Aspects of Mathematical Understanding

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Introduction

"Teaching for understanding" is often considered to be an important educational objective. For instance, the Singapore Elementary Mathematics Syllabus (1981) states that ". . . pupils should know and understand mathematical ideas and principles, including the techniques and skills in mathematical computation" (p.2). As the computer is used more frequently in education and society, the aim of education should shift from training for specific skills to understanding. However, "understanding" may mean different things to different people. This paper provides an analysis of the concept of understanding and reports on a survey about mathematical understanding.

A Theoretical Analysis of Understanding

First, we have to distinguish between being understanding and understanding something. In the cognitive aspect, understanding has to be linked to some content or knowledge. For each specific type of content, understanding may be conceived as a mental state of cognition, a psychological process, or an ability. These four aspects of understanding are discussed briefly below with special emphasis on mathematics.

Content of understanding

The content of understanding may consist of "knowing that," "knowing how" and "knowing why" (Woods and Barrow, 1975).

In mathematics, "knowing that" refers to knowledge of facts, concepts, and principles. These knowledge items involve the acquisition of meaning. Three kinds of meaning are evident in mathematics: semantic, syntactic, and pragmatic (Van Engen, 1953). For example, the semantic meaning of the symbol "2" refers to the concept of two-ness. Its syntactic meaning depends on where it is used; for example, "2" has different meanings in "2x" and in "x^2". The pragmatic meaning evokes certain emotional tones that may affect learning. For instance, "2" may be linked to the idea of "smallness"; this idea can effectively interfere with the learning of the concept of limit or infinitesimal.

"Knowing how" refers to skills, which are operations and procedures carried out according to prescribed sets of rules, instructions, or algorithms. This is intimately related to the common notion that "if you understand the rule, then you should know how to do it." Skemp (1976, 1979) described knowing what to do without reasons as "instrumental" understanding. He pointed out that this kind of understanding is likely to be rote, to be compartmentalised into numerous rules, to involve direct application, and to be short term in effect. The advantage of "instrumental" understanding is that the pupil can get the correct answer quickly. Normal classroom teaching tends to stress this kind of understanding.

"Knowing why" is justifying "that" and "how." Three kinds of "why" in mathematics were proposed by Jones (1969): chronological, logical, and pedagogical. "Chronological why" communicates the sense of why results hang together as they do from a historical perspective. "Logical why," in expunging all the mistakes and non-logical processes involved in the creation of mathematical knowledge, is necessary for understanding the structures of mathematics. Wagner (1980) argued that knowledge of logical deduction is the more central aspect of mathematical understanding than knowledge of meaning or knowledge of application. Finally, "pedagogical why" is not restricted to rigorous proofs. Quite often, acceptance of authority, observation of cases, and personal experiences may be used heuristically at school level, since many pupils do not recognise the need for formal proofs of results that appear "obvious" to them (Pearson, 1980). On the contrary, pupils should be encouraged to ask questions about mathematical rules. Otherwise, they may apply these rules in a mindless manner.

Skemp (1979) considered "knowing what to do and why" as "relational" understanding, and "conforming to accepted forms of presentation" as "logical" understanding. Byers and Herscovics (1977) proposed a tetrahedral model, whose vertices correspond to the four kinds of understanding: instrumental, relational, formal, and intuitive. Haylock (1982) suggested that these distinctions are not necessary. What is important is to make suitable connections among the four kinds of experiences: words, pictures, concrete situations, and symbols. These conceptions, though rather imprecise, highlight the fact that understanding can mean different things to the student and to the teacher. Any mismatch in what is to be achieved by "learning for understanding", if not attended to, can lead to learning difficulties (Skemp, 1976).

Processes of understanding

A mental process "goes on in time, has different phases, and may be interrupted . . . . and sometimes resumed at the point of interruption" (Baker & Hacker, 1983, p.327).

From the corpus of psychological studies on learning, we may deduce the following inferences about the possible processes of understanding.

