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TAKING A CLOSER LOOK AT MATHEMATICAL PROBLEM SOLVING

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In this paper, we will discuss what appropriate problems are and how to make use of heuristics to solve mathematical problems. We will also examine the affective domain of teachers and students as they engage in problem solving and finally we will discuss what is in store for us in the future of mathematical problem solving.
Problem solving, heuristics, teaching practices.

INTRODUCTION

Central to our Singapore mathematics framework is mathematical problem solving (Ministry of Education [MOE], 2006a, 2006b) where its importance is echoed by many. The influential Cockcroft report (1982, p. 73) has stated that "the ability to solve problems is at the heart of mathematics" and the National Council of Teachers of Mathematics (2000, p. 52) standards also stressed that "problem solving is an integral part of all mathematics learning". Yet, the true integration of problem solving into our curriculum has not been realized due to a multitude of intricately linked concerns and difficulties. Hence the intent of this paper is to introduce concrete ways to create meaningful classroom discourse to our teachers to aid them in making authentic problem solvers out of their students, through the discussion of problems, heuristics and pedagogies.

APPROPRIATE PROBLEMS

Smith (1986) wrote that a suitable problem used for teaching and learning problem solving in classrooms should be one which is challenging but not too difficult or intimidating. As Foong (2006) pointed out, students must have the mentality that there is no immediate way to arrive at the solution and they must be prepared to work out the problem using the set of heuristics that is available to them.

Figure 1 shows a classification scheme for different types of mathematical tasks (Foong, 2006). Typical textbook problems are not considered as they are usually solved by the mere application of a specific algorithm or formula. They are categorised under exercises or routine sums where the main aim is to reinforce a certain procedure via repetition. Problems used in the problem-solving setting are sub-divided into two categories, closed and open-ended structure. The former are "well-structured" problems, each with a single accurate answer, whereas the latter are “ill-structured” in the sense that there are missing information and each can have more than one acceptable answer.

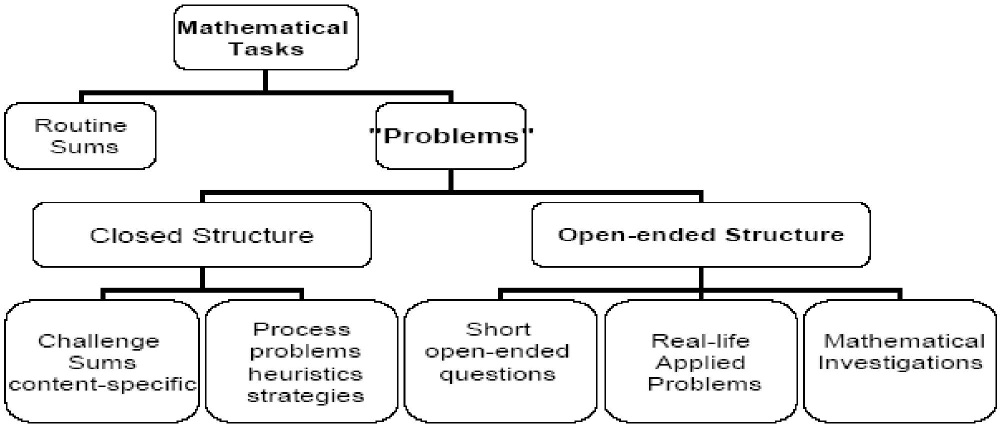

Figure 1. A classification scheme for different types of mathematical tasks

Under closed structured problems, there are challenge sums whose key purpose is to test higher-order thinking skills. An example of this problem type is “Two trains travel in the same direction on parallel tracks. Train *A* starts out 50 km ahead of Train *B*. Train *A* travels 90 km/h and Train B travels 100 km/h. In how many hours will the two trains be in the same place?” Next, process problems otherwise known as non-routine problems are usually not topic-specific, like finding the number of squares on a n x n chessboard.

