Information theoretic approach to single-particle and two-particle interference in multi-path interferometers

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We propose entropic measures for the strength of single-particle and two-particle interference in interferometric experiments where each particle of a pair traverses a multi-path interferometer. Optimal single-particle interference excludes any two-particle interference, and vice versa. We report an inequality that states the compromises allowed by quantum mechanics in intermediate situations, and identify a class of two-particle states for which the upper bound is reached. Our approach is applicable to symmetric two-partite systems of any finite dimension.

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Interference effects of two kinds can be observed in quantum processes that involve pairs of particles. There are the usual single-particle interference fringes and there are phase-dependent coincidence probabilities that constitute two-particle interferences (see, e.g., [1, 2]). As Horne and Zeilinger noted already in 1985 [1], there is a certain complementarity between the two types of interference: Optimal interference of one kind excludes any interference of the other.

The various quantitative studies of the possible trade-off between single-particle and two-particle interference that are on record (see [3, 4] in particular) are dealing with two-path interferometers for the single particles. It is then possible to measure the strength of the single-particle interference by the familiar Michelson fringe visibility —the difference of maximal and minimal probabilities in one output divided by their sum— and introduce a matching measure for the two-particle interference.

There is, however, no unique analog for multi-path interferometers, even for single-particle interference [5, 6]. Dürr has recently compiled a list of desirable properties of any multi-path generalization of Michelson's two-path visibility [7], but quite a few of equally plausible definitions meet the criteria [8], and the extension to two-particle interference is largely unexplored territory.

In an attempt to close this gap, at least partly, we introduce here an information-theoretic definition of the measure of the strength of two-particle interference. It exploits the mutual information contained in the coincidence probabilities. The corresponding single-particle measure is, in some sense, the fringe-contrast analog of the entropic measure for path knowledge that was first used by Wootters and Żurek [9] in 1979 in the context of a two-path interferometer (Young's double slit), and later also by others (see, e.g., [10, 11]). But entropic measures of this kind have never been popular and have been criticized occasionally [4, 7].

Yet, we think that they possess certain advantages. In particular, there is the immediate benefit that the dimensionality of the systems is irrelevant (although we assume, for the purposes of this paper, that the two systems are of the same dimension).

Let us close these introductory remarks by mentioning that the problem studied here is very closely linked with the relations between single-particle fringe visibility and which-path knowledge in quantitative studies of wave-particle duality. Going back to Einstein's 1905 paper on the photo-electric effect, this intriguing aspect of quantum mechanics is as old as quantum theory itself. Recent studies include [3, 4, 7, 12, 13, 14] where also the earlier literature can be found.

Consider an interferometric experiment of the kind that is sketched in Fig. 1. A source emits pairs of entangled particles, one of each pair propagating to observer Alice (A) on the left, the other to Bob (B) on the right, N paths being available on either side. Before its registration by the respective detectors, each particle undergoes a unitary transformation consisting of the action of phase shifters (phases $\phi_k^{(\mathrm{A})}$ and $\phi_l^{(\mathrm{B})}$, respectively) and an unbiased symmetric multiport beam splitter. The latter effects the unitary discrete Fourier transformation, F, that is specified by the matrix elements

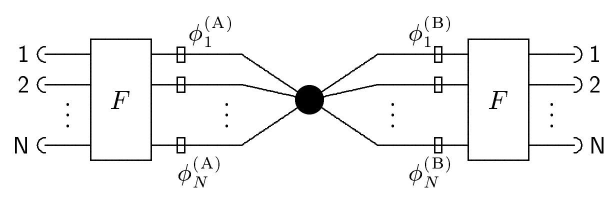

FIG. 1: Sketch of the interferometric setup. The source at the center emits paired particles. One of them propagates to the left, the other to the right, whereby both have a choice between N paths. After traversing the respective arrays of phase shifters (phases $\phi_k^{(\mathrm{A})}$ on the left, $\phi_l^{(\mathrm{B})}$ on the right) the particles pass through N-port beam splitters that effect a discrete Fourier transformation (F) of the probability amplitudes for the paths, and are then detected.