1. The plausible phases of the process of understanding are (a) establish learning goals, (b) activate relevant prior knowledge, (c) make new connections, and (d) look ahead. Various teaching models seem to place different emphases on the various phases: learning hierarchies specify the goals in small steps (Gagne, 1977); advance organisers activate the relevant prior knowledge (Lesh, 1976); games and activities provide concrete experiences to facilitate linking of knowledge (Dienes, 1971); and a spiral approach takes cognizance of future development at a higher level of abstraction and generality (Bruner, 1960). Writing about the processes of mathematical thinking, Davis and McKnight (1979) noted the sequence: (a) the student learns a visually-moderated sequence (VMS), i.e., he or she sees something, which leads to the retrieval and execution of some procedure; the execution of this procedure yields a modified visual input, which leads to the retrieval and execution of the next segment of procedures, and so on. (p.95) (b) with sufficient practice, the VMS becomes a frame that encodes the whole sequence as a gestalt unit; (c) with more experience, additional instruction, and possibly deep contemplation, metalanguage for the frame is created. This metalanguage consists of appropriate descriptors for the frame.

2. The process can take place at different levels leading to different degrees of understanding. Hence, there is no complete understanding (Buxton, 1978; Byers, 1980; Skemp, 1971). Polya (1965) identified four levels of understanding a rule as: mechanical, inductive, rational, and intuitive (when one is totally convinced that the rule is true). By careful analysis of pupils' performance on mathematical tasks, it is possible to identify hierarchical levels of understanding (Hart, 1981).

3. The process is active. Even though the relevant information has been organised in an appropriate way, the student must still take an active role in processing the information: analyse it into its constituent parts, set it into a broader perspective, check for special cases, make comparisons, and so on (Michener, 1978). However, in order that this active process can take place, the student must recognise the value of understanding in contrast to reliance on memorised rote procedures. Without such a realisation, the student may continue learning mathematics by rote, imagining that everybody learns mathematics

in this way (Davis, Jockusch & McKnight, 1978, p.283). This active processing is a sort of re-invention (Resnick & Ford, 1981). It requires an alternation of meaningful discussion and quiet contemplation (Davis & McKnight, 1979; Skemp, 1979).

4. The process is dynamic and erratic. The student may understand one week, forget the next, and remember again (Tall, 1978). Each new encounter with the same material is another chance to promote new understanding.

5. The process is error-prone. Any theory of understanding must take into account the patterns and sources of errors made by students. Errors can arise from non-understanding (not having any appropriate schema) or mis-understanding (activating the inappropriate schema). Quite often, misunderstandings arise out of an active process. For instance, Evans (1982) suggested that students use reconciliation strategies to negotiate through difficult areas of understanding when they solve routine mathematical problems. Brown and VanLehn (1980) argued that, when children are faced with an impasse after using incorrect procedures to solve arithmetic problems, they tend to "repair" the impasse by inventing new procedures that allow them to continue with their task executions, albeit in a potentially erroneous way.

6. The process may be gradual or sudden. Leaps of understanding are often reported, but they are not always true (O'Hear, 1981; Ziff, 1972).

State of understanding

The common notion of a state of something refers to its conditions. Some educators believe that understanding is a mental state from which various applications follow (e.g., Woods & Barrow, 1975), while others reject this state notion of understanding (Baker & Hacker, 1983).

In general, information-processing theories assume that the degree of understanding of an object depends on the nature of the internal representation of the object. Greeno (1978) provided three characteristics of good understanding of mathematical knowledge and procedures: coherence, connectedness, and correspondence between the internal representation and the knowledge to be understood. Coherence and connectedness refer to how individuals organise their knowledge. Such individual differences can give rise to different degrees of understanding. In contrast to this private connection, the third criterion of "correctness" implies a publicly agreed version. The recognition of understanding in others involves coming up to some appropriate public standards, and eventually, as O'Hear (1981) argued, sharing the living traditions of the discipline. Thus, a "correct" understanding of mathematics is not only getting the "correct" answers, or obtaining a "good" match between the learner's cognitive structure and that of the experts', but also learning about the historical development of mathematical ideas (Byers, 1982; Polya, 1965; Wilder, 1972).