Short open-ended problems are used to enhance students’ understanding of mathematical concepts and communication, like given 12 oranges to put into bowls, such that each bowl must hold the same number of oranges, demonstrate how you could put oranges into bowls (Foong, 2006). Open-ended real-world problems could be to design the best cylindrical size for a beverage can. Lastly, mathematical investigations can allow students to gain insight into the inherent nature of mathematics. For example, students could be introduced to palindromic numbers, like 31413, numbers that read the same both forward and backward. In Eq.(1) given below, a number (96) is added to its reverse (69), the sum (165) is then added to its reverse (561), and so forth. Repeating this process four times yields a palindromic number. Will this process always result in a palindromic number?

$$\begin{array}{r} 96 \\ \underline{+\ \ 69} \\ 165 \\ \underline{+\ 561} \\ 726 \\ \underline{+\ 627} \\ 1353 \\ \underline{+\ 3531} \\ \underline{4884} \end{array} \qquad (1)$$

It is insufficient just knowing the different problem types available for the fostering of problem-solving skills. The teacher needs to source for problems of appropriate level for the class too, which is an uphill task. Not only does the teacher has to consider the disparity in mathematical abilities within the same class, he or she will also have to take into consideration, issues like students' linguistic abilities and their attitudes and conceptions towards learning mathematics. Teachers must take care to avoid two types of problems. First, problems that seems to be easy to handle, but are in fact cognitively challenging; and second, problems that look impossible (Montague & Applegate, 1993). The former may lure students to persevere in the belief that they can do it, till they become irritated and quit prematurely, whereas the latter may simply discouraged them to even make an attempt.

When choosing problems for students, there are some factors to bear in mind. First, it would be preferred to keep the choice of words simple and the presentation of the problem concise. Words that may mislead the students should not be used and redundant or useless data should be removed. Silver and Thompson (1984) also highlighted that students are more interested in tackling problems with familiar context or have practical value to them, like planning for a trip. Teachers of younger students could start off with simple one-step problems before moving on to multi-step problems. It is definitely less intimidating for the young learners if teachers could lead them through the problem solving process by breaking down the original problems into smaller parts that are more manageable and accessible for them (Smith, 1986). This particular group of students would also prefer problems that require or could be represented by concrete materials, problems that are presented in a more lively and interesting manner and problems that involve data gathering so that children can feel that they are nearing the solution. Teachers can be assured that with practice and experience, they will become better in selecting the right problems.

PROBLEM SOLVING HEURISTICS

There are lists of problem solving heuristics that can be found in many mathematics curriculum documents or provided for by researchers or mathematicians (MOE, 2006a; MOE, 2006b, Smith, 1986). Some of these strategies are:

- Pattern recognition
- Work backwards
- Guess and test
- Organise data
- Simplify the problem
- Solve part of the problem
- Make a systematic listing
- Logical Deduction
- Visual Representation – graph, equation, algebraic expression, table, chart, diagram
- Adopt a different point of view
- Consider extreme cases
- Act it out

Pattern Recognition

We always look for or create our own set of patterns to help us remember figures and items. For instance, we tend to remember telephone numbers by clustering the digits into two groups, by linking our postal codes to our block numbers, or by forming pin numbers using numbers that are familiar to us, like our age or birth dates. A simple illustration is find the next two numbers in the sequence, 1, 8, 64, __, ___, and for such questions, it is rather unlikely that this can be solved in any other method, other than using pattern recognition. A more suitable illustration to best exemplify the power of pattern recognition is to find the last digit of 7^{105}. For powers of 7, we have

$$
\begin{aligned}
7^1 &= 7\\
7^2 &= 49\\
7^3 &= 343\\
7^4 &= 2401\\
7^5 &= 16807\\
7^6 &= 117649
\end{aligned}
\qquad (2)
$$

It is noticeable that the last digit of powers of 7 cycle through 7, 9, 3, 1, 7, 9, 3, 1, in cycles of 4 (Eq.(2)). Therefore, since 105 divided by 4 has a remainder of 1, the last digit will be 7. To further extend this question, we could check if all the powers of single-digit numbers are cyclical and if so, what the cycles are.