$F_{mn} = \gamma_N^{mn}/\sqrt{N}$ where $\gamma_N = \exp(2\pi i/N)$ is the basic N-th root of unity. Depending on the two-particle state emitted by the source, the resulting paired detector clicks may exhibit correlations that originate in the path interference that is tested by the Fourier transformation in conjunction with the phase shifters. It is our objective here to quantify the strength of the interference observed in experiments of this kind.

The experimental data are the N^2 coincidence probabilities $p_{mn}^{(\mathrm{AB})} = p_{mn}^{(\mathrm{AB})}(\phi_1^{(\mathrm{A})},\dots,\phi_N^{(\mathrm{A})};\phi_1^{(\mathrm{B})},\dots,\phi_N^{(\mathrm{B})})$, of unit sum, between Alice's m-th detector and Bob's n-th detector ($m,n = 1,2,\dots,N$). The relative frequencies of Alice's and Bob's individual detection events are given by the marginal sums of $p_{mn}^{(\mathrm{AB})}$,

$$
\begin{aligned}
p_m^{(\mathrm{A})}(\phi_1^{(\mathrm{A})},\dots,\phi_N^{(\mathrm{A})}) &= \sum_{n=1}^{N} p_{mn}^{(\mathrm{AB})}\,,\\
p_n^{(\mathrm{B})}(\phi_1^{(\mathrm{B})},\dots,\phi_N^{(\mathrm{B})}) &= \sum_{m=1}^{N} p_{mn}^{(\mathrm{AB})}\,.
\end{aligned}
\tag{1}
$$

In their dependence on the local interferometric phases, $p_m^{(\mathrm{A})}$ and $p_n^{(\mathrm{B})}$ contain the pertinent information about the phase relations between the N paths on Alice's and Bob's side, respectively, and information about the N^2 path pairs can be extracted from the coincidence probabilities $p_{mn}^{(\mathrm{AB})}$.

We have mentioned above that Michelson's standard definition of the fringe visibility for two-path interferometers has no unique generalization for multi-path interferometers. This is so for a simple reason: a single number cannot do full justice to the complex pattern displayed by $p_m^{(\mathrm{A})}(\phi_1^{(\mathrm{A})},\dots,\phi_N^{(\mathrm{A})})$ or $p_n^{(\mathrm{B})}(\phi_1^{(\mathrm{B})},\dots,\phi_N^{(\mathrm{B})})$. Thus one must make a judicious choice among the various possible definitions that meet the reasonably obvious criteria on Dürr's list [7].

Here we opt for measuring the various interference strengths in terms of the respective Shannon entropies. We shall continue to speak of "visibility" although the analogy with the usual two-path visibility is a bit remote. Alice's single-particle visibility $V^{(\mathrm{A})}$ is then defined in accordance with

$$
V^{(\mathrm{A})} = \max_{\phi_1^{(\mathrm{A})},\dots,\phi_N^{(\mathrm{A})}} \bigl[1 - H_N\bigl(p^{(\mathrm{A})}\bigr)\bigr]\,, \tag{2}
$$

where

$$
H_N\bigl(p^{(\mathrm{A})}\bigr) = -\sum_{m=1}^{N} p_m^{(\mathrm{A})} \log_N p_m^{(\mathrm{A})} \tag{3}
$$

is the normalized Shannon entropy of her marginal probability distribution, and Bob's single-particle visibility $V^{(\mathrm{B})}$ is defined analogously. For convenient normalization, we take the logarithm to base N so that, by construction, both $H_N\bigl(p^{(\mathrm{A})}\bigr)$ and $V^{(\mathrm{A})}$ are nonnegative numbers that do not exceed unity.

The definition of $V^{(\mathrm{A})}$ in (2) exploits the fact that path knowledge after a Fourier transformation corresponds to interference strength before the transformation [8]. The largest amount of terminal path knowledge that is potentially available for optimally set phase shifters is thus a self-suggesting measure for the interference strength at the input, and the Shannon entropy is one natural quantitative measure of knowledge about a probability distribution—the one we find particularly useful here. This a priori justification of our definition of $V^{(\mathrm{A})}$ is, of course, no more than an appeal at plausibility; the definition is ultimately justified by the consequences that we report below.