The internal representation is often described in terms of organisation of knowledge in the semantic memory. Propositional knowledge (such as "7 is a prime number") is generally organised as nodes representing concepts and complex network made up of associations among these meaningful concepts. Procedural knowledge (such as constructing a triangle with compasses) is usually stated in the formalism of "production system" made of IF-THEN pairs (Anderson, 1982). Various techniques have been proposed to assess memory structure, but they are still fraught with problems (Preece, 1978; Stewart, 1980). Hence, Greeno's criteria cannot be applied precisely. In spite of such difficulties, the role of memory in mathematical understanding should be examined in the light of recent research in information-processing theories (Byers and Erlwanger, 1985).

Describing understanding as a mental state emphasises its static aspect in terms of coherence among knowledge items (structure). This complements its dynamic aspect in terms of processes (how it operates), as outlined in the previous section. Both conceptions are necessary in understanding "understanding."

Understanding as an ability

The most prevalent notion about understanding is that of an ability. This idea is closely linked to the practical problem of assessing objectively and reliably the attainment of understanding as a learning outcome.

Several kinds of ability have been mentioned, but none of them can be taken to be the sole criterion of understanding. These include the ability to

1. See relationships
2. Apply knowledge or skill to appropriate situations
3. Relate to logical thinking and structure
4. Know the meaning
5. Explain to others
6. Detect errors
7. Make inferences
8. Be creative
9. Be appreciative

With so many abilities linked to understanding, it is no wonder that Ziff (1972) wrote, "one can no longer avoid the dismal conclusion that to understand understanding is a task to be attempted and not to be achieved today, or even tomorrow" (p.20). However, it is clear that this list can be used to design test items to measure the various aspects of understanding.

The above review summarises the four aspects that must be considered in thinking about understanding. To what extent these conceptions are held by educators remains to be investigated in a systematic way. The next section describes a survey into the various conceptions of mathematical understanding held by a sample of mathematicians, mathematics teachers and Dip. Ed. students in Brisbane, Australia.

A Survey of Mathematical Understanding

Questionnaire, subjects and administration

A specially designed "Understanding Questionnaire" (UQ) was developed for this study (Wong, 1984). It consisted of four parts. The first part asked for biographical information about the subjects. The second part consisted of 50 items about the nature of understanding. The third part consisted of a list of 35 tasks that could be used to assess a person's understanding of mathematics. The final part consisted of five routine mathematical problems. For each problem, several fictitious pupils' answers were given. For each pupil's answer, the subjects were asked to judge its level of understanding.

TABLE 1: MEANS OF CONCEPTION ITEMS ARRANGE BY TOTAL MEANS

No	Item	T	Ma	Te	St	
Nature of Understanding						
6	Conceptual understanding without details	4.2	4.5	4.0	4.3	*
48	There are central ideas in understanding a topic	3.9	4.1	4.0	3.8	
17	Can have complete understanding in context	3.9	3.9	4.0	3.9	
10	Able to judge answers to problems as reasonable	3.8	3.9	3.7	3.8	
11	See relationships between an idea and other ideas	3.7	3.6	3.7	3.8	
12	Knowing Why more important than knowing How	3.5	3.4	3.4	3.8	
20	To understand is to discover or to reconstruct	3.3	3.1	3.4	3.4	
8	We have little knowledge of how people understand	3.3	3.6	3.1	3.4	
15	Understanding is a tested generalised insight	3.2	2.9	3.3	3.1	
13	Cannot define understanding	3.2	2.8	3.2	3.2	
2	No complete understanding, only different degrees	3.2	3.3	3.1	3.2	
9	Understanding is a slow process	3.1	3.5	3.0	3.3	
4	Understanding arises automatically	2.9	2.8	2.8	3.0	
7	Understanding accompanied by flash of insight	2.7	2.8	2.7	2.7	
1	Understanding is an all or none process	2.0	1.5	2.1	2.0	

