Protecting two-qubit quantum states by π-phase pulses

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We study the state decay of two qubits interacting with a common harmonic oscillator reservoir. We find both a decoherence error and the error caused by the amplitude change of the superradiant state. We show that frequent π-phase pulses can eliminate both types of errors and therefore protect a two-qubit odd-parity state more effectively than the frequent measurement method. This shows that the methods using dynamical decoupling and the quantum Zeno effects actually can give rather *different* results when the operation frequency is finite.

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I. INTRODUCTION

The interaction between a quantum system and its environment inevitably leads to the decoherence [1,2] of a quantum state. Such quantum decoherence can often cause severe distortion to a quantum state rendering many quantum systems in the real world useless [3–10]. In order to protect a quantum state, many methods against decoherence have been studied. Among the existing proposals, most of them are for single-qubit state protection [11–16]. Recently [17], a scheme for the protection of quantum entanglement of two qubits at 0 K was proposed using the quantum Zeno effect (QZE), i.e., via frequent measurement of the environment photon number for the Jaynes-Cummings (JC) model. However, as shown below, besides the decoherence error, the amplitude of the superradiant state can also be changed. It decreases with evolution. Intuitively speaking, the superradiant state changes into $|00\rangle$ gradually in the evolution, therefore the initial odd-parity state can be severely distorted after a long time evolution. The frequent measurement method cannot eliminate such distortion efficiently because it actually removes the term $|00\rangle$ at every step. After a long evolution time with a fixed measurement frequency, the amplitude of the superradiant state decreases a lot and therefore severely distorts the initial unknown state. On the other hand, given the existing technologies, it seems that the measurement of the photon number of the environment of all modes remains a challenging task. It is therefore an interesting problem to study how one could protect a two-qubit state with techniques which have been demonstrated already, for example, the dynamical decoupling scheme [11–16,18], which has been demonstrated experimentally recently [19–21]. It is well known that in the limit of infinitely frequent operations, QZE and dynamical decoupling are unified and can have the same results [22–24]. The two methods are not compared in the more realistic condition when the operation frequencies are finite. Here we show that the dynamical decoupling scheme achieved through a frequent application of π-phase pulses can protect a two-qubit state more effectively. The scheme not only protects the state from decoherence error, but also prevents the amplitude changing of the superradiant state.

This paper is arranged as follows: We first review the existing results of the JC model [17,25] for two qubits, in particular, the time evolution of the odd-parity state [17] under zero temperature. We point out why the amplitude of the superradiant state changes in a frequent measurement scheme. We then show how to protect the state by π-phase pulses and why the π-phase pulse scheme can prevent the amplitude change. The consequences of the finite frequency and duration of each pulse are also presented.

II. AMPLITUDE CHANGE OF SUPPERRADIANT STATE IN JC MODEL

Consider the following Hamiltonian for a two-qubit system and its environment as used in [17]:

$$H = H_s + H_e + H_i, \tag{1}$$

where

$$H_s = \omega_0(\sigma_1^+\sigma_1^- + \sigma_2^+\sigma_2^-) \tag{2}$$

is the Hamiltonian of the two-qubit system,

$$H_e = \sum_k \omega_k b_k^\dagger b_k \tag{3}$$

is the Hamiltonian of the environment, and

$$H_i = (\alpha_1\sigma_1^+ + \alpha_2\sigma_2^+)\sum_k g_k b_k + \text{H.c.} \tag{4}$$

is the Hamiltonian of the interaction between the system and the environment. The notations b_k and $b_k^\dagger$ are the annihilation and creation operators of the environment with frequency ω_k; ω_0 is the atomic transition frequency between the ground state $|0\rangle$ and the excited state $|1\rangle$; $\sigma^+ = |1\rangle\langle 0|$ and $\sigma^- = |0\rangle\langle 1|$. The solution of such a model for the case of zero environmental temperature is well known and it can be found in Refs. [25,26].