We note that $V^{(\mathrm{A})}$ of (2) reaches its upper bound of unity only for certain pure states of Alice's particle, namely those for which it is possible to adjust the phases $\phi_k^{(\mathrm{A})}$ such that the particle is surely directed to one particular detector, which is to say that the arriving particle must follow each path with the same probability of $1/N$ and that there must be definite relative phases between the paths. It is equally important that $V^{(\mathrm{A})}$ vanishes if Alice's particle arrives along one particular path, in which situation there is no room for relative phases between the paths. Accordingly, $V^{(\mathrm{A})}$ vanishes also if the statistical operator $\rho^{(\mathrm{A})}$ of Alice's particle is a convex sum of pure states that refer to definite paths. In the case of the totally mixed state $\rho^{(\mathrm{A})} = 1/N$, we face the extreme situation of utter ignorance in which nothing is predictable about the particle's path before or after the Fourier transformation.

The single-particle visibility of (2) also meets the indispensable criterion of convexity: The visibility of a convex sum of states cannot exceed the convex sum of the individual visibilities. This is an immediate consequence of the concavity of the Shannon entropy.

As a preparation for the definition of the two-particle visibility $V^{(\mathrm{AB})}$ in (5) below, we now consider the interferometric experiment from the perspective of information theory. Imagine that the source and beam splitters are contained in a black box and only the detectors are accessible. We observe paired clicks of the detectors with the coincidence probabilities $p_{mn}^{(\mathrm{AB})}$ and, on either side, individual clicks with the marginal probabilities $p_m^{(\mathrm{A})}$ and $p_n^{(\mathrm{B})}$, respectively. We then wonder about the strength of correlations between clicks of pairs of detectors at opposite sides of the experiment. Obviously, the quantitative measure of two-particle interference should vanish if there are no correlations between the clicks for any setting of the phase shifters and it should be maximal if perfect correlations can be achieved for some setting.

A good candidate for a measure having these properties, and the natural one to accompany the single-particle visibility of (2), is the maximal value of the mutual information $I\bigl(p^{(\mathrm{AB})}\bigr)$ contained in the $p_{mn}^{(\mathrm{AB})}$'s, which is the relative Shannon entropy between the probability distributions $p_{mn}^{(\mathrm{AB})}$ and $p_n^{(\mathrm{A})}p_m^{(\mathrm{B})}$, that is: between the actual coincidence probabilities and the corresponding products of the marginal probabilities.

For a particular setting of the phase shifters, we have

$$I\big(p^{(\mathrm{AB})}\big) = H_N\big(p^{(\mathrm{A})}\big) + H_N\big(p^{(\mathrm{B})}\big) - H_N\big(p^{(\mathrm{AB})}\big) = \sum_{m,n=1}^{N} p_{mn}^{(\mathrm{AB})} \log_N \frac{p_{mn}^{(\mathrm{AB})}}{p_n^{(\mathrm{A})}p_m^{(\mathrm{B})}} \,. \qquad (4)$$

If the two distributions happen to be equal, $p_{mn}^{(\mathrm{AB})} = p_n^{(\mathrm{A})}p_m^{(\mathrm{B})}$, then the stochastic variables n and m are statistically independent, the detector clicks are not correlated, and the mutual information vanishes accordingly. Since it is also bounded, $0 \le I\big(p^{(\mathrm{AB})}\big) \le 1$, we are invited to define the two-particle visibility as

$$V^{(\mathrm{AB})} = \max_{\phi_k^{(\mathrm{A})},\,\phi_l^{(\mathrm{B})}} I\big(p^{(\mathrm{AB})}\big) \,. \qquad (5)$$

Before proceeding we should not fail to mention the alternative proposal by Jaeger, Horne, and Shimony [3]. In the context of two-path interferometers, they define a two-particle visibility by a Michelson-type formula for the extremal values of the *difference* $p_{mn}^{(\mathrm{AB})} - p_n^{(\mathrm{A})}p_m^{(\mathrm{B})}$, and it appears that a systematic study of the possible generalization to multi-path interferometers is lacking. By contrast, our definition involves (the logarithm of) the *ratio* $p_{mn}^{(\mathrm{AB})}/\big(p_n^{(\mathrm{A})}p_m^{(\mathrm{B})}\big)$ and is immediately applicable to multi-path interferometers.