Contd. TABLE 1: MEANS OF CONCEPTION ITEMS ARRANGE BY TOTAL MEANS

No	Item	T	Ma	Te	St	
Teaching and Learning related to Understanding						
46	Train students to ask own questions	4.2	4.0	4.1	4.3	
25	Explaining to other helps to foster understanding	4.1	4.1	4.0	4.1	
33	Shows different ways of solving same problem	3.8	3.8	3.8	3.6	
29	Interact with concrete materials then abstraction	3.8	3.9	3.8	3.6	
16	Students need to rearrange materials in own ways	3.7	3.5	3.8	3.8	
49	Do a lot of similar exercises to master skill	3.6	3.6	3.7	3.4	
47	Students understand better through self-discovery	3.5	3.5	3.3	3.8	
45	Insist on neat work develops understanding	3.5	3.5	3.6	3.4	
34	Different ways confuse weak students	3.5	3.2	3.8	3.3	*
40	Teacher explanation is best way to understanding	3.4	3.6	3.4	3.4	
32*	Know principles before practice skill	3.2	2.8	3.1	3.8	**
38	History of maths helps understanding	2.8	3.4	2.6	2.6	*
44	Memorise exactly facilitates understanding	2.7	3.3	2.8	2.0	**
35	Textbook explanations little help to students	2.7	2.4	2.9	2.7	
24	Student explanations more helpful than teacher's	2.7	2.7	2.6	3.0	
23	Perfect skill before knowing underlying concepts	2.6	2.7	2.8	2.4	
22	Simplify maths leads to superficial understanding	2.6	2.7	2.8	2.4	
14	Students reminded that problem has correct answer	2.1	1.8	2.0	2.3	
Conditions and Barriers to Understanding						
18	Prerequisite knowledge is essential	4.2	4.0	4.3	4.1	
41	Positive attitude to maths is essential	4.1	4.1	3.9	4.3	
43	Students have trouble because of lack of effort	3.7	3.5	3.8	3.5	
26	Low motivation a major cause in failure	3.7	3.6	3.6	3.8	
31	Lack of confidence in numbers a barrier	3.6	3.2	3.6	3.9	
37	Ability to visualise is essential	3.5	3.1	3.5	3.8	
5	Understand maths requires hard thinking	3.5	4.3	3.4	3.4	*
42	Student can't understand maths because no innate ability	3.0	2.8	3.2	2.8	
28	Understand maths independent of language ability	3.0	3.0	2.7	3.3	
21	Too many symbols hinder understanding	2.7	2.9	2.8	2.4	
19	Great memory for details is essential	2.4	2.1	2.5	2.4	
30	Girls have more difficulty in understanding maths	2.2	2.6	2.2	2.0	
3	Harder to understand maths as one gets older	2.2	2.2	2.2	2.4	
50	Can or cannot do maths, work hard no difference	2.0	2.2	2.1	1.8	
Compare Mathematical Understanding with other contexts						
36	Different abilities from understanding story	3.5	3.6	3.3	3.7	
39	Similar to playing chess	3.0	2.8	3.0	3.2	
27	Similar abilities to understand music	3.0	2.6	3.0	3.0	

Note: T = Total (N = 90), Ma = Mathematicians (N = 16),
Te = Teachers (N = 46), St = Dip Ed students (N = 28).
Items range from 1 (Strongly disagree) to 5 (Strongly agree).
Items with significant F ratios are indicated. *$p < .0.5$, **$p < .01$.

The UQ was answered by 16 mathematicians at the Mathematics Department of the University of Queensland, Australia, 49 secondary mathematics teachers in Brisbane, and 28 Dip. Ed. students at the University of Queensland, who took mathematics as one of their teaching areas.

Conception of understanding

The means of the 50 Conception items are given in Table 1. Only the abbreviated forms of the items are given here, the full statements being available from the author.

Based on the Total means, it is clear that the subjects agreed strongly that understanding is related to central ideas rather than to details about rules and formulae. The ability to judge whether answers are reasonable or not and seeing relationships were also important aspects of understanding. Neutral opinions were expressed about the elusiveness of understanding as described by its automatic occurrence, accompanied by a flash of insight, little knowledge about how people understand, and difficulty in defining understanding. Differing opinions were expressed about the completeness of understanding: an apparent rejection of a simple dichotomy between all or nothing in understanding (item 1), strong agreement on complete understanding in certain context (item 17), but divided opinions on different levels of understanding (item 2, SD = 1.3).

With respect to the instructional processes, the subjects generally agreed that students should ask own questions about the materials, explain things to others, interact with concrete materials followed by abstraction, and arrange materials in own ways. The teachers more than the mathematicians and the trainees believed that learning different ways of solving the same problem is confusing to the weaker students.