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In particular, for an odd-parity two-qubit initial state, there exists a dark state

$$|\mu\rangle = \frac{\alpha_2}{\alpha}|1\rangle_1|0\rangle_2 - \frac{\alpha_1}{\alpha}|0\rangle_1|1\rangle_2 \tag{5}$$

and a superradiant state [27–30]

$$|\nu\rangle = \frac{\alpha_1}{\alpha}|1\rangle_1|0\rangle_2 + \frac{\alpha_2}{\alpha}|0\rangle_1|1\rangle_2. \tag{6}$$

The dark state $|\mu\rangle$ does not change with time under the JC model, while the superradiant state changes with time according to

$$|\nu\rangle \otimes |0\rangle_e = \eta(t)|\nu\rangle \otimes |0\rangle_e + |0\rangle_1|0\rangle_2 \otimes \sum_k [c(t)_k|1_k\rangle_e] \tag{7}$$

and

$$\eta(t) = e^{-\lambda t/2}\left[\cosh\left(\frac{\Omega t}{2}\right) + \frac{\lambda}{\Omega}\sinh\left(\frac{\Omega t}{2}\right)\right], \tag{8}$$

where $\Omega = \sqrt{\lambda^2 - 4R^2}$, $\alpha = \sqrt{\alpha_1^2 + \alpha_2^2}$, and $R = \alpha W$. Given any odd-parity initial state $|\psi_0\rangle = \beta_1|\mu\rangle + \beta_2|\nu\rangle$, the time evolution is

$$\beta_1(t) = \beta_1, \tag{9}$$

$$\beta_2(t) = \beta_2\eta(t), \tag{10}$$

and $\eta(t)$ is given in Eq. (8),

$$\begin{aligned}|\psi(t)\rangle = {} & [\beta_1|\mu\rangle + \beta_2\eta(t)|\nu\rangle] \otimes |0\rangle_e \\ & + \sum_k |0\rangle_1|0\rangle_2 \otimes c(t)_k|1_k\rangle_e.\end{aligned} \tag{11}$$

The decoherence error comes from the term $\sum_k |0\rangle_1|0\rangle_2 \otimes c(t)_k|1_k\rangle_e$. As shown in Ref. [17], by frequently measuring the environment, one can remove the the term $|00\rangle$ and protect the initial state from the decoherence error, as long as one does not find a photon coming from the reservoir. Intuitively speaking, such a frequent measurement works like a state filter, which removes $|00\rangle$ during the stage when its probability is small. However, even though one can always remove the term with $|00\rangle$ successfully by measurement, one cannot protect the initial state for a long time with the scheme because $\eta(t)$ decreases significantly with time. Suppose the environment is measured after every time interval Δt, and we continue to find no photon. At time t, the state is changed into

$$|\psi(t)\rangle = [\beta_1|\mu\rangle + \beta_2 r(t)|\nu\rangle] \otimes |0\rangle_e \tag{12}$$

and

$$r(t) = [\eta(\Delta t)]^{t/\Delta t}. \tag{13}$$

Since each measurement removes the photon in the environment, $|\nu\rangle \otimes |0\rangle_e$ restarts the evolution from the initial state again in each time interval. To protect the two-qubit state of the system more effectively, we can use the dynamical decoupling scheme through applying π pulses frequently instead of a filtration scheme with frequent measurement. The main idea is that whenever the state decays a little bit, i.e., $\eta(t)$ decreases a little bit and the term with $|00\rangle$ appears with a small amplitude, a π pulse is applied and as a result, $\eta(t)$ will rise and the amplitude of the term with $|00\rangle$ decreases. That is to say, a π pulse does not simply remove the term with $|00\rangle$, it changes the term with $|00\rangle$ back to the superradiant state. Therefore it differs from the state filtration of the measurement-based scheme—it is really a more effective scheme of state recovery.

III. ELIMINATION OF DECOHERENCE WITH FREQUENT π-PHASE PULSES

We show here that π-phase pulses can eliminate the decoherence on the one hand and prevent amplitude changing of the superradiant state on the other hand. A π-phase impulse takes a phase-shift operation as

$$|0\rangle \to -|0\rangle \; |1\rangle \to |1\rangle. \tag{14}$$

Apply a π pulse to each qubit and the two-qubit unitary operation is then

$$\begin{aligned}|1\rangle_1|0\rangle_2 &\to |1\rangle_1|0\rangle_2,\\ |0\rangle_1|1\rangle_2 &\to |0\rangle_1|1\rangle_2,\\ |0\rangle_1|0\rangle_2 &\to -|0\rangle_1|0\rangle_2.\end{aligned} \tag{15}$$

