Aspects of Non-Cognitive Factors in Problem Solving

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Introduction

The way problem solvers' perceive problems and problem solving can be influenced by non-cognitive factors such as beliefs and metacognition. Beliefs look at "[t]he individual's subjective knowledge of self, mathematics problem solving, and the problems dealt with in problem statements" (Lester, Garofalo and Kroll, 1989). Evidence (Lampert, 1990; Schoenfeld, 1985) shows that individuals construct such beliefs as a result of the teaching and learning experiences in mathematics classroom.

Metacognition considers the executive control exercised by problem solvers as they consider the information provided and what solution path to follow or to abandon. There is evidence to suggest that problem solvers with more mathematical knowledge might not be better problem solvers than those with less. Schoenfeld (1987) showed that the latter were able to execute effective control over whatever knowledge they had to solve the problem. The former, however, squandered their chances as they lacked effective control over their skills. "What made the difference was how the problem solvers made use of what they did know" (Schoenfeld, op. cit., p. 195). Schoenfeld's (1992) article "Learning to Think Mathematically: Problem solving, Metacognition, and Sense making in Mathematics" is a comprehensive review of relevant literature looking at how such non-cognitive factors affect problem solving by students in conventional mathematics classrooms as well as by mathematicians in non-mathematical situations.

In this paper I will be presenting the findings of a study where I engaged sixteen volunteers in problem solving situations. Of these sixteen volunteers, fifteen were postgraduate students in a British University and the sixteenth was a head of department at the same university. I was interested to find out what they perceived as a problem, and how they manage the resources they had to solve the problems. Three of the four questions posed to the participants required basic mathematical knowledge.

Procedure

The participants were interviewed individually at a time determined by them. They were given the problems, one a time, with the request that they "think aloud". The instruction "think aloud" is not identical with the instruction to introspect which has been common in experiments on thought processes. While the instrospector makes himself as thinking the object of his attention, the participant who is thinking aloud remains immediately directed to the problem, allowing his activity to become overt. When someone, while thinking, says to himself "Maybe I could try to cut the square ... " or "Now is A a square?" one could hardly call this introspection, yet in such remarks something is revealed of the way the participant approaches the problem (Duncker, 1945).

The participants' verbal reports as they worked on the problems were recorded and the recordings transcribed. These transcripts and the written work produced by the participants constituted the data. The participants were emphatically urged not to leave unspoken the most fleeting or foolish idea. They were told that where they did not feel completely informed, they were free to question me and that no previous specialised knowledge was necessary to solve the problems.

The participants were reassured that time was not a limiting factor. This strategy on time was adopted based on the study by Rokeach (1950). He tested the effect of perception time upon rigidity of thinking. He hypothesised that the person in a hurry would be forced to resort to behavioural supports of a concrete nature to solve the problem. One consequence of not having enough time for perceiving-thinking was inflexibility or rigidity of behaviour. The results of his study confirmed his expectation that rigidity would decrease with increase perception time, except for an important proviso: there seemed to be an optimal time allowance for each task which gave maximum returns but further delay did not result in increased flexibility. Hence participants terminated their attempts to solve the problems when they indicated that they could not proceed any further.

As far as possible, I did not provide direct answers to the participants' questions, but rather encouraged the participants to find the answers to their questions by asking leading questions. In order to understand how and why the participants employed a certain step they were asked to retrace their solution after it was completed. By doing this I hoped to elucidate what had occurred to the participants. This has an added advantage as the participants might be able to relate their performance to the whole problem. I took note of the participants' overt behaviours, for example, their manipulations of the material and emotional responses.

At the end of the session I attempted to fit into a coherent pattern, the problem-solving behaviours as reflected in the articulations, the overt movements and the participants' additional introspective interpretations of what had transpired.

The study of this small sample was not designed to provide data for statistical analysis. Rather the aim was to provide a glimpse into the very multidimensional world of problem solving where many factors influence the problem-solving performances of individuals. Because the sample was small there would be no analysis of the responses according to gender.