Memorisation has always been a difficult issue in mathematics learning. The trainees considered memorisation not important to achieve understanding, but some mathematicians saw merits in this approach (item 44). Although the substitution of meaning by memorisation can lead to a lack relational understanding,there are certain advantages in memorising definitions and algorithms: the exactness leading to fewer errors and its availability so that the thinking process is not unduly disrupted by having to look things up. (See Cockcroft, 1982, p.69 & p.179; Byers, & Erlwanger, 1985). The danger, according to Wilder (1968), is that, although mathematicians engage in "symbolic initiative" behaviours when they develop formulae and short-cut procedures as labour-saving devices, the pupils generally memorise these formulae (via the drill type of teaching) at the "symbolic reflex" level without knowing the purpose of these formulae or why they work.

On the sequence of skill mastery before understanding principles or the reverse (items 23 and 32), the trainess agreed that principles should be taught before practising skills. On teaching the history of mathematics, the mathematicians had slightly more positive responses than the other two groups. These subjects did not quite share the enthusiasm of some mathematics educators on the use of history of mathematics in teaching. Finally, the subjects generally agreed that pre-requisite knowledge, positive attitude, effort, and motivation are important conditions for achieving understanding.

Assessment of Understanding

The subjects rated how important each of the 35 tasks was in assessing understanding on a 5-point scale from 0 (not important) to 4 (essential). The means for these tasks are given in Table 2. The result thus provides an empirical ranking of these tasks in order of importance for assessing understanding.

According to this sample, the most important assessment task was the ability to write down the conditions under which the result is valid/can be used. Knowing when the result cannot be used was also important. Other evidence, however, suggests that these two abilities have not been given the necessary attention in school practice (Galbraith, 1982). Next was the ability to translate results from one form to another, an important ability for comprehending mathematics problems. The routine problem solving task, used very frequently in school tests, was also considered important by these subjects. However, solving problems within a certain time limit was considered less important, thus suggesting that

TABLE 2: MEANS OF ASSESSMENT ITEMS ARRANGE BY TOTAL MEANS

No	Item	T	Ma	Te	St	
Nature of Understanding						
27	Know valid conditions of using result	3.2	3.1	3.1	3.4	
12	Use result to solve 4 routine problems	3.1	3.1	3.0	3.1	
4	Translate result from one form to another	3.1	3.2	3.1	3.2	
16	Think of appropriate ways to solve problems	2.9	2.9	3.0	2.8	
3	Describe result in own words	2.9	2.9	2.7	3.2	*
28	Know invalid conditions of using result	2.9	3.0	2.6	3.4	**
19	Estimate a numerical answer to a problem	2.8	2.7	3.0	2.4	**
14	Use result to solve 4 novel problems	2.8	2.7	2.9	2.6	
11	Explain result to others	2.7	2.6	2.6	2.8	
32	Write down a generalisation of result	2.7	2.2	2.7	2.8	
5	Identify examples and non-examples	2.6	2.2	2.5	2.7	
18	Solve novel problems under open-book condition	2.5	2.8	2.4	2.4	
24	Illustrate proof with numerical example	2.5	2.6	2.2	2.9	*
10	Describe use of result	2.5	2.1	2.5	2.7	
35	Recall, recognise, use result after a long time	2.5	2.8	2.4	2.5	
22	Write down proof in own way	2.5	2.8	2.4	2.5	
34	State contrapositive, converse or inverse	2.5	2.6	2.4	2.6	
13	Use result to solve 4 routine problems with time limit	2.5	2.3	2.5	2.4	
26	Illustrate result with concrete materials	2.4	2.2	2.4	2.5	
31	Detect errors	2.4	2.3	2.4	2.5	
17	Think of ways to solve problems with time limit	2.4	2.0	2.6	2.0	*
9	Describe relationships in own words	2.4	2.3	2.1	2.8	**
33	Write down a conjecture of result	2.3	2.2	2.3	2.4	
6	State 3 examples of a result	2.2	2.6	2.1	2.2	
29	State reasons why to learn result	2.2	2.0	2.0	2.4	
8	Select best description of relating results	2.1	2.3	2.1	2.1	
1	Select best description of a result	2.1	1.4	2.1	2.1	
2	Write down result as learnt	2.1	2.8	2.2	1.7	*
15	Use result to solve 4 novel problems with time limit	2.0	1.6	2.4	1.7	*
21	Write down proof as learnt	1.9	2.1	1.9	1.9	
23	Arrange given steps of proof in correct sequence	1.9	0.6	2.0	2.1	**
20	Select most elegant solution	1.6	2.0	1.8	1.3	*
7	State 3 non-examples of a result	1.6	1.9	1.5	1.6	
30	Describe how result arose historically	1.4	1.4	1.4	1.3	
25	Write down at least 2 proofs of same result	1.2	1.1	1.1	1.5	