Work Backwards

We often unknowingly used this strategy to schedule our time. For example, when we are given a task and its deadline, we often sub-divide the task into smaller portions, and "work backwards" to assign a time slot to each sub-task. In traffic investigations, when an automobile accident happened, the police must work backwards to seek the causes (Posamentier & Krulik, 1998). From the two scenarios provided, it is not hard to realise that to work backwards, we must first have the unique goal in mind. A problem to consider on the technique of working backwards is:

> "You are given two jugs, one holds 5 gallons of water when full and the other holds 3 gallons of water when full. There are no markings on either jug and the cross-section of each jug is not uniform. Show how to measure out exactly 4 gallons of water from a fountain."

If we wish to post this question to students locally, it will be good to shorten the question and to use terms that they could relate better with like using litres instead of gallons. A suggestion is to shorten the question to "Tom has a 5-litre can and a 3-litre can, with no markings on either can. How can he measure out exactly 4-litre of water?" This problem can be resolved using the work backwards strategy by having 4-litres in the 5-litre can. But how do we get the one empty litre? To obtain one empty litre, Tom will need to have two litres in the 3-litre can, leaving exactly one litre, to be poured in from the 5-litre can. This can be done by filling up the 5-litre can and pouring three litres into the 3-litre can, leaving exactly two litres in the 5-litre can. Next, pour away the water in the 3-litre can, and transfer the two litres from the 5-litre can to the 3-litre can, and we will have that desirable 2-litres in the 3-litre can!

Guess and Test

A typical real-life usage of the "guess and test" technique is when we are cooking, we will usually try out different combination of the seasonings, vary the timings, and the different ways of cooking. Posamentier and Krulik (1998) gave the following example in their book,

> Evelyn has five boxes of apples. When she weighs them two at a time, she obtains the following weights (in pounds):
>
> 110 112 113 114 115 116 117 118 120 121.
>
> What are the weights of the five individual boxes of apples, assuming that there is no duplication of weights?

This can be solved either by forming 10 algebraic equations or with some intelligent guessing and testing. We can begin by guessing one half of 110 but keeping in mind that since there is no duplication in weights, we shall consider 54 and 56 first. Next, consider 54 and 58 to get 112 and since 56 and 58 gives 114, it seems that we have gotten the first 3 answers. To get the first odd number, there is no point in trying 55 or 57, because either of these two when added to 54 will not give the smallest odd number 113. Thus, we should try 59 instead, since 54 and 59 will give us 113. Using 59 with 56 and 59 with 58 will give us 115 and 117, the next two odd sums. To find the last number, use the smallest sum remaining, which is 116, and therefore, the number which is to be added to 54 is 62. Testing out 62 with 56, 58 and 59 will gives us the remaining sums of 118, 120 and 121.

Organise Data

The re-organisation of data could to re-state the problem, to use pictorial representations, or to group the given information. Let us take a look at the classic problem of finding the sum of all the numbers from 1 to 99. It will look silly and tedious if a student starts by keying in all the numbers from 1 to 99 into the calculator to find the total. Instead, if the student re-organises the data, he or she may realise that by grouping the first number 1 with the last number 99, the sum is 100, and by grouping the second number 2 with the second last number 98, the sum is 100 too. Since it is an odd number, only the number 50 will not have a partner. There will therefore be 49 pairs, and a single 50, hence the total is

$$49 \times 100 + 50 = 4950 \qquad (3)$$

For students in elementary grades, a simplified version to find the sum of numbers from 1 to 10 could be given. This problem will not be suitable for post-secondary students though as they will treat this problem as an arithmetic progression and simply apply the algorithm.

Simplify the Problem

By solving the simpler problem, the technique discovered could be used to resolve the original one. For instance, instead of finding the sum of all the digits of the numbers in the sequence 1, 2, 3, …., $10^n - 1$, we could let *n* be 4 and reduce the sequence to 10,000 numbers by counting 0 in. This simplified the whole problem to a more manageable one. Since each number has 4 digits (by filling all the digits in front of 1-digit, 2-digit and 3-digit numbers with zeros), there will be 40, 000 digits in total. Each digit will have equal probability of occurrence and hence each will appear 4000 times. Thus, the sum of all the digits from 1 to 9999 is 4000×45 (Eq. (4)), or $4 \times 10^{4-1} \times 45$, giving the general solution as $n \times 10^{n-1} \times 45$.

