Applying the Van Hiele Theory to theTeaching of Secondary School Geometry

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Introduction

The study of mathematics is the organization and abstraction of reality in order that reality may be studied in a structured way and Geometry is the branch of mathematics which seeks to structure space. Even as ancient civilizations sought to make sense of the space in which they lived, mankind has had to understand and apply spatial concepts. To the ancient Egyptians, Babylonians and Chinese, the study of geometry was for practical purposes of mensuration, construction and navigation. However, with the flourishing of the Greek civilization, geometry became removed from practical ideas and developed into a science of reasoning, a system of deduction to be valued for the intellectual rigour it required.

Today's geometry in secondary schools is a combination of a simplified version of Euclidean geometry together with transformation geometry and trigonometry. Most of the geometrical topics deal with properties of plane figures and the application of such properties in computing lengths and angles. Students find the rigour of proofs particularly difficult, especially when many steps are required and in fact, Euclidean-type proofs are not required by the syllabus although included in most textbooks.

It has been found that there can be barriers to learning geometry when the teacher and pupils are not communicating or understanding each other. The author once had the difficult task of convincing Secondary one pupils that a square was a rectangle and, being an untrained relief teacher from the university, resorted simply to explaining that the square "satisfied" the definition of a rectangle. The pupils agreed that yes, the properties were satisfied but remained fully unconvinced – "No, a rectangle looks like this – two long sides and two short sides." The teacher and pupils were not communicating at the same level.

Today's teachers have the advantage of having a theory of geometry learning to help them understand their pupils' difficulties. The theory was developed by two Dutch mathematics teachers, Pierre and Dina van Hiele, in the late 50's and this theory, known as the van Hiele theory, seeks to explain how children learn geometrical concepts. The theory has been found useful by many mathematics educators and has given rise to projects in many parts of the world. The van Hiele theory is also now included in the mathematics methodology courses at the National Institute of Education. The objective of this article is to briefly describe the van Hiele theory and to discuss its implications for mathematics teachers. Although the theory can be applied to the learning of geometrical concepts at all levels, this article will concentrate on geometrical concepts learned at the secondary school level.

The Van Hiele Theory

Basically, the van Hiele theory describes the learner of geometrical concepts as progressing through five levels of thinking. It also suggests phases of learning by which means a teacher can help pupils move from one level to the next. The five levels are described below:

Level 1 (Recognition)

At level 1, the learner recognises figures according to their appearance alone and sees shapes as a whole. Properties of the figure play no part in the recognition of the figure.

Level 2 (Analysis)

At this level, the properties of the figures become important to the learner and serve as a means of identifying figures. However, the learner sees properties discretely and do not relate the properties to each other.

Level 3 (Ordering)

The relationships between the various properties of the figure are now understood. It is only at this level that definitions make sense (although they may not be expressed in minimum terms). At this level, the learners do not understand the distinction between necessary and sufficient conditions. Deduction methods begin to appear but at an elementary level.

Level 4 (Deduction)

At level 4, the pupil understands the significance of deduction and can understand the logical development of a proof. Definitions involving necessary and sufficient conditions are understood.

Level 5 (Rigour)

The essence of geometry and necessity for rigour is appreciated. Pupils can accept logically correct proofs even if the concepts are counter to intuition. At this level, the learner can understand non-Euclidean systems based on purely axiomatic structures.

The theory describes the learner of geometrical concepts as progressing from one level to the next. Each level must necessarily be attained before the learner can progress to the next level. Learners can also show the appearance of having attained a higher level by learning rules or definitions by rote or by applying routines mechanically without true understanding. Students may be at different levels for different concepts but once one concept has been raised to a higher level, it takes less time for other concepts to reach that level.

People reasoning at different levels in general cannot understand each other. For example, in trying to convince my students that a square was a rectangle because it satisfied the conditions of the definition, I was communicating at at least level 3 whereas my students were probably somewhere between levels 1 and 2.

A similar example would be a teacher saying

"Look at Figure 1.
Since OA = OB, (radii),
therefore OAB is an
isosceles triangle."