A. Some iteration formulas

Our method involves simultaneously applying π pulses to each qubit. To obtain the results of the method, we need some iteration formulas first. Suppose the interval between two consequent impulses is τ and $n = [\frac{t}{\tau}]$; we can calculate the coefficients r_1 and r_2 by

$$\begin{aligned}\dot{r}_1(t) = {} & -\int_{n\tau}^{t} d\kappa f(t-\kappa)\big[\alpha_1^2 r_1(\kappa) + \alpha_1\alpha_2 r_2(\kappa)\big]\\ & - \sum_{m=0}^{n-1}(-1)^{n-m}\int_{m\tau}^{(m+1)\tau} d\kappa f(t-\kappa)\\ & \times \big[\alpha_1^2 r_1(\kappa) + \alpha_1\alpha_2 r_2(\kappa)\big],\\ \dot{r}_2(t) = {} & -\int_{n\tau}^{t} d\kappa f(t-\kappa)\big[\alpha_2^2 r_2(\kappa) + \alpha_1\alpha_2 r_1(\kappa)\big]\\ & - \sum_{m=0}^{n-1}(-1)^{n-m}\int_{m\tau}^{(m+1)\tau} d\kappa f(t-\kappa)\\ & \times \big[\alpha_2^2 r_2(\kappa) + \alpha_1\alpha_2 r_1(\kappa)\big],\end{aligned} \tag{16}$$

where we know that $c(t)_k$, the coefficient of $|1_k\rangle_e$ in Eq. (11), changes the signal because of the two phase impulses corresponding to $t = m\tau$ (m is an integer).

Under the interaction with a Lorentzian spectral density, we get a similar result as the free evolution, namely that there exist one dark state $|\mu\rangle$ and one superradiant state $|\nu\rangle$. We next study the condition that $\beta_1(t)$ and $\beta_2(t)$ should satisfy within one time interval.

When t is between $m\tau$ and $(m+1)\tau$,

$$\beta_1(t) = \beta_1, \tag{17}$$

$$\ddot{\beta}_2(t) + \lambda\dot{\beta}_2(t) + R^2\beta_2(t) = 0. \tag{18}$$

We know that the general solution of Eq. (18) is

$$\begin{aligned}\beta_2(t) = e^{-\frac{\lambda(t-m\tau)}{2}}\bigg[& A_m\cosh\left(\frac{\Omega(t-m\tau)}{2}\right)\\ & + B_m\sinh\left(\frac{\Omega(t-m\tau)}{2}\right)\bigg],\end{aligned} \tag{19}$$

where $t \in [m\tau,(m+1)\tau]$. A_m and B_m are the constant coefficients of the solution $\beta_2(t)$ at each time interval. Our task now is to determine the relation of all A_m and B_m.

At $t=(m+1)\tau$ using the boundary condition, $\beta_2((m+1)\tau^-) = \beta_2((m+1)\tau^+)$ and $\dot{\beta}_2((m+1)\tau^-) = -\dot{\beta}_2((m+1)\tau^+)$, we can get the relation

$$A_{m+1} = e^{-\lambda\tau/2}\left[A_m \cosh\left(\frac{\Omega\tau}{2}\right) + B_m \sinh\left(\frac{\Omega\tau}{2}\right)\right],$$
$$B_{m+1} = -e^{-\lambda\tau/2}\left[A_m \sinh\left(\frac{\Omega\tau}{2}\right) + B_m \cosh\left(\frac{\Omega\tau}{2}\right)\right] + \frac{2\lambda}{\Omega}e^{-\lambda\tau/2}\left[A_m \cosh\left(\frac{\Omega\tau}{2}\right) + B_m \sinh\left(\frac{\Omega\tau}{2}\right)\right]. \quad (20)$$

Here $\beta_2(m\tau^-)$ means the value of $\beta_2(m\tau)$ before the pulse and $\beta_2(m\tau^+)$ is the value of $\beta_2(m\tau)$ after the pulse.