We note that the two-particle visibility (5) is maximal, $V^{(\mathrm{AB})} = 1$, if the source emits a maximally entangled state such as

$$\rho_{\text{max. ent.}}^{(\mathrm{AB})} = \frac{1}{N}\sum_{j,k=1}^{N} |jj\rangle\langle kk| \,, \qquad (6)$$

where $|jk\rangle$ is the ket vector for one particle in Alice's j-th path and the other in Bob's k-th path. And if the source emits a product state,

$$\rho^{(\mathrm{AB})} = \rho^{(\mathrm{A})}\rho^{(\mathrm{B})} \,,\;\; \rho^{(\mathrm{A})} = \mathrm{tr}_{\mathrm{B}}\rho^{(\mathrm{AB})} \,,\;\; \rho^{(\mathrm{B})} = \mathrm{tr}_{\mathrm{A}}\rho^{(\mathrm{AB})} \,, \qquad (7)$$

then $p_{mn}^{(\mathrm{AB})} = p_n^{(\mathrm{A})}p_m^{(\mathrm{B})}$ and $V^{(\mathrm{AB})} = 0$. This limit is also reached in the case of the maximally mixed, or chaotic, state

$$\rho_{\text{chaos}}^{(\mathrm{AB})} = \frac{1}{N^2}\sum_{j,k=1}^{N} |jk\rangle\langle jk| = \frac{1}{N^2} \,. \qquad (8)$$

All these features are in complete agreement with what one would expect from a reasonably defined two-particle visibility.

The stage is now set for presenting the central result: For correlations generated by quantum mechanics (arbitrary joint probabilities $p_{mn}^{(\mathrm{AB})}$ do not have this property), the inequalities

$$V^{(\mathrm{A})} + V^{(\mathrm{AB})} \le 1 \,, \quad V^{(\mathrm{B})} + V^{(\mathrm{AB})} \le 1 \qquad (9)$$

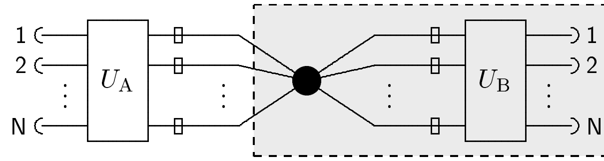

FIG. 2: Another view of the experiment. It is *as if* Bob prepared Alice's particle on the left in a particular state by detecting his particle on the right. Upon regarding the components inside the shaded area as Bob's effective source, the setup can be viewed as realizing a certain communication protocol, and can thus be interpreted and assessed from the point of view of information theory.

are obeyed by the single-particle visibilities $V^{(\mathrm{A})}$, $V^{(\mathrm{B})}$ and the two-particle visibility $V^{(\mathrm{AB})}$, and there are families of pure two-particle states, $\rho_\lambda^{(\mathrm{AB})} = |\lambda\rangle\langle\lambda|$ with $0 \le \lambda \le 1$, for which the equal signs hold with each visibility covering the whole range from 0 to 1. In other words, the inequalities can be saturated.

In view of the complete symmetry between Alice and Bob, it suffices to prove one of the inequalities, say the first one. Consider the experiment of Fig. 2, where we have two observers who can apply arbitrary $U(N)$ transformations, U_A and U_B, to their respective incoming particles, which arrive in some state $\rho^{(\mathrm{AB})}$ that could be pure or mixed. Since there is no restriction here to Fourier transformations, the more general single-particle visibility $\widetilde{V}^{(\mathrm{A})}$ bounds $V^{(\mathrm{A})}$ from above,

$$V^{(\mathrm{A})} \le \widetilde{V}^{(\mathrm{A})} = \max_{U_\mathrm{A}}\big[1 - H_N\big(p^{(\mathrm{A})}\big)\big] = 1 - S_N\big(\rho^{(\mathrm{A})}\big) \,, \qquad (10)$$

where $S_N(\rho) = -\mathrm{tr}\big\{\rho\log_N\rho\big\}$ is the scaled von Neumann entropy of the reduced state $\rho^{(\mathrm{A})}$. The maximum in (10) is reached for those U_A that diagonalize $\rho^{(\mathrm{A})}$.