To reduce some of the variables that could influence the participant's success or failure at problem solving, problems that were relatively content-free were chosen. This might help ease any difficulty that might be encountered as a result of specific resources needed. To search for different responses, four problems were chosen to focus on various problem-solving skills.

The Problems

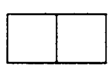
The four problems chosen were:

Problem 1 Given a tin can what can you do with tin cans such as this? List as many uses as you can find for these tin cans.

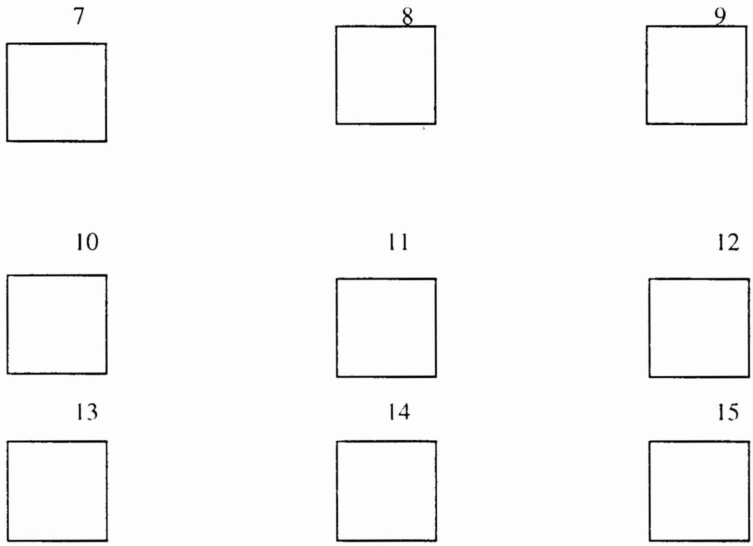
Problem 2 You are the officer-in-charge of issuing number plates to newly released cars. Determine how many different number plates you can form if you are given the letters N, U, M and the numbers 1, 2, 3. There is a law in your country that forbids number plates with repeating letters and numbers.

Problem 3: A little child comes up to you and asks why $\frac{20}{35}=\frac{4}{7}$. How would you explain to this child why these two fractions are the same?

Problem 4 : Here's a square cut into four smaller squares.



Find ways to cut squares into the number of smaller squares indicated below.



Findings: Responses and the Behaviours Displayed by the Participants

Problem 1 did not require any mathematical knowledge. It was used to help ease the participants into the interview situation and to get them use to the idea of articulating their thought processes for the following three problems. The tin was used because it was a common object and thus the participants would have uses of such a tin can.

The scoring of the responses was based on a modified form of the Torrance's (1965) study. Two basic measures of "fluency" and "flexibility" were obtained for each individual. Fluency was determined simply by the raw number of acceptable answers the participant produced. Bizarre, repetitious and irrelevant responses were considered unacceptable and were thus not counted. The scoring system was subjective but consistent. The flexibility score was equal to the number of different categories of usage suggested by the participant. The responses of the participants were recorded in scatter diagrams, an example of which is given below.

Categories of increasing creativity

C		⊗	⊗		⊗ ⊗
B					
A	⊗			⊗ ⊗	
	I	II	III	IV	V

Categories of increasing flexibility

Scatter diagram II of Participant Y with a fluency of 7 and a flexibility of 5

Here is a set of responses of a participant with fluency 7 and flexibility of 5. (1) use the tin can as a vase (IA); (2) use the base of the can to measure the mathematical constant of pi;

(IIC) (3) cut out the base of the can to use as a toy Frisbee (IIIC); (4) string the tin cans together to build a bird scare (IVA); (5) convert 2 tin cans to form a makeshift telephone (IVA); (6) use it as toy (VC); and (7) pile up many tin cans and use it for target practice (VC).

Only one participant spent more than six minutes to consider this problem. The rest did not consider this is as a proper problem because it did not involve any mathematical calculation. The participant who spent more than six minutes on this question motivated herself by "Come on if you are in a desert, you can find more uses for the tin can. You need to be creative to find uses for these tin cans." Perhaps it is when we are faced with extreme difficulties we are forced to think of more creative uses of ordinary objects around us. All the participants found uses for the tin as a container. Only one participant considered melting down the tin can into its raw material and making objects from the raw material.

