AN EXPLORATORY STUDY OF SINGAPORE PROSPECTIVE MATHEMATICS TEACHERS' PROBLEM-SOLVING ABILITIES

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Abstract

This paper reports on an exploratory study that was conducted to examine the types of heuristics selected by twenty-five prospective mathematics teachers when they solved four mathematical problems during a one-hour written test. The findings revealed that the prospective teachers were able to apply the appropriate heuristics to solve successfully three of the four problems given. They were, however, less successful on the problem that required logical deduction. Some of them even used multiple heuristics to solve the problems.

Introduction

Mathematical problem solving has been the core of the Singapore mathematics curriculum framework since 1991. Emphasis has been placed on it so as to prepare our present students to become better thinkers, good problem solvers and learners to meet the challenges ahead of them. During the last decade, problem solving has spawned numerous studies in Singapore, exploring factors such as the choice of heuristics (Ramasubban, 1997; Wong, 2002), analytic reasoning ability (Yeap, 1997), approaches to enhance problem-solving abilities (Ho, 1997; Chen, 2001; Seoh, 2002) and affects that contributed to the high levels of success in mathematical problem solving (Kuppusamy, 1992). The subjects in these studies are mainly students. However, problem-solving studies conducted on teachers in Singapore are relatively scant (e.g., Foong, 1994).

The call for such studies also draws its impetus from the current changing profile of prospective mathematics graduate teachers in Singapore. In recent years, prospective mathematics teachers in the Post-graduate Diploma in Education (PGDE) (Secondary) programme at National Institute of Education (NIE) hail from different disciplines. Since teachers' experience in problem solving may influence their teaching pedagogy, which in turn will shape the students' learning, so how are they at problem solving? It may be timely now for us to conduct studies on teachers to examine their mathematical knowledge and skills in problem solving so as to gain an insight into their problem-solving abilities.

For these reasons, I set out to investigate how twenty-five prospective mathematics graduate teachers from the PGDE (Secondary) programme approached and solved four mathematical problems in this study. This study addresses the following research question:

> Which problem-solving heuristics do the prospective mathematics graduate teachers of the PGDE (Secondary) programme use to solve the four mathematical problems?

Method

The study used the quantitative approach which involved the prospective teachers sitting for a one-hour test made up of four mathematical problems after introducing the various heuristics to them. The test aimed to examine the types of heuristics used to solve each of the four problems. In what follows, details of the prospective teachers, the test instrument and the procedures will be presented.

Subjects

Twenty-five prospective mathematics graduate teachers from the January 2003 intake of the PGDE (Secondary) programme participated in the study. All of them, whose first teaching subject was Mathematics, came from the same class and had just embarked on their formal pre-service teacher training after completing their teaching experience in various secondary schools.

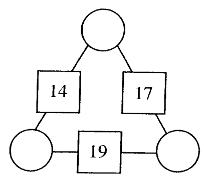
The academic qualifications of the prospective teachers were quite diverse. They were mostly Science and Engineering graduates whilst a few were Arts graduates. All, except for one who held a Master's, had a Bachelor's degree. They also had outstanding academic results for mathematics at both the O-level and A-level. At the O-level, at least 80% of them attained distinctions (Grades A1 and A2) for Elementary Mathematics and for Additional Mathematics. At the A-level, about 70% of them attained distinctions (Grade A) for Mathematics C.

Instrument

The test instrument used in this study comprised four mathematical problems which were non-isomorphic to any of those that had been discussed and presented during tutorials. Neither were they typical problems found in any textbooks used in the local schools. These problems, which were deliberately selected because each could be solved by at least one heuristic, were useful to study the prospective teachers' choices of heuristics when they solved unfamiliar problems.

Three of the four problems, namely "Car Wash", "Arithmagon" and "Library", were adapted from the instrument used by Malloy and Jones (1998) in their study of African-American students whilst the fourth problem, "Golden Apples", was shared by a NIE colleague during a mathematics course. The four problems are presented in Table 1 below.