Note: T = Total (N = 83), Ma = Mathematicians (N = 9),
Te = Teachers (N = 46), St = Dip Ed students (N = 28).
Items range from 0 (not important) to 4 (essential).
Items with significant F ratios are indicated. *p<.0.5, **p<.01.

time was not an important factor for assessing understanding.

In mathematics teaching, the use of examples and non-examples is a common practice. Although the ability to *identify* examples and non-examples was moderately important (item 5, mean 2.6), the ability to *state* examples (item 6, mean 2.2) and non-examples (item 7, mean 1.6) was less important.

At the other end, three unimportant tasks were: select an elegant solution, history of mathematics, and know at least 2 proofs of the same result. The lack of importance for knowledge of alternative proofs may reflect a common teaching approach that deals only with "the" proof in the deductive fashion. Historical knowledge was considered unimportant, hence not supporting the philosophical view that "really" understanding a discipline includes sharing the cultural traditions of that discipline. Finally, the search for elegance resides more in the aesthetic aspect of understanding than in the cognitive aspect.

Levels of Understanding

For this part of the questionnaire, the subjects were asked to award an integral score between 0 (no understanding) and 4 (good understanding) to each of 23 pupils' answers to 5 mathematical problems. Responses to only two problems are reported here. These answers are arranged below in descending order by the Total means. Total standard deviation and means for the mathematicians, teachers, and trainees are also given in parentheses at the end of each answer. Significant F-ratios for differences in group means are indicated: $^{*}p<.05$, $^{**}p<.01$.

Case 2

What is the next term of the sequence {7, 16, 25, 34, 43, 52. . .} ?

Answer (i)

This is an AP of common difference = 9. The next term is 61.
(3.8, 0.4; 3.3, 3.8, 3.9)**

Answer (iii)

If it is an AP, the next term is 61. It can also be 59 because the sum of the digits of each term is divisible by 7.
(3.5, 0.8; 3.6, 3.4, 3.6)

Answer (ii)

It can be anything. You can't tell from six terms only. You must be given the rule.
(0.9, 1.1; 2.6, 0.7, 0.8)**

Answer (iv)

This is a set. In set, you can arrange the terms in any way you like, say { 7, 34, 52, 16, . . . }. Don't know the next term.
(0.5, 0.7; 0.7, 0.4, 0.5)

Answer (i) is the "expected" one after the pupil has studied arithmetic progression, and hence was given the highest overall score. However, compare the group means for (i) and (iii). The mathematicians considered the extra solution given in (iii) as better than (i), while the teachers and the trainees had the opposite view. The last two groups tended to treat answers other than the expected one as a sign of poorer understanding rather than flexibility in problem solving. Indeed, some subjects may have the wrong concept of sequence, as one trainee wrote, "Sequences are concerned with integers and not with digits."

Answer (ii) is correct theoretically, but was not accepted by most of the subjects. One teacher wrote, "He/she is right but is failing to see the point of the exercise." Others commented that the pupil was trying to hide his ignorance, or that 6 terms were sufficient to determine the rule. The latter comment shows misunderstanding among some subjects of sequences. A similar misunderstanding could arise from the use of similar questions in IQ tests.

Answer (iv) illustrates a potential source of misunderstanding when the same symbol is used to represent different mathematical concepts.

Case 4

In a certain town, there are 6 families each of which has 2 radios, and there are 8 families each of which has 4 radios. Find the average number of radios per family of all the families taken together.