Problem 2 was used to determine answers for the following questions

(i) Did the participants consider this as an administrative problem for the licensing authority or a mathematical problem?

(ii) If the participants knew of a particular formula would they apply it directly to solve the problem?

The question was phrased in a non-specific manner to give the participants the freedom to determine their own patterns. Would these participants, given this freedom, take the opportunity to do so? Or would they still believe that their solutions should resemble those that they have learned in schools? Would they be confident enough to believe that they can impose their own interpretations on the question and then to look for the appropriate solutions?

This problem highlighted the difficulties the participants had in trying to interpret the problem and to impose their own suitable solution to it. Only three participants considered this to be an administrative problem that needed mathematics to solve. Thirteen of the participants seemed to want reassurance that they were doing the right thing. They seemed to be acting out the belief that their solution would be acceptable if an external authority confirmed its suitability. The following list are questions frequently raised by the participants as a means to seek approval and reassurance for their problem-solving attempts.

(i) For example, could they form number plates of the form MUN213 or was it permissible to have N1M2U3?

(ii) "Is this correct?"

(iii) "Is this the right answer?"

The thirteen participants frequently raised questions (i) and (ii). They carried on with their solutions only when they received confirmations that they were on the right track. They applied the formula they had learned when they were in school.

Memory and confidence of one of the participants (identified as M) played a very important part in helping him to solve the problem. When M began the search, he did not attempt to recall the formula directly. He applied his own heuristic very successfully to his problem. M broke the problem down to various sub-problems. By reasoning, he first established the solution for choosing one character out of six different ones; and then two out of six.

Satisfied with these solutions, he proceeded to compare them with those that he obtained by substitution into a formula he faintly recalled. His confidence that he was on the right path was affirmed when the solutions matched. He then re-checked the formula by comparing the answer for choosing three with that obtained by reasoning. Satisfied that the formula he recalled was correct, he proceeded to use it to calculate number plates for four, five and six characters.

What seemed to have happened was M retrieved information from his long-term memory to construct and answer a more specific question. His memory appeared to be a looping, questioning activity. The search was active and constructive. When it could not go directly from one point to another (in this case retrieving the right formula and applying it to the problem), the problem was broken into a series of sub-problems. For each sub-problem the questions were: "Can it be solved?" Will the solution move me nearer to the actual formula?" When one sub-problem got solved, new ones were defined and the search continued.

M's success in reaffirming that the formula was correct was not a matter of simple recall. Rather it was a mixture of logical re-constructions of what he understood of the relevant concepts with the fragmentary recollections of what he had learned and experienced in school (Lindsay and Norman, 1972). He displayed good control over the resources he had and applied the right heuristic to the problem. M went back thirty-five years to retrieve his formula! When questioned why he needed to look for the formula, he put forward two reasons. First, when he was in school, he was taught to use the formula to solve such problems. Second with a formula one could proceed with any number of characters, thus "making life easier for you".

The way the question was phrased could have influenced the arrangements proposed by the participants. None of the participants started the arrangements with the numbers. All the participants in this group when confronted with this, admitted that the order of the letters and the numbers in question affected their approach to the solution. The participants' observation give credence to Torrance's (1983) proposal "In warming up pupils for creative thinking avoid giving examples or illustrations which will freeze or unduly shape their thinking. ... [T]he giving of examples establishes expectations which are difficult to break. This makes it difficult for pupils to get away from the obvious and commonplace in their thinking" (p. 27). Even though no specific examples were given in this question the fact that the letters preceded the numbers in the question had a strong influence on the participants' thinking.

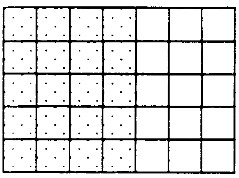
The participants were very confident of their solutions. The familiar structure of the question could have inspired such confidence in those participants who admitted to being familiar with such questions. The formal learning experience of school seems to have a tenacious hold on the thinking of these participants. As seen, one of the participants went back thirty-five years to seek out a formula he knew would work for him in this question.