Table 1
Mathematical problems in the test instrument

Problems
Car Wash Adam, Ben, Chen and Dan had a car wash on Saturday. Adam washed twice as many cars as Ben, Ben washed 1 fewer than Chen, Chen washed 6 more than Dan. Dan washed 6 cars. How many cars did Adam wash?
Arithmagon In the three-sided figure below, the number in each square must be equal to the sum of the number in the two circles on either side of the square. Find the numbers that go inside the circles. 
Library During the *Reading Week* in school, the teacher librarian plans to recommend three different books each day by placing the book numbers on a board in the front of the library. The teacher librarian must buy plastic cards to put on the board. Each card has one large digit on it. The teacher librarian wants to buy as few cards as possible. The books are numbered from 1 to 632. What is the fewest number of cards that must be purchased to make sure it is possible to display any selection of three different titles each days?
Golden Apples A prince picked a basketful of golden apples in the enchanted orchard. On his way home, he was stopped by a troll who guarded the orchard. The troll demanded payment of one-half of the apples plus two more. The prince gave him the apples and set off again. A little further on, he was stopped by a second troll guard. This troll demanded payment of one-half of the apples the prince now had plus two more. The prince paid him and set off again. Just before leaving the enchanted orchard, a third troll stopped him and demanded one-half of his remaining apples plus two more. The prince paid him and sadly went home. He had only two golden apples left. How many apples had he picked?

Procedures

In Foong's (1994) study of local prospective teachers regarding their problem-solving processes, she discovers that problem solvers' failure to solve an unfamiliar problem is often due to their lack of knowledge of a set of useful heuristics to help them approach it. But knowing a set of heuristics does not necessarily ensure they will know how to apply each of them. According to Thompson's (1989) study of teachers on their conceptions and beliefs of problem solving, modelling the use of heuristics in solving problems is equally important as well. Hence, the instructional approach adopted in this study is basically similar to what many mathematics teachers are used to teach in schools: introduction of heuristics, then demonstration of examples, followed by independent practice, finally presentation of solutions on the board and checking of answers. More details are outlined in the remaining section.

Three tutorials were spent on problem solving. The prospective teachers were firstly introduced to the Polya's problem-solving model. Next, since many of them were not exposed to the Model Approach which was a unique problem-solving heuristic used in Singapore primary schools, several examples were illustrated to demonstrate how some arithmetical problems could be solved by this approach. This was then followed by an individual class exercise to give them some opportunities to practise and reinforce what they had learned. Finally, some prospective teachers were called upon to present their solutions on the board whilst the rest checked their solutions.

After teaching the Model Approach, other commonly used heuristics, listed in Table 2 below, were also introduced to the prospective teachers. Each heuristic was illustrated with an example to demonstrate its application. Additional problems were then given to the prospective teachers as homework for individual practice to reinforce what they had learned. A week later, some prospective teachers were called upon during a tutorial to present their solutions on the board whilst the rest checked their solutions.

Table 2
List of heuristics taught to the prospective teachers

Heuristics
(a) use a diagram or tabulation
(b) guess and check
(c) make a systematic list
(d) look for pattern
(e) work backwards
(f) use logical deduction
(g) make suppositions
(h) restate the problem in another way
(i) simplify the problem
(j) solve part of the problem

A week after the prospective teachers had presented and shared the additional problems with the class, they sat for the test during an extra tutorial outside their curriculum time. Without the use of a calculator, they had to complete the four problems in the test instrument within an hour, as well as to show all their workings clearly on the test scripts. Their scripts were then marked and their choice of heuristics was tabulated.

Findings

The findings from the study have revealed that the prospective teachers were successful on three of the four problems: *Car Wash*, *Arithmagon* and *Golden Apples*. At least twenty-one of them had correctly solved each of these problems. However, they were less successful on the "Library" problem which had been correctly solved by only four of them. A summary of the number of prospective teachers who had used the heuristics in correct and incorrect solutions for each problem is presented in Table 3 below. In the remaining section, the answer to the research question is presented.

Table 3
Number of subjects who had used the heuristics in correct and incorrect solutions