Answer (ii)

$$\text{Average} = \frac{6 \times 2 + 8 \times 4}{6 + 8} = 3\frac{1}{7}.$$

(3.9, 0.4; 4.0, 3.9, 3.8)

Answer (iv)

$$\text{Average} = \frac{6 \times 2 + 8 \times 4}{6 + 8} = 3.1429.$$

(3.7, 0.5; 3.7, 3.7, 3.6)

Answer (iii)

$$\text{Average} = \frac{6 \times 2 + 8 \times 4}{6 + 8} = 3\frac{1}{7}.$$

Since you can't have fractional radio, the average is 3.
(3.1, 0.8; 2.7, 3.0, 3.4)*

Answer (v)

Should take the median. Average = 4.
(1.2, 1.1; 1.7, 1.5, 0.6)**

Answer (vi)

There are more families having 4 radios each. Average = 4.
(0.9, 1.0; 1.2, 1.0, 0.6)

Answer (i)

There are two types.
Average = (2 + 4)/2 = 3.
(0.8, 0.5; 0.5, 0.8, 0.8)

From the first three responses, it appears that the mathematicians tended to expect more accurate results. In (iii), where the fraction was approximated to a whole number, the interpretation was more acceptable to the trainees than to the mathematicians. A trainee commended the pupil by writing, "He was really thinking about the context of the problem." On the other hand, some mathematicians, while not considering the interpretation completely wrong, felt that averages can take fractional values irrespective of the "real world" situations.

On the other hand, using median or mode as an interpretation of "average" was not well accepted by these subjects, especially the trainees. The Modern Maths approach to statistics typically stresses that there are three types of averages, namely mean, median, and mode, and that pupils should understand which is the more appropriate one to use in any given situation. Despite nearly twenty years of modern mathematics in schools, the above responses suggest that the predominant meaning attached to "average" may still be the arithmetic mean.

To summarise, the subjects were in general agreement in awarding marks to expected answers. However, when confronted with less clear-cut situations, such as different interpretations of the problem situation or solution, there was greater variation in giving partial credit. Although these subjects tended to believe in a relational interpretation of understanding, these responses seem to suggest that judging the levels of understanding is based on an instrumental criterion of what is to be "expected" under certain context. If this interpretation of the results is valid, then it points to an apparent gap between theoretical thinking and practical judgement of understanding.

Implications

Although the sample size is small, it is still possible to draw several implications for research and classroom instruction from this study.

The survey has identified some consensual agreements among mathematicians, mathematics teachers, and teacher trainees in their thinking about understanding. In particular, understanding is closely linked to seeing relationships in a global sense rather than being restricted to details about formulae and definitions. Hence, to assess understanding, more emphasis should be placed on higher mental processes like knowing when to use or not to use a rule, translating information from one form to another, judging the reasonableness of obtained answers, and making generalisations. However, the ability to apply learnt rules to routine problems is still essential since without such a fundamental ability, the higher processes may not be developed. Thus, mathematics teachers should encourage their pupils to actively process mathematical information so that the pupils do not always consider mathematics to be rote manipulation of rules and symbols. To ensure that relational understanding shall prevail, the teacher should design tests that reflect this perception. The empirical ranking of tasks given in Table 2 provides a possible guide to the construction of such a test.

Results in judging the levels of understanding point to a probable gap between theoretical thinking and practice. One possible cause for such a gap is that the concept of understanding is commonly used without much deliberation of its nature. As a start, the above review and the survey data can be used to design a workshop on the nature of understanding for teacher trainees. The results also point to a possible lack of mastery of certain mathematical concepts among teacher trainees. Hence, to teach for understanding, teacher educators should ensure that the trainees possess the necessary mastery of the subject matter.

For theoretical analysis of understanding, the review suggests an approach for viewing understanding under four components: content, process, state, and ability. This approach helps to provide an organising model that can link diverse ideas about understanding expounded by philosophers, psychologists, mathematicians, and mathematics educators. A similar approach may be applied to other subject matter.

This research provides an example of using a statistical approach to analyse a theoretical construct. As such, it is a novel way of supplementing rational analysis with empirical findings. With concerted efforts in both rational analyses and statistical studies, we may gain a better understanding of "understanding" so that it becomes a meaningful educational construct for all disciplines.

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