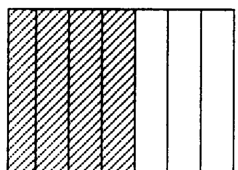
Thus the promise of solutions being contained within the framework of a formula could have given added confidence to these participants. This could also have prevented all but three participants from venturing into other possible combinations. These three participant considered different arrangements for the number plates but they evaluated their arrangements so that more cars could be coded and that the public would have no difficulty reading the numbers on the number plate. The third participant was more flexible in his thinking as he considered different spatial arrangements for the number plates but he rejected the vertical arrangement as space might be a problem. For these three participants they did not adhere to the task of finding an answer to the problem but they evaluated their solutions to the problem, a quality emphasised by Polya (1957).

Problem 3 was not to test the participants' knowledge of equivalent fractions. It was to ascertain if the participants would assess the situation as how they would help the child understand why these two fractions were equivalent. How would the participants manage two differing demands, one being that this is a young child and secondly how would they garner their mathematical knowledge to help the child.

Only six of the sixteen participants asked for the age of the child and if the child could do fractions. With respect to the second question, the six participants were told that the child was capable of applying algorithms for reducing fractions. What the child wanted to know was why the two fractions are equal.

Fourteen participants treated the child's problem as only a problem with $\frac{20}{35} = \frac{4}{7}$. Three of the fourteen participants explained to the child using the algorithm for reduction of fractions. The rest of this group drew the standard fraction diagrams to demonstrate the equality



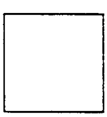


$$\frac{20}{35} = \frac{4}{7}$$

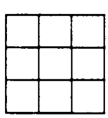
Only two developed their explanations by drawing upon other simpler equivalent fractions $\frac{2}{4} = \frac{1}{2}$ and then addressing the child's question.

The fourteen participants who went straight to the problem seemed to be explaining to an imaginary teacher. The simplicity and familiarity of the question evoked such strong memories of their school mathematics learning that it was not surprising to hear them say "That's how I learned it", "That's how my teacher taught me" at the end of their explanations. Only one of the two participants who started from simpler solutions questioned the suitability of using concrete materials to explain the equivalence. It might not be possible for larger numbers.

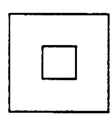
In Problem 4 participants were offered two sets of squares to solve the problem. They could use squares drawn on blank paper (blanks) or squares drawn on grids (grids) or both to solve the problem. The size of the squares was arbitrary. The solution of squares within squares was not acceptable.



'Blanks'



'grids'



Squares within squares

Below is an example of a possible solution set.

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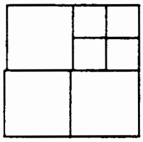

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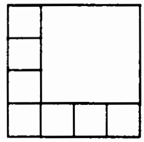

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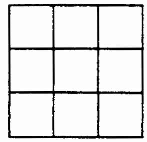

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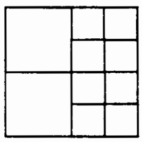

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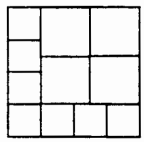

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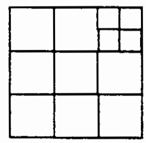

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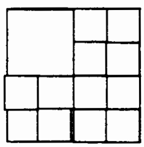

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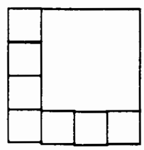

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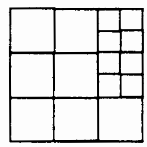
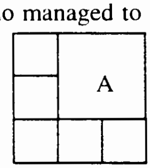


To facilitate discussion of the participants' responses, the solutions were divided into two sets.

Normal Set	Solutions '7', 9', '10', '12', '13' and '15'
Off-set	Solutions '8', '11' and '14'

When the participants could not proceed at all they were asked if another example would help. The '6' was then given. If they still could not proceed the '7' was used as an additional prompt. For those participants who managed to produce solutions in the normal set but not those of the off-set, the '6' was given.