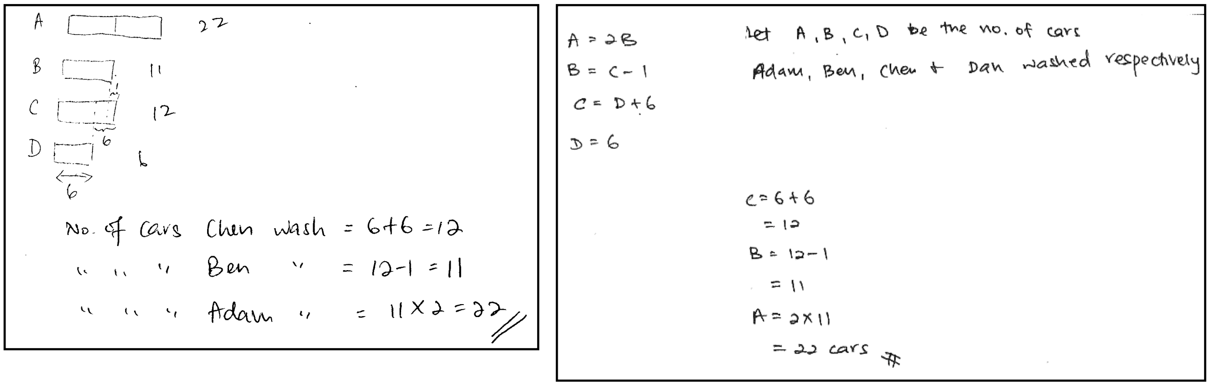
Heuristic	**Car Wash**		**Arithmagon**		**Library**		**Golden Apples**	
	Correct n = 24	Incorrect n = 1	Correct n = 25	Incorrect n = 0	Correct n = 4	Incorrect n = 21	Correct n = 21	Incorrect n = 4
Draw a diagram/model	12	-	-	-	-	-	5	1
Work backwards	23	1	-	-	-	-	11	-
Guess & Check	-	-	12	-	-	-	-	-
Make a systematic list	-	-	6	-	1	2	-	-
Use logical deduction	-	-	-	-	4	16	-	-
Use equation	4	-	13	-	-	1	10	3
Did not attempt	-	-	-	-	-	4	-	-

Research question: *Which problem-solving heuristics do the prospective teachers use to solve the four mathematical problems?*

The quantitative analysis of data has revealed the following results:

(i) For "Car Wash", the prospective teachers used "Work backwards" and "Draw a diagram/model". For "Arithmagon", "Guess and Check" and "Make a systematic list" were used. For "Library", "Use logical deduction" and "Make a systematic list" were used. For "Golden Apples", "Work backwards" and "Draw a diagram/model" were used. In summary, five different heuristics were used in this study: *Work backwards*, *draw a diagram/model*, *guess and check*, *make a systematic list* and *use logical deduction*.

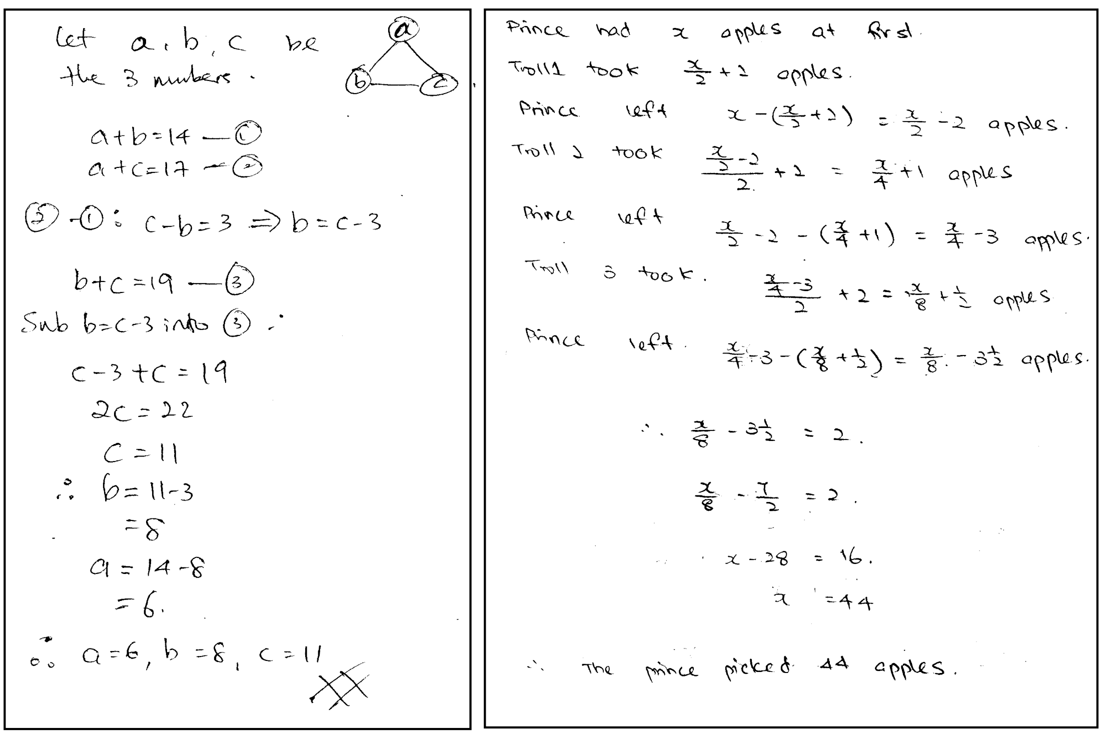
(ii) Some prospective teachers resorted to solving the problems by equations despite the availability of other heuristics. For instance, "Car Wash" could actually be solved simply by "Draw a diagram/model" and "Work backwards" without having to involve any equations (see Figure 1a below). Yet four prospective teachers solved it by using equations. Figure 1b below presents an example in which the problem was solved by equations. Other than "Car Wash", equations were also used in solving "Arithmagon" and "Golden Apples" (see Figure 2 below).