We know that if the initial state is a superradiant state, $\beta_2(t)=1$, the initial value should be $A_0 = 1$ and $B_0 = \frac{\lambda}{\Omega}$. By using the relationship between A_m, B_m, A_{m+1}, and B_{m+1}, we can get an analytical function $\xi(t)$ which can express the fidelity $\xi(t) = \sqrt{\langle\nu|\rho_\nu(t)|\nu\rangle}$ of the superradiant state,

$$\xi(t) = e^{-\frac{\lambda(t-m\tau)}{2}}\left[A_m \cosh\left(\frac{\Omega(t-m\tau)}{2}\right) + B_m \sinh\left(\frac{\Omega(t-m\tau)}{2}\right)\right], \quad (21)$$

for $t \in [m\tau,(m+1)\tau]$. The fidelity $F(t) = \sqrt{\langle\psi|\rho(t)|\psi\rangle}$ is given by

$$|\psi\rangle = (\beta_1|\mu\rangle + \beta_2|\nu\rangle), \quad (22)$$
$$F(t) = |\beta_1|^2 + |\beta_2|^2\xi(t). \quad (23)$$

With this iteration formula for $\xi(t)$ above, we see that the fidelity oscillates about a horizontal line within a small range after a pulse is applied, first increasing and then descending, as shown in Fig. 1. In contrast, the fidelity based on the quantum Zeno effect method descends almost monotonously with time.

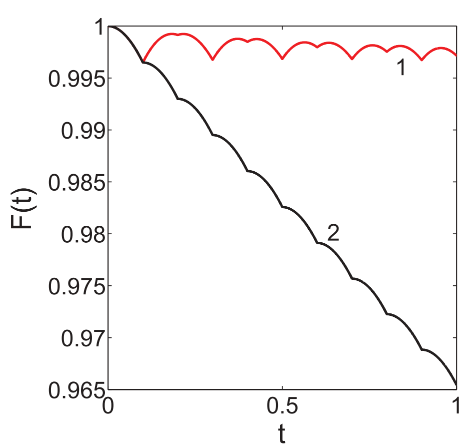

FIG. 1. (Color online) Comparison of the fidelity evolution during $t=[0,1]$ of two methods for the superradiant state. We set $\tau = 0.1$, $\lambda = 2$ and $\Omega = 1$. Line 2 is for dynamical decoupling of this work and Line 1 is for QZE of Ref. [17]. The fidelity only oscillates in a small range around a horizontal line in the dynamical decoupling method.

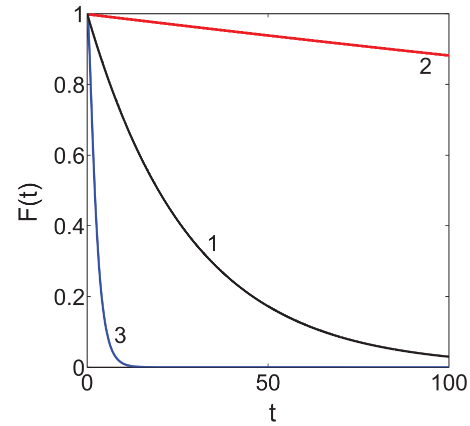

FIG. 2. (Color online) The initial state is the superradiant state. Line 2 is the fidelity under the sequential double-π-phase method (dynamical decoupling). Line 3 is the fidelity under QZE. Line 1 is the fidelity of free evolution under the interaction between system and environment. From the result, we can easily find that the sequential-pulse method is much better than the QZE.

B. Effect of finite duration of double-π-phase operation

In practice, one cannot set the duration pulse time to be infinitely small. Here we consider the more realistic case that the duration time of the double-π phase is finite and we study the effect on fidelity. We only consider the sequential pulse method and we suppose that the pulse duration is $\frac{\tau}{N}$ in each time interval τ. We know that the decoherence coefficient of the superradiant state is Eq. (19) when $t \in [m\tau,(m+1-\frac{1}{N})\tau]$. So the effective Hamiltonian for the sequential double-π-phase operator is $H_{\rm phase} = \frac{N\pi}{\tau}(\sigma_1^+\sigma_1^- + \sigma_2^+\sigma_2^-)\sum_{m\geqslant 1}[\Theta(m\tau - \frac{1}{N}\tau) - \Theta(m\tau)]$ and $\Theta(x)$ is the step function. Under the Hamiltonian $H_{\rm all} = H_s + H_e + H_i + H_{\rm phase}$, we can write down the state during the time of $[(m-\frac{1}{N})\tau, m\tau]$:

$$|\psi(t)\rangle = e^{-i(N\pi/\tau)\{t-[m-(1/N)]\tau\}}[r(t)_1|1\rangle_1|0\rangle_2 + r(t)_2|0\rangle_1|1\rangle_2]\otimes|0\rangle_e + \sum_k |0\rangle_1|0\rangle_2 \otimes c(t)_k|1_k\rangle_e. \quad (24)$$

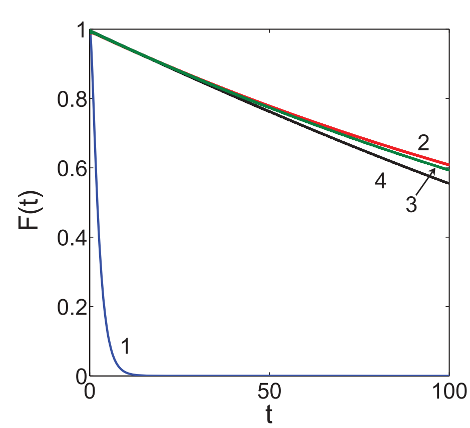

FIG. 3. (Color online) The initial state is the superradiant state. Line 2 is the method with the instantaneous pulse. Line 4 is for $N=20$ and line 3 is for $N=10$. Line 1 is for the free evolution.

In the same way as shown above, we also can get an integral equation describing the coefficient $\dot{r}_1 = -\int_0^t d\tau f'(t-\tau)[\alpha_1^2 r_1(\tau) + \alpha_1\alpha_2 r_2(\tau)]$ and the new correlation function of the duration time should be $f'(t) = W^2 e^{[-\lambda + i(N\pi/\tau)]t}$. Using the boundary condition of $t = (m - \frac{1}{N})\tau$ and $t = m\tau$, we eliminate the coefficient in the operator's duration time and get coupled relationships between A_m, B_m and A_{m+1}, B_{m+1} [Eq. (19)]

$$e^{-(\lambda/2)[1-(1/N)]\tau}\left\{A_m \cosh\left[\frac{\Omega}{2}\left(1-\frac{1}{N}\right)\tau\right] + B_m \sinh\left[\frac{\Omega}{2}\left(1-\frac{1}{N}\right)\tau\right]\right\} = \left(1+\lambda\frac{e^{\lambda\tau/N}+1}{4i\gamma-2\lambda}\right)A_{m+1} - \frac{e^{\lambda\tau/N}+1}{4i\gamma-2\lambda}\Omega B_{m+1},$$

$$\frac{\Omega}{\lambda}e^{-(\lambda/2)[1-(1/N)]\tau}\left\{A_m \sinh\left[\frac{\Omega}{2}\left(1-\frac{1}{N}\right)\tau\right] + B_m \cosh\left[\frac{\Omega}{2}\left(1-\frac{1}{N}\right)\tau\right]\right\}$$
$$= \left(1+e^{\lambda\tau/N}+\frac{\lambda}{4i\gamma-2\lambda}\right)A_{m+1} - \left(\frac{\Omega}{\lambda}e^{\lambda\tau/N}+\frac{\Omega}{4i\gamma-2\lambda}\right)B_{m+1}, \tag{25}$$

where $\gamma = \frac{N\pi}{2\tau}$.

The time evolution of the system is given by $|\psi(t)\rangle = [\beta_1|\mu\rangle + \beta_2\xi(t)|\nu\rangle]\otimes|0\rangle_e + \sum_k |0\rangle_1|0\rangle_2 \otimes c(t)_k|1_k\rangle_e$ and $\xi(t)$ is defined by Eq. (19) with the new relation of A_m and B_m above. Also, the fidelity of the state under decoherence with the original state is

$$|\psi\rangle = (\beta_1|\mu\rangle + \beta_2|\nu\rangle), \tag{26}$$

$$F(t) = |\beta_1|^2 + |\beta_2|^2\xi(t). \tag{27}$$

C. Numerical calculation of dynamical decoupling and free evolution

In case 1, we set $\tau = 0.1$, $\lambda = 2$, and $\Omega = 1$ and we compare the fidelity of the free evolution, sequential double-π-phase operator, and the random one. The result is shown in Fig. 2. In case 2, we set $\tau = 0.2$, $\lambda = 2$, $\Omega = 1$, $N_1 = 10$, and $N_2 = 20$ to compare how the duration time of pulses affect the fidelity. The result is shown in Fig. 3.