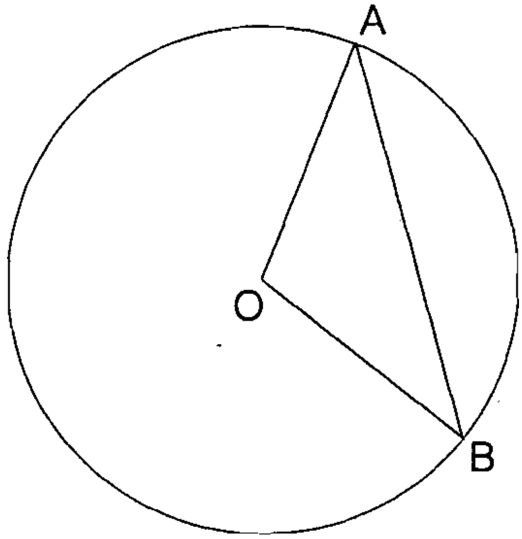


Figure 1

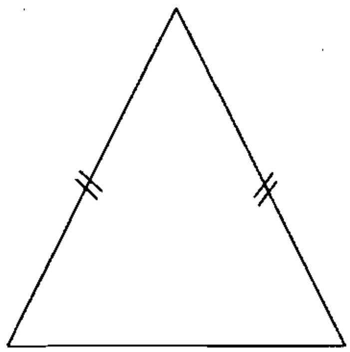


Figure 2

The student at level 1 is thinking, "No, it doesn't look like an isosceles triangle" because to him an isosceles triangle looks like Figure 2.

Implications for teachers

The above examples show how students can have difficulty in understanding when they are not at the van Hiele level which is expected for the topic or at the level at which the teacher is communicating. It is therefore necessary for the teacher to be aware of the level of reasoning of his pupils and to work to bring pupils up to the expected level for that topic.

Let us now consider the type of activities teachers can use to help pupils progress through the levels.

At level 1, instruction should be based on **perception** of the concept involved. It is crucial that pupils should have experience in recognizing the particular concept, especially in *non-standard* orientations. Concrete materials or transparencies which can be moved or turned to highlight different orientations are obviously very useful. Pupils and teachers alike are "conditioned" to seeing one "standard" orientation by textbooks. For example, perpendicular lines tend to be always drawn horizontal and vertical and parallel lines are often horizontal.

For another example, in introducing the angle properties of circles, teachers often do not spend sufficient time letting pupils recognize "angles subtended at the circumference", "angles subtended at the centre", "angles in the same segment" etc. In most presentations, there is a brief introduction of the concept before proceeding quickly to the properties themselves. Furthermore, only one or at most two examples are given to introduce the concept and angles at centre and circumference almost invariably point upwards and are subtended by minor arcs in such introductions like in Figure **4** rather than in Figure 5.

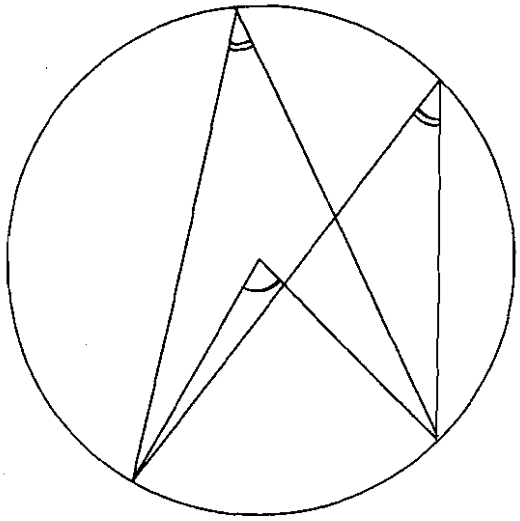


Figure 4

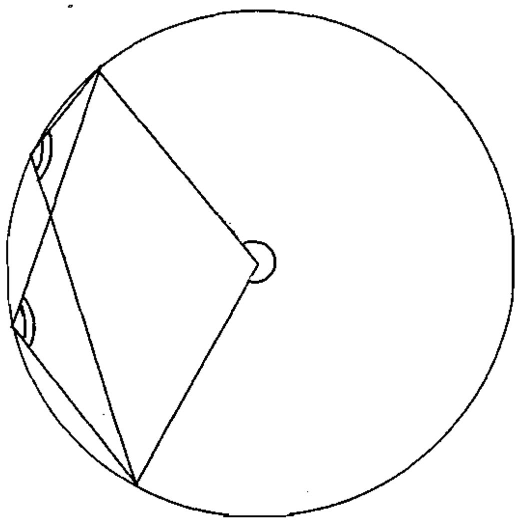


Figure 5

It is no wonder that some pupils have trouble applying the properties to problems given later where angles are "disguised" in non-standard orientations or in complicated figures. They simply did not recognize the angles. When the angles are pointed out, pupils or even teachers would exclaim, "Oh, I didn't *see* it/them!"