$$4000 \times (1+2+3+4+5+6+7+8+9) = 4000 \times 45 \qquad (4)$$

Logical Deduction

This strategy can help to anticipate problems before they arise and improve inter-personal relationships. A logic question would be to find the sum of squares in an $n \times 3$ array, where we can only obtain 1×1 squares, 2×2 squares, and 3×3 squares from an $n \times 3$ array. The total number of 1×1 squares is $n \times 3$ and that of 2×2 squares is $(n-1) \times 2$, since with the addition of 1 row, there are always 2 extra 2×2 squares. To calculate the sum of 3×3 squares, simply use the same logic for the 2×2 squares, giving $(n-2) \times 1$. Therefore the sum of squares is

$$3n + 2(n-1) + (n-2) = 6n - 4\,. \qquad (5)$$

Visual Representation

All electronic devices, like mobile phones, televisions come with a manual that always have a picture of the device to demonstrate how to use the item. In schools, to teach young children the concept of fractions, we use a picture of a pizza cut into equal number of pieces. It is always easier to depict concepts with pictorial representations, although we must be careful not to equate the illustration with the concept definition. A seemingly difficult problem on how many ways can we seat eight people in a table, with two on each side can be easily answered with the help of diagrams. An example using this strategy will be provided for in the later part of this paper.

Adopt a Different Point of View

All teachers practise this daily when they take the attendance of their classes. Instead of counting how many students are present, it is often easier to see how many are absent. Another case would be to find the probability that of five people in a room, at least two will have the same birth date (assuming the exclusion of February 29th). Instead of finding the probability that at least two have the same birth date, why not find the probability where none has the same birth date, and thereafter subtract it from one. In this way, the problem is significantly simpler and easier to manage.

Consider Extreme Cases

Extreme cases or worst-case scenarios (Posamentier & Krulik, 1998) are used in manufacturing lines like when a company decides on the expiry date of a food item or the warrantee to be granted for an electrical device. A problem suitable for the usage of this strategy is “How many people must I put in a room such that at least two of them are born in the same month?” We now apply the extreme case reasoning, such that the worst-case scenario is having 12 people where all are born in different months. Thus the answer is 13.

TEACHERS' BELIEFS AND CONCERNS

For problem-solving to truly take place in classrooms, teachers are the deciding factor. It is thus necessary to explore what are some of their beliefs and concerns. First and foremost, many believe that teaching problem solving is a time-consuming task. For students to even understand the problem means that a great deal of time is needed just to discuss the problem statement, which is only the first stage in Polya's model. Moreover, some problems require students to have certain pre-requisites so that they can apply the required concepts and skills appropriately. Even as educators, we had experienced moments when we were confounded by the problem or were led down the wrong route. Even when solutions were presented to us, we might still take time to understand them and even longer time to reproduce the solution.

Another concern teachers may have is on assessment. Due to the nature of problem solving, grading the students' work could be a headache, since students can utilise a wide range of strategies, present their solutions in varied ways and gives many possible answers. Hence apart from searching for appropriate problems, teaching them skills, concepts, and heuristics, teachers still need to devise a rigorous set of rubrics for fair assessment. This situation can be further worsened, if teachers are unable to differentiate what exactly has students attained, since some problems only require very minimal mathematical skills.

Although problem solving seems to be the main focus of many mathematics curriculums, the weightage given to problem solving in national examinations are negligible. It thus appears to many, that it is pointless and a pure waste of time and effort to teach problem solving strategies. Therefore, although it is advocated by many that emphasizing on mathematical problem solving will improve students' metacognitive and cognitive abilities which are essential for them, especially so in this era (Posamentier & Krulik, 1998), it is not sufficient to convince teachers. This is especially true in Singapore, as we have a rather fixed curriculum structure with national examinations, where students' results, schools' reputations, and many other interests at stake. Unless there is a major change in the nationwide assessment, mathematical problem solving might have problem taking flight. Moreover, there is a lack of proposed problems or investigations from teaching guides and textbooks.