IV. CONCLUDING REMARKS

We present a strategy to protect the odd-parity states of two qubits under the 0 temperature environment by frequently applying the π pulses. Comparison between this method and the method based on frequent measurement is done; it seems that the frequent π-pulse method is more effective in protecting the states. Our result clearly shows that the quantum Zeno effect and the dynamical decoupling can have rather *different* results when the operation frequency is finite, though the two methods give essentially the same results in the limit of infinite operation frequency as shown in Refs. [22–24].

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[1] P. Štelmachovič and V. Bužek, Phys. Rev. A **64**, 062106 (2001).
[2] A. G. Kofman and G. Kurizki, Phys. Rev. Lett. **93**, 130406 (2004).
[3] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[4] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[5] D. P. DiVincenzo, Phys. Rev. A **51**, 1015 (1995).
[6] D. P. DiVincenzo and J. Smolin, *Proceedings of the Workshop on Physics and Computation, PhysComp'94* (IEEE Comp. Soc. Press, Los Alamitos, 1994), p. 14, e-print arXiv:cond-mat/9409111v1.
[7] Y. Makhlin, G. Schön, and A. Shnirman, Rev. Mod. Phys. **73**, 357 (2001).
[8] Y. P. Huang and M. G. Moore, Phys. Rev. A **77**, 062332 (2008).
[9] L. M. Duan, M. D. Lukin, J. I. Cirac, and P. Zoller, Nature (London) **414**, 413 (2001).
[10] J. Z. Hu, Z. W. Yu, and X. B. Wang, Eur. Phys. J. D **51**, 381 (2009).
[11] L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).
[12] H. P. Breuer, B. Kappler, and F. Petruccione, Phys. Rev. A **59**, 1633 (1999).
[13] D. Vitali and P. Tombesi, Phys. Rev. A **59**, 4178 (1999).
[14] G. S. Uhrig, Phys. Rev. Lett. **98**, 100504 (2007); **102**, 120502 (2009).
[15] W. Yang and R. B. Liu, Phys. Rev. Lett. **101**, 180403 (2008).
[16] F. F. Fanchini and R. d. J. Napolitano, Phys. Rev. A **76**, 062306 (2007).
[17] S. Maniscalco, F. Francica, R. L. Zaffino, N. Lo Gullo, and F. Plastina, Phys. Rev. Lett. **100**, 090503 (2008).
[18] J. R. West, B. H. Fong, and D. A. Lidar, Phys. Rev. Lett. **104**, 130501 (2010).
[19] J. Du *et al.*, Nature (London) **461**, 1265 (2009).
[20] M. J. Biercuk *et al.*, Nature (London) **458**, 996 (2009).
[21] H. Uys, M. J. Biercuk, and J. J. Bollinger, Phys. Rev. Lett. **103**, 040501 (2009).
[22] P. Facchi, D. A. Lidar, and S. Pascazio, Phys. Rev. A **69**, 032314 (2004).

[23] P. Facchi, S. Tasaki, S. Pascazio, H. Nakazato, A. Tokuse, and D. A. Lidar, Phys. Rev. A **71**, 022302 (2005).

[24] J. Busch and A. Beige, e-print arXiv:1002.3479.

[25] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, England, 1997).

[26] M. Tavis and F. W. Cummings, Phys. Rev. **170**, 379 (1968).

[27] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004); **97**, 140403 (2006).

[28] M. P. Almeida *et al.*, Science **316**, 579 (2007).

[29] G. M. Palma, K.-A. Suominen, and A. K. Ekert, Proc. R. Soc. London, Ser. A **452**, 567 (1996).

[30] P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997).