a. “Draw a diagram/model” and “Works backwards

b. By “Use equations” and “Work backward”

*Figure 1. Strategie*s used to solve “Car Wash”

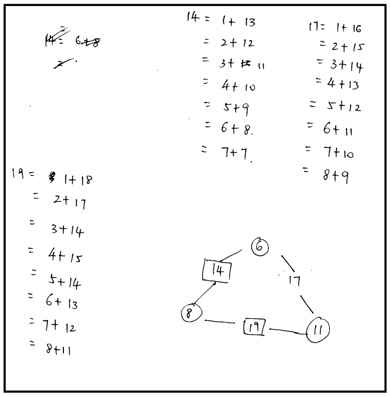


a. Arithmagon

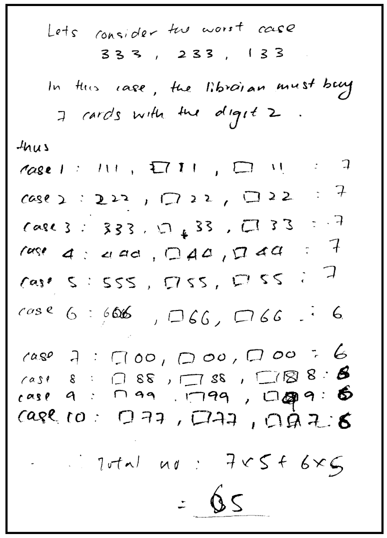
b. Golden Apples

Figure 2. Problems solved by equations

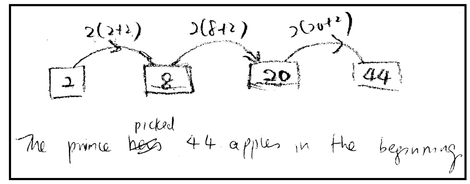
(iii) Multiple heuristics were used by some prospective teachers to solve the problems. For instance, some used the following combinations of two heuristics to correctly solve “Car Wash”: “Draw a diagram/model” with “Work backwards” (see Figure 1a), and “Use equation” with “Work backwards" (see Figure 1b). For "Arithmagon", "Guess and check" was used together with "Make a systematic list" (see Figure 3a). For "Library", "Use logical deduction" was combined with "Make a systematic list" (see Figure 3b). For "Golden Apples", "Work backwards" was used in conjunction with "Draw a diagram/model" (See Figure 3c).



a. Arithmagon



b. Library



c. Golden Apples

Figure 3. Problems solved by multiple heuristics

Discussion

As clearly indicated in Table 3 above, many prospective mathematics teachers in this study have demonstrated their understanding of a few heuristics by showing their ability to apply these heuristics appropriately to solve the following three problems correctly: *Car Wash*, *Arithmagon* and *Golden Apples*. In contrast to the high success rates for these problems, the unexpectedly low success rate for "Library" exposes their shortcoming in logical deduction and inadequacy in solving such a problem.

Only four prospective teachers have successfully solved "Library" by logical deduction. They have adopted an analytic approach by considering the various cases, then counting the number of cards required for each digit in each case before totalling up the number of cards required for the entire set to display the three different titles simultaneously (see Figure 3b). On the other hand, many of those who have failed to solve it appear to have misinterpreted the problem. Instead of counting the minimum number of cards required to display three titles concurrently, they have counted the number of cards required for each digit to display only one title. For instance, to display any number from 1 to 632, the minimum number of cards required for each digit from 1to 5 is three whilst that for each digit from 6 to 9 and the digit 0 is two. Hence, fewer cards are required in this case. An example of this incorrect solution is presented in Figure 4.