To find a corresponding upper bound for the two-particle visibility, we first note that

$$V^{(\mathrm{AB})} \le \widetilde{V}^{(\mathrm{AB})} = \max_{U_\mathrm{A},\,U_\mathrm{B}}\big[H_N\big(p^{(\mathrm{A})}\big) + H_N\big(p^{(\mathrm{B})}\big) - H_N\big(p^{(\mathrm{AB})}\big)\big] \qquad (11)$$

and then observe that

$$H_N\big(p^{(\mathrm{AB})}\big) - H_N\big(p^{(\mathrm{B})}\big) = -\sum_{m,n=1}^{N} p_{mn}^{(\mathrm{AB})} \log_N \frac{p_{mn}^{(\mathrm{AB})}}{\sum_{m'} p_{m'n}^{(\mathrm{AB})}} \ge 0 \,. \qquad (12)$$

As a consequence, we get $V^{(\mathrm{AB})} \le \widetilde{V}^{(\mathrm{AB})} \le S_N\big(\rho^{(\mathrm{A})}\big)$. In conjunction with (10) this implies the first inequality in (9), and then the second inequality follows from the symmetry of the setup.

Alternatively, we can estimate the two-particle visibility by an information-theoretic argument. When Bob performs his unitary transformation U_{B} and then registers a click of his n-th detector, it is *as if* he prepared Alice's particle in the state described by

$$\rho_n^{(\mathrm{A})} = \frac{1}{p_n^{(\mathrm{B})}} \mathrm{tr}_{\mathrm{B}} \left\{ P_n^{(\mathrm{B})} U_{\mathrm{B}}^{\dagger} \rho^{(\mathrm{AB})} U_{\mathrm{B}} \right\} , \tag{13}$$

where $P_n^{(\mathrm{B})}$ is the projector corresponding to a click of his n-th detector. In effect, Alice applies her U_{A} to $\rho_n^{(\mathrm{A})}$ and then detects her particle.

The process of finding the two-particle visibility in the experiment can now be understood as a communication protocol where Alice and Bob collaborate to obtain the maximal mutual information in the B-to-A quantum channel with the restriction that only projective measurements are allowed. The well known Holevo bound [15] limits the available mutual information in situations of this kind. In the particular case under consideration, the Holevo bound reads

$$I\bigl(p^{(\mathrm{AB})}\bigr) \le S_N\bigl(\rho^{(\mathrm{A})}\bigr) - \min_{U_{\mathrm{B}}} \left\{ \sum_{n=1}^{N} p_n^{(\mathrm{B})} S_N\bigl(\rho_n^{(\mathrm{A})}\bigr) \right\} . \tag{14}$$

Since the subtracted term is nonnegative, it follows that $V^{(\mathrm{AB})} \le S_N\bigl(\rho^{(\mathrm{A})}\bigr)$, and the argument concludes as above.

To prove the second assertion at (9), namely that the upper bound for the sums of single-particle and two-particle visibilities in (9) is tight, we consider the following one-parameter family of pure states

$$\rho_\lambda^{(\mathrm{AB})} = |\lambda\rangle\langle\lambda| , \qquad |\lambda\rangle = \frac{|0\rangle(1-\lambda) + |1\rangle\lambda\sqrt{N}}{\sqrt{1+(N-1)\lambda^2}} , \tag{15}$$

where we have, for $\lambda = 0$, the ket vector

$$|0\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} |kk\rangle \tag{16}$$

of the maximally entangled state (6) and, for $\lambda = 1$, the ket vector

$$|1\rangle = \frac{1}{N} \sum_{j,k=1}^{N} |jk\rangle = \bigl(F_{\mathrm{A}} F_{\mathrm{B}}\bigr)^{-1} |NN\rangle \tag{17}$$

of the product state that is obtained from $|NN\rangle$ by twofold inverse Fourier transformation. The normalizing denominator in (15) takes $\langle 0|1\rangle = 1/\sqrt{N}$ into account.

