A Problem Solving Strategy : Look for Patterns in Sequences

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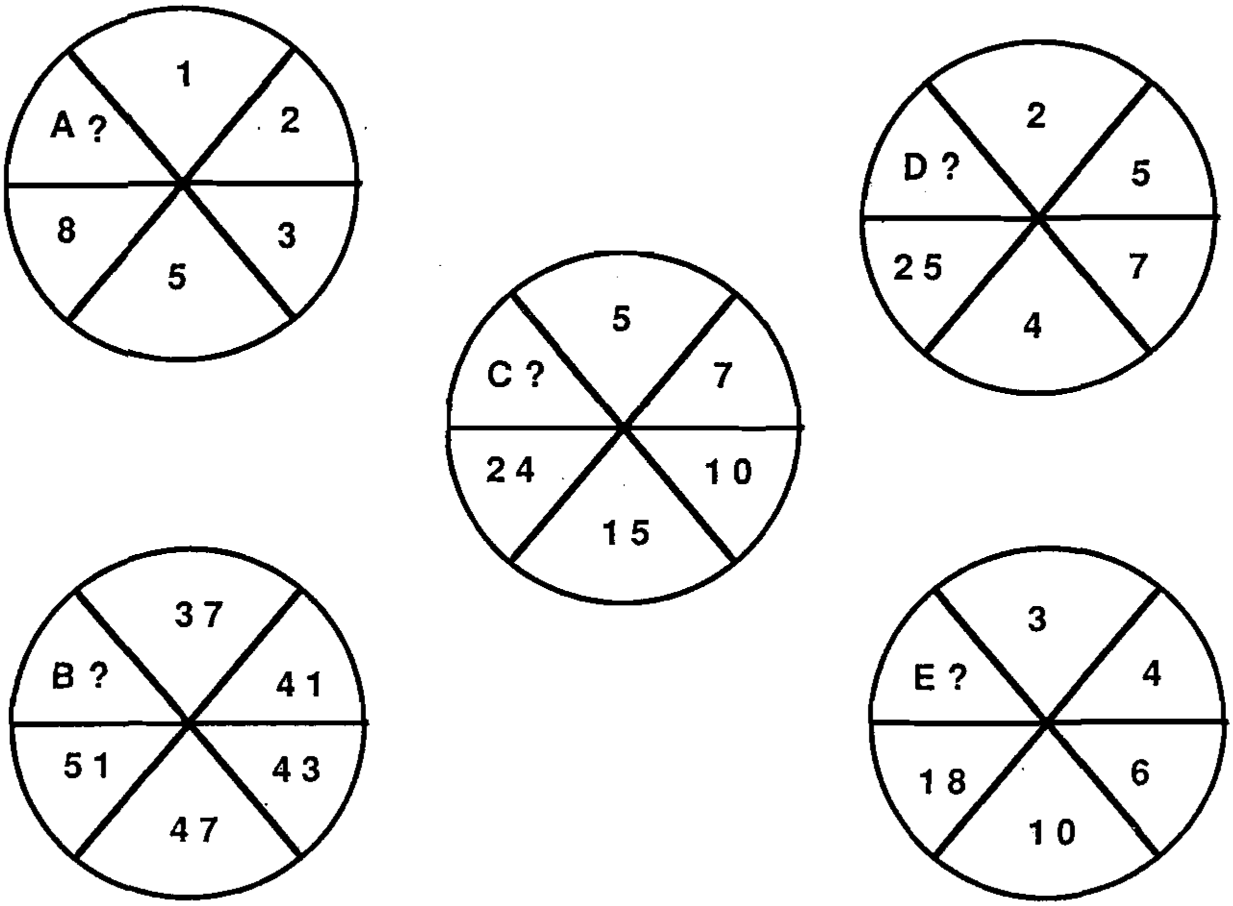
Problem Solving is a major theme in the mathematics curriculum in Singapore schools and it is suggested that problem solving strategies be explicitly taught to students. One of the most important problem solving strategies in mathematics is to look for patterns from a sequence of numbers.

