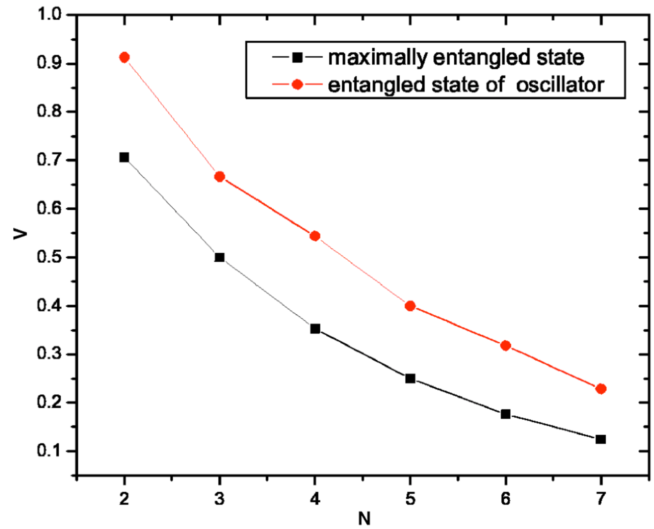

FIG. 1. Critical visibility of N-qubit Bell inequality ($N=2,3,4,5,6,7$) for both maximally entangled state and the entangled state of oscillator.

Our study shows that the multipartite entangled state in the Wigner representation exhibits nonlocal realism and this violation of local realism can be observed using the N-mode NOPA state. The violation of local realism for the NOPA state can be manifested through the violation of the N-particle Bell inequality for a state described by the Wigner function. This provides an exciting possibility to test the violation of local realism for the N-mode entangled state experimentally for the general case using the quantum N-mode squeezed state.

We thank Professor B. G. Englert and Professor D. Kaszlikowski for valuable discussion. This work was supported by NUS Academic Research Grant No. WBS: R-144-000-089-112. J.L.C acknowledges financial support from Singapore Millennium Foundation and (in part) by NSF of China (Grant No. 10201015).

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).
[3] J. F. Clauser and M. A. Horne, Phys. Rev. D **10**, 526 (1974).
[4] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[5] D. M. Greenberger, M. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[6] A. Peres, Found. Phys. **29**, 589 (1999).
[7] J. L. Chen, D. Kaszlikowski, L. C. Kwek, and C. H. Oh, Mod. Phys. Lett. A **17**, 2231 (2002); J. L. Chen, D. Kaszlikowski, L. C. Kwek, C. H. Oh, and M. Żukowski, Phys. Rev. A **64**, 052109 (2001); D. Kaszlikowski, L. C. Kwek, J. L. Chen, M. Żukowski, and C. H. Oh, *ibid.* **65**, 032118 (2002); L. B. Fu, J. L. Chen, and X. G. Zhao, *ibid.* **68**, 022323 (2003); L. B. Fu, Phys. Rev. Lett. **92**, 130404 (2004); D. Kaszlikowski, D. K. L. Oi, M. Christandl, K. Chang, A. Ekert, L. C. Kwek, and C. H. Oh, Phys. Rev. A **67**, 012310 (2003); A. Acín, J. L. Chen, N. Gisin, D. Kaszlikowski, L. C. Kwek, C. H. Oh, and M. Żukowski, Phys. Rev. Lett. **92**, 250404 (2004); J. L. Chen, C. F. Wu, L. C. Kwek, and C. H. Oh, *ibid.* **93**, 140407 (2004).
[8] M. Żukowski and Č. Brukner, Phys. Rev. Lett. **88**, 210401 (2002).
[9] E. Wigner, Phys. Rev. **40**, 749 (1932).
[10] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984).

[11] See, for example, J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, England, 1993).
[12] K. Banaszek and K. Wódkiewicz, Phys. Rev. A **58**, 4345 (1998); Phys. Rev. Lett. **82**, 2009 (1999); Acta Phys. Slov. **49**, 491 (1999).
[13] M. D. Reid and P. D. Drummond, Phys. Rev. Lett. **60**, 2731 (1998).
[14] B. G. Englert, J. Phys. A **22**, 625 (1989); B. G. Englert, S. A. Fulling, and M. D. Pilloff, Opt. Commun. **208**, 139 (2002).
[15] H. Y. Fan and N. Q. Jiang, J. Opt. B: Quantum Semiclassical Opt. **5**, 283 (2003).
[16] A. Royer, Phys. Rev. A **15**, 449 (1977);H. Moya-Cessa and P. L. Knight, *ibid.* **48**, 2479 (1993).