The '6' L-shaped pattern



Group BO (7 participants): 'Blanks' only	They saw the patterns in the '7' and the '9' column.
Group GB (3 participants): Grids and then Blanks	2 changed over with prompting.
	1 changed over on own initiative.
Group GP (4 participants): Grids only	They saw patterns as in group BO
Group GR (2 participants): 'Grids' only	Random divisions along lines

Only one participant of Group BO solved the entire question on his own. Three others could not proceed beyond the '7' and '9' motif despite being shown the '6'. Two of the participants of Group GB made the change over after some prompting from me. All but one of the participants in the other groups had problem solving the problem despite receiving the '6' pattern.

All the participants raised this question. "Do the squares have to be the same size?" and fourteen of the participants asked if there were solutions to the '8', '11' and '14'.

All the participants thought that they would have difficulty solving for the squares in the '7' and '9' column. However they found that it was the '8' column that was most problematic. The reason was they applied the same division pattern of the '7' and '9' columns to the '8'. When they could not proceed to find the solution to the '8' most were unable to break out of that pattern until the '6' was shown. Perhaps these participants were caught in a cognitive tunnel vision and they had difficulty managing their resources to get out of this tunnel.

Participants who chose the grids thought the grid lines would help them in the divisions of the squares. Unfortunately they found that the grid lines hampered and confused them in their divisions. They were forced to divide squares within the grid lines and this confused them further as they thought that the dimensions of the squares had to be of integral values. As a result of their confusion they were not sure if A was a square. Rather than they managing the effects of the lines the participants ended up being controlled by them.

Participants who chose the 'blanks' did not experience the difficulty faced by those who chose the 'grids'. When the former divided a blank square into smaller squares it was clear to them that the remaining parts of the larger square was another smaller square. Unlike those who chose the 'grids' they were aware that the dimensions of the squares need not be of integral values.

Discussion

Though the four problems were very different, there were some common findings based on the responses and the behaviours of the participants in their attempts to solve the problems.

The participants were uncomfortable with Problem 1 and Problem 4. Problem 1 had no fix answers and Problem 4 was unfamiliar to all of the participants. They seem to believe that a problem is a 'proper' problem when it requires mathematical formulae to solve it. Problem 1 was thus not a real problem because there was no specific formula that they could apply to solve it. They did not see the value of engaging in divergent thinking. Problem 2 was a problem as it was similar to questions they had encountered in school and the quickest way to solve it was to apply the formula. The positive finding from Problem 2 was that many adults do remember the formula they learned while in school and would search for such a formula when confronted with a related problem. Most of the participants referred to me for confirmation that they were on the right track because I introduced the problem to them. I believe that this was a carry over effect of their school experience. In schools either the teacher or the textbook was the authority to affirm that their solutions were correct. However it could be that the participants found it easier to work with a partner on a problem. In this case I was the partner and my presence made it easier for them to articulate their thoughts.

Successful problem solvers exercised good managerial control in their problem solving exercise. Good problem solvers not only exercise control over numerical facts but they also consider non-numerical information presented in the problem. This is indicated in Problems 2 and 3 where they considered the relevance of their solutions to the context of the problems. Good problem solvers were able to predict the effect the choice of materials had on their solution. This is demonstrated in Problem 4 where participants who chose the 'blanks' had to cope with fewer problems than those who chose the 'grids'. While some problem solvers were happy to finish with the task, good problem solvers were more likely to evaluate their solutions. Perhaps the former were so taxed by the problems that they were glad to have completed the task. Some of the participants who were successful in solving for Problem 4 expressed their displeasure that there was an off-set pattern to the solution. One of the participants expressed his displeasure when he said "I hate it".

Some of the participants denied themselves a fair chance at problem 4 right at the outset of the problem. One of the participants believed that because it was unfamiliar she would be unable to solve it: "I have never been able to do such things."

Conclusion

Individuals are strongly influenced by their school experiences and they tend to be uncomfortable with unfamiliar problems. Many individuals seem to work better with another individual as the presence of a partner encourage problem solvers to articulate their thoughts. Perhaps schools could encourage group problem solving among their students by providing students with more unfamiliar problems. This practice of publicly articulating their thoughts during problem solving might help problem solvers transfer their public problem-solving speech into either self-regulatory private or inner speech.

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