It is suggested that in introducing the concepts of angles in a circle, as pupils are only at level 1, this being a new concept, many examples and non-examples in many different orientations need to be experienced by the pupils. Activities such as the following could be given.

Name all pairs of "angles in the same segment" you can find in Figure 6.

O is the centre of the circle

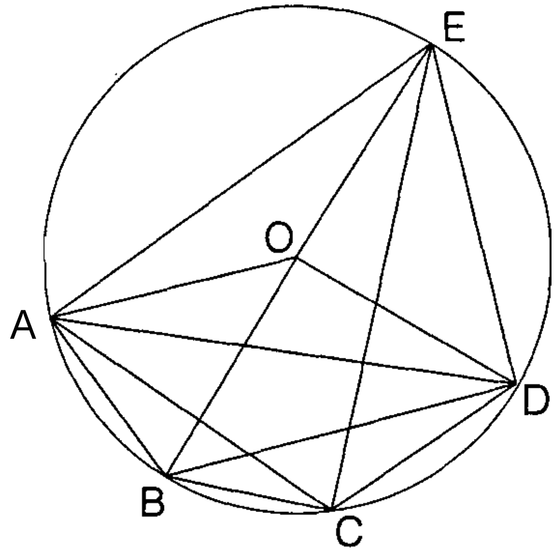


Figure 6

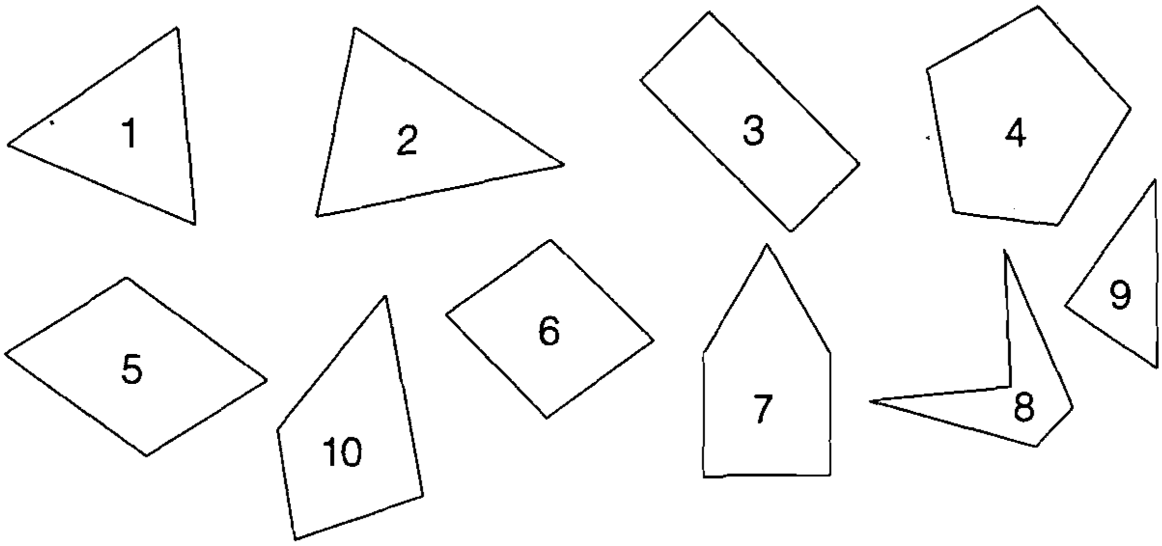
We can also use diagrams like Figure 6 for students to identify angles at centre/circumference and get them to also specify the arcs (both major and minor) which subtend the angles concerned.

At the secondary level, most students would have attained at least level 2 for concepts which they encountered at primary school e.g. they would know the properties of special quadrilaterals or of special triangles. However, they wouid begin at level 1 for new concepts as mentioned above but, having already attained level 2 in other concepts, it is not too difficult nor time consuming for teachers to help students also attain level *2* for such concepts by providing suitable recognition acitivies as illustrated above.