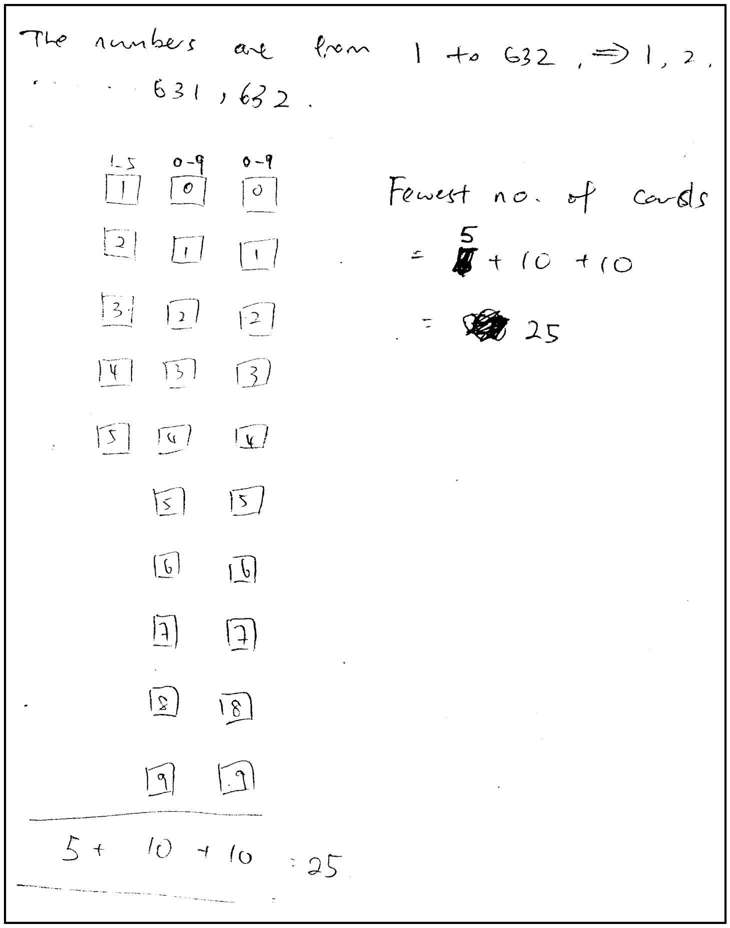


Figure 4. An incorrect solution of "Library"

Another interesting finding in this study is the use of equations to solve some of the given problems. Its use, most prevalent in "Arithmagon" and "Golden Apples", is hardly surprising especially when the subjects are prospective teachers who have mastered the essential algebraic knowledge and manipulative skills. Drawing from the anecdotal evidence provided by them, there are two reasons for using equations. First, some teachers believe that "Use equations" is a very powerful and the most familiar heuristic, and that many problems can seemingly be solved by equations. Second, some had chosen it at that time because they had no idea which other appropriate heuristic to use.

Although "Use equations" is a powerful heuristic, prospective teachers should realise that its use can sometimes result in not only a long and complicated, but also tedious working. Figure 2b above is a manifestation of such an example for "Golden Apples". As Figure 3c above has made clear, a more efficient approach is to solve it by "Work backwards". Interestingly, all those who used only "Work backwards" had correctly solved this problem. On the other hand, not all who used equations were successful. Three of them had incorrectly solved it. Figure 5 below presents an example showing how a prospective teacher has attempted to solve "Golden Apples" by equations.

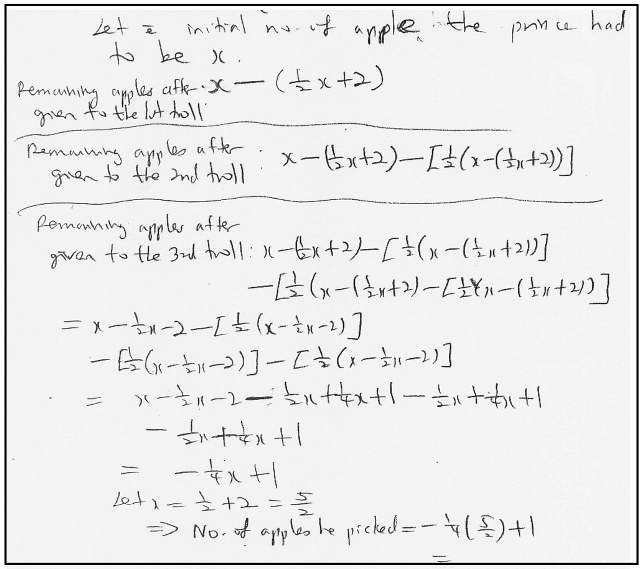


Figure 5. An incorrect solution of "Golden Apples"

Finally, the use of multiple heuristics to solve the problems indicates that it is usually necessary to use more than one heuristics to solve a problem. Additionally, the prospective teachers also display their flexibility in their strategies to solve the problems when they use multiple heuristics in their solutions.