The importance of patterns in mathematics can be seen from the words of W. W. Sawyer, "Mathematics is the classification and study of patterns". Recognising patterns is a powerful way to give meaning and understanding to mathematical symbols or data. It is a useful skill that students should be encouraged to develop. Pattern recognition permits relationships to be determined from the given data. Formulas are then used to summarise or generalise the relationships. In other words, to look for a pattern from a sequence of numbers is to find a general formula.

Teachers in giving examples to illustrate the strategy of looking for a pattern use often sequences such as 2, 4, 6 ... asking students to give the next term and also the nth term of the sequence. However, teachers should be careful in giving such sequences in class and be prepared for some unexpected answers unless restrictions are imposed on such sequences.

Before we discuss it any further, let us use the strategy to solve the following problem which appeared at the Orchard MRT station in 1990. I was told that the same problem could still be found at the Bishan MRT station in 1992.

I showed the problem to several classes of pre-service and in-service teachers of mathematics including heads of department. The problem is to find the missing numbers in each of the five sequences, shown as follows:



How many can you solve before the next train comes?

The teachers could easily obtain the answers in less than five minutes but nearly all of them gave the "expected" answers:

A = 13, B = 53, C = 41, D = 49, E = 34.

When I told them that the answers could well be

A = 100, B = 100, C = 100, D = 100, E = 100,

they were very surprised and found it hard to accept the answers until the following explanation was given.

Consider the first sequence.

1, 2, 3, 5, 8, A?

The "expected" answer is A = 13 because each of the last four numbers is formed by the addition of the two preceding numbers.

For A to be 100, the formula that generates the numbers in the sequence is

$$T(n) = \frac{89}{120}n^5 - \frac{67}{6}n^4 + \frac{509}{8}n^3 - \frac{508}{3}n^2 + \frac{3122}{15}n - 91$$

If readers do not believe it, they can do the following arithmetic to check it.

T (1) = 1

T (2) = 2

T (3) = 3

T (4) = 5

T (5) = 8

T (6) = 100

For the other sequences, answers are also given below:

37, 41, 43, 47, 51, B?

B = 53 is an expected answer.

B = 100

$$T(n) = \frac{53}{120}n^5 - \frac{55}{8}n^4 + \frac{977}{24}n^3 - \frac{905}{8}n^2 + \frac{2957}{20}n - 32$$

5, 7, 10, 15, 24, C?

C = 41 is an expected answer.

C = 100

$$T(n) = \frac{1}{2}n^5 - \frac{179}{24}n^4 + \frac{169}{4}n^3 - \frac{2677}{24}n^2 + \frac{549}{4}n - 56$$

2, 5, 7, 4, 25, D?

D = 49 is an expected answer.

D = 100

$$T(n) = -\frac{4}{15}n^5 + \frac{43}{8}n^4 - \frac{445}{12}n^3 + \frac{893}{8}n^2 - \frac{2893}{20}n + 67$$

3, 4, 6, 10, 18, E?

E = **34** is an expected answer.

E = 100

$$T(n) = \frac{67}{120}n^5 - \frac{25}{3}n^4 + \frac{1133}{24}n^3 - \frac{374}{3}n^2 + \frac{4567}{30}n - 64$$

Teachers should therefore be cautious about the teaching of the problem solving strategy: To look for a pattern.

If the following sequences are given as

2, 4, **6,** .... (This sequence is an A. P.)

2, 4, **8,** .... (This sequence is a G. P.)

1^2, 2^2, 3^2, ... (This is a sequence of squares.)

The answers to the next term and the nth term of the sequences will be unique.

However, if the sequence 2, 4, 6 .... is given without any restriction, then the answers to the next term and the nth term would not be unique. The sequence could be

2, 4, 6, 8, ... , T(n) = 2n

or 2, 4, 6, 10, ... ,

$$T(n) = \frac{1}{3}(n^3 - 6n^2 + 17n - 6)$$

or 2, 4, 6, 100, ... ,

$$T(n) = \frac{1}{30}(184n^3 - 828n^2 - 1256n - 552)$$

etc. (Readers may verify the formulas for themselves).

Teachers should be aware that number sequences such as the above can have many unexpected answers!

One way to improve the skill of looking for a pattern is to attempt to find both expected and unexpected patterns from the same set of data. Hence, readers are left to derive the above or other formulas for the given sequences.