The transition from level 2 to level 3 is more difficult. Pupils at level 2 are not able to follow deductive proofs which are at level 4. They may be able to follow each individual step under the guidanceof the teacher but are unable to have an overview of the proof. At level 2, teachers need to give pupils much experiencing in discussing the properties concerned. Rather than expecting pupils to follow rigorous but more abstract proofs, pupils will accept the truth of the properties more readily if they can "discover" the properties in an experiential way.

By getting pupils to constantly justify their reasoning, teachers can help pupils to establish level 3 understanding. There should be many experiences for pupils to examine relationships between properties of given figures and properties between similar but related figures. In fact, the concepts/properties learnt at secondary level such as concept of polygon, concept of symmetry etc allow pupils to relate different "types" of figures. Classification is an important cognitive activity here and teachers should beware of being restricted and restricting their pupils to only the obvious means of classification. Consider the following open-ended question which could be given at Secondary two level:

Example: Sort the following figures into groups explaining why you sorted them that way.



Different classifications could include

A. The obvious classification is {1,2, 9}, (3,5, 6, 10} and {4,7, 8} according to number of sides.

B. {1, 3, 4, 6}, those which have rotational symmetry and {2, 5, 7, 8, 9, 10}, those which do not have rotational symmetry.

C. {9, 10} those with no line symmetry,
{2, 7, 8} those with one line of symmetry
{3,5} those with 2 lines of symmetry
{1}, {6} and {4} with 3, 4, 5 lines of symmetry respectively.

There are certainly other classifications, for example whether the figure contains obtuse angles, right angles, etc.

It should be the goal of our geometry curriculum that students attain level 3 at the end of their secondary school mathematics without which they will only have a superficial understanding of geometry and will regard geometry as a parcel of unrelated concepts, rules and properties. When they attain level 3, they will be able to follow simple deductions and understand logical deductions between consecutive steps. Although we should not expect students at level 3 to have a comprehensive understanding of the significance of proofs, yet with some understanding of the meaningfulness of deductions and of definitions and relationships between concepts, students can have an appreciation of the structure of the space which is what geometry is all about.

The attainment of level 4 represents the higher goal of Secondary and Junior College students. Teachers whose students have fully reached level 3 can encourage students to understand and explain whole proofs or to sketch overviews of proofs in order that they may see the "totality" of the proof. When or if they attain level 4, students would be able to comprehend and even to construct proofs for themselves. The use of problem-solving strategies such as "working backwards" and "identifying sub-goals" will be very essential in providing students with tools to construct proofs.

The scope of our geometry syllabus at secondary level will not permit the development of level 5 and only the most mathematically capable students should be expected to attain it. Such attainment in learning geometry is certainly desirable but will be confined to Further Mathematics at 'A' level and for mathematics at University level and will not concern our readers overmuch.

Conclusion

The above discussion shows that the van Hiele theory is most useful for teachers because it guides them in planning instruction such that the classroom activities will be effective for learning. Determining a particular student's or group's level of reasoning is not important in itself but what is important is how the teacher is then able to select and use activities appropriate to pupils at theirlevel of thinking. It is only by doing so that geometry can be meaningful to the student and that pupils can progress from one level to the next. To ignore this will mean miscommunication and will result in pupils' resorting to rote memory and mechanical application of rules.

There are naturally other learning theories which can be applied to the learning of mathematics in general and geometry in particular and teachers seeking meaningful learning for their pupils will plan lessons which will relate geometrical concepts to the world around them. Lessons from history teach us that man will learn and be creative when there is a practical need for that learning and geometry which studies the space in which we live can certainly be useful as well as mathematically and abstractly beautiful.

With the knowledge of the van Hiele theory and its implications for teaching geometry and with the use of practical activities to motivate students, mathematics teachers can lead students to appreciate both the usefulness of geometry as well as some of its structural beauty.

Reference

Fuys, D., Geddes, D. & Tischler, R. (1988), *The van Hiele model of thinking in geometry among adolescents,* Journal for Research in Mathematics Education Monograph No 3, NCTM.

Pegg, J. (1985), *How children learn geometry: the van Hiele theory,* The Australian Mathematics Teacher, 41(2).