If the source emits the two-particle state $\rho_\lambda^{(\mathrm{AB})}$ and all phases $\phi_k^{(\mathrm{A})}$, $\phi_l^{(\mathrm{B})}$ are set to zero, then the coincidence probabilities are $p_{mn}^{(\mathrm{AB})} = \bigl|\langle mn|\lambda\rangle\bigr|^2$. The ingredients of the probability amplitude are

$$\begin{aligned} \langle mn|F_{\mathrm{A}} F_{\mathrm{B}}|0\rangle &= \frac{\delta_{m+n,N}^{(N)}}{\sqrt{N}} , \\ \langle mn|F_{\mathrm{A}} F_{\mathrm{B}}|1\rangle &= \langle mn|NN\rangle = \delta_{m,N}^{(N)} \delta_{n,N}^{(N)} , \end{aligned} \tag{18}$$

where the periodic Kronecker delta $\delta_{j,k}^{(N)}$ is 1 if $j = k$ modulo N, and 0 otherwise. The resulting probabilities

$$p_{mn}^{(\mathrm{AB})} = \frac{\Bigl[(1-\lambda)\delta_{m+n,N}^{(N)} + N\lambda\delta_{m,N}^{(N)}\delta_{n,N}^{(N)}\Bigr]^2}{N[1+(N-1)\lambda^2]} \tag{19}$$

exhibit perfect correlations inasmuch as for each m value there is only one n value for which $p_{mn}^{(\mathrm{AB})} \ne 0$. As a consequence then, all summands vanish in the double sum of (12), which in turn implies

$$1 = \bigl[1 - H_N\bigl(p^{(\mathrm{A})}\bigr)\bigr] + I\bigl(p^{(\mathrm{AB})}\bigr) \le V^{(\mathrm{A})} + V^{(\mathrm{AB})} \le 1 , \tag{20}$$

so that equal signs must hold throughout. Indeed, the upper bound in (9) is reached for all λ, with $V^{(\mathrm{AB})} = 1$ for $\lambda = 0$ and $V^{(\mathrm{AB})} = 0$ for $\lambda = 1$, and all intermediate values in between.

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[1] M. A. Horne and A. Zeilinger, Proceedings of the Symposium on the Foundations of Modern Physics, eds. P. Lahti and P. Mittelstaedt (World Scientific, Singapore, 1985).
[2] D. M. Greenberger, M. A. Horne, and A. Zeilinger, Physics Today **46**(8), 22 (1993).
[3] G. Jaeger, M. A. Horne, and A. Shimony, Phys. Rev. A **48**, 1023 (1993).
[4] G. Jaeger, A. Shimony, and L. Vaidman, Phys. Rev. A **51**, 54 (1995).
[5] G. Weihs, M. Reck, H. Weinfurter, and A. Zeilinger, Phys. Rev. A **54**, 893 (1996).
[6] M. Żukowski, A. Zeilinger, and M. Horne, Phys. Rev. A **55**, 2464 (1997).
[7] S. Dürr, Phys. Rev. A **64**, 042113 (2001).
[8] B.-G. Englert, D. Kaszlikowski, and L. C. Kwek, unpublished.
[9] W. K. Wootters and W. Żurek, Phys. Rev. D **19**, 473 (1979).
[10] P. Mittelstaedt, A. Prieur, and R. Schieder, Found. Phys. **17**, 891 (1987).
[11] P. J. Lahti, P. Busch, and P. Mittelstaedt, J. Math. Phys. **32**, 2770 (1991).
[12] B.-G. Englert, Phys. Rev. Lett. **77**, 2154 (1996).
[13] B.-G. Englert and J. Bergou, Opt. Commun. **179**, 337 (2000).
[14] M. Jakob and J. A. Bergou, eprint arXiv/quant-ph/0302075 (2003).
[15] A. S. Holevo, *Statistical problems in quantum physics* in: Proceedings of the Second Japan–USSR Symposium on Probability Theory, ed. by G. Maruyama and J. V. Prokhorov, Lecture Notes in Mathematics **330** (Springer-Verlag, Berlin 1973), pp. 104–119.