Implications for teachers

The findings have suggested that the prospective mathematics teachers are familiar with the use of some problem-solving heuristics. As a whole, they also seem to be quite successful in solving problems, perhaps except for those that involve logical deduction. So a good start for any teacher who wishes to improve on his or her reasoning skills is to work on non-routine mathematical problems found in mathematics quiz or competition books, mathematical journals or even mathematics textbooks. This is important because solving problems continually on his or her own has some of the following benefits: honing his or her problem-solving and reasoning skills, and becoming more flexible, as well as more confident, in mathematical problem solving. In the long term, he or she will become a more successful problem solver.

When teaching mathematical problem solving in schools, it is crucial for teachers to know and accept a variety of strategies to tackle a non-routine problem so as to build up their students' confidence in problem solving, rather than insisting on using prescribed strategies which they have learned (Vacc & Bright, 1999; Perkins & Flores, 2002). However, a teacher may not think of all the possible strategies to solve a given non-routine problem. Thus, it is important to expose prospective teachers to as many different strategies as possible to such a problem during their pre-service teacher training. One way to do it is to get one of them from the class to compile the various strategies to a non-routine problem, then make the compilation available to all the rest in the class. Similarly for teachers in schools, their sharing at departmental meetings can expose fellow mathematics teachers to the various strategies of solving non-routine problems.

Given a mathematical non-routine problem, teachers may work out its solution first to determine which level of students the problem is suitable for. When making this decision, teachers may like to contemplate a few issues, particularly when they have solved the problem by equations. The first is whether their students have learned the essential mathematical content knowledge to understand the solution. Take solving "Arithmagon" by equations for instance. If the solution (see Figure 2a above) were to be shown to lower secondary students in Singapore, many of them may probably encounter some difficulties in understanding it, especially when it involves solving three equations in three unknowns as follows: reduce the three equations to two by eliminating one of the three unknowns first, then solve the remaining two equations in two unknowns simultaneously. This is because solving simultaneous equations in two variables is not taught in secondary one, but in secondary two. Even if the solution for Arithmagon by equations is shown to secondary two students after they have learned to solve simultaneous equations, it may still be abstract for some to understand fully.

The second issue teachers have to contemplate is whether every student has an equal opportunity to work on the problems. This concern has arisen because teachers tend to give problems, which they have solved by equations, to the more able students, thereby neglecting the less able ones. To illustrate, a teacher who solves "Arithmagon" by simultaneous equations may deem it unsuitable for the less able students, thinking that they will not be able to manipulate the equations. Hence, the problem may be given to the more able students when it can actually be given to the less able students as well, who may solve it by a different heuristic other than by equations.

Conclusion

This study was an exploratory investigation into the prospective mathematics teachers' choices of heuristics when they solved four mathematical problems. The prospective teachers were successful on three of the four problems given, except for the one that required logical deduction. As the quantitative data analysis had clearly indicated, many of them were able to choose the appropriate heuristics to solve the problems. Some even used multiple heuristics to solve the problems while some chose to solve a couple of problems by equations.

However, it should be noted that some of the findings of this study may not be generalisable to a wider population of prospective mathematics teachers in the PGDE (Secondary) programme, due to the use of only one intact class, as well as the small sample size. The prospective teachers in this study were clearly successful in mathematics in accordance with their mathematics academic results in both O-level and A-level examinations. Additionally, they were expected to have a strong foundation in mathematics, given that their first teaching subject is mathematics. So they do not necessarily reflect the abilities typical of prospective mathematics teachers in the PGDE (Secondary) programme.

In conclusion, it must be remembered that students' learning may be shaped by their teachers' pedagogical approaches. So if the current emphasis on preparing and training students to apply and translate their learning into handling and solving real-world problems is to continue, then more studies need to be conducted on teachers in Singapore schools to gain an insight into their problem-solving abilities. Therefore, the present study can be extended to include a bigger sample of prospective mathematics teachers in the PGDE (Secondary) programme, as well as trained mathematics teachers who are presently practising in schools. Furthermore, the test instrument can be extended to include the task of solving a problem in as many ways as possible. Finally, the future study should also include interviews with teachers to probe their reasons for using a particular heuristic, as well as their attitudes towards problem solving.

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