Lastly, some teachers may have reservations for teaching problem solving as they are concerned about the shift of teacher role, from being the knowledge source to a facilitator. The uneasiness is heightened for those who may lack confidence in their problem-solving abilities and are apprehensive about dealing with unfamiliar or uncomfortable situations that are likely to arise when students solve problems. Thus, other than having a good repertoire of heuristics, the teachers must also adjust their mindsets and be accustomed to being baffled by problems or accepting the fact that their students' solutions might outshine theirs.

STUDENTS' ISSUES

Schoenfeld (1983, cited in Mason, 2003, p. 74) stated that students' beliefs can be so powerful that it can hinder their abilities and reduce their motivation and volition in solving problems. These beliefs which include the usefulness of mathematics, their roles as learners, their mathematical abilities, and the importance of acquiring problem-solving skills, are deciding factors that contribute to the attitude, the amount of time, effort, and commitment students are willing to put into problem solving (Montague & Applegate, 1993). In fact, some researchers commented that students believed that under normal circumstances, mathematical problems should be solved in less than 12 minutes, else it is unlikely that they can do it.

More than often, students have been put through the experience that there is only one single and best way to solve a problem. Even if they can solve the problems, they seldom check or extend their understanding. Hence, as a rule of thumb, teachers should always encourage students to discuss and share their solutions, think of alternative solutions, or even pose their own similar problems (Posamentier & Krulik, 1998). Some problems lend themselves well to various solution methods. For instance, if in a room of ten people and everyone shakes hand with everybody else exactly once, to find out how many handshakes there are, there are at least five paths a problem solver can take. One can choose to "draw a diagram" or "use a table" to list down all the possible answers. One can also consider sub-goals, by regarding handshake *A*-*B* as being different from *B*-*A*, so that 10 people will shake nine other people's hands, giving 90 handshakes each follow by halving the answer to arrive at the final result of 45. Of course, this is simply a combinatory problem, such that for a handshake to happen, I must select only two people. Therefore, the solution is simply $\binom{10}{2}$. For students at lower grade levels, they could even employ the strategy, "Act it out".

The solutions that I have provided here are by no means exhaustive. The purpose is to highlight what has been mentioned earlier, that there is often more than one unique route to solve a problem. Hence it is crucial that we teach our students to reflect upon their thinking processes and if they should get lost in the midst of solving problems, they can always abandon their plans and attack a new line of thought. This is provided that we also emphasize that changing tactics should only be done when they have given the previous plan enough time and thought.

Lastly, on the cognitive side, researchers found that the average students are less-abled in terms of using heuristics to solve problems and monitoring their metacognition as compared to those who are intellectually gifted (Montague & Applegate, 1993). This means that teachers must be committed not only to teach, but to motivate the majority in persevering through the problem solving process.

GUIDE STUDENTS TO BECOME BETTER PROBLEM SOLVERS

To be able to guide our students to become better problem solvers, we, teachers, must become one first. Teachers must be willing to devote time to build their own repertoire of problems solving heuristics by attempting many varied mathematical problems, reflect on the problem solving processes, and anticipate the potential pitfalls, misconceptions, and difficulties students may be faced with while solving problems.

Planning the appropriate classroom setting for problem solving is unlike that of a traditional class. One preferred method is to encourage students to work in pairs or groups in a cooperative learning setting as it tends to lend support to their learning. Foong (1992) suggested for teachers to get their students to verbalise their thoughts as they solve problems. Not only does "thinking aloud" help students to be more conscious of their thinking processes, it also provides teachers with an avenue to "hear" students' thinking. The classroom environment must be supportive and non-intimidating so that students will not be afraid to make mistakes, and provocative for students to be creative in tackling the problems. To increase students' confidence level, teachers should permit the use of calculators too (Silver & Thompson, 1984).

The skill of problem solving could and should be taught to students explicitly, so that they will be more informed and conscious of their own problem-solving devices and thinking processes. The most well-known problem-solving approach being Polya's four phase problem-solving model (Silver & Thompson, 1984), understand the problem, devise a plan, carry out the plan, and look back. Although the linear structure of Polya's model does not really reflect the true essence of problem solving as it oversimplifies the entire problem solving process, it is valuable as a structure to organize instruction and as a guide for many mathematicians and researchers as they devise their own problem-solving frameworks. For instance, Kulik (Smith, 1986) constructed his model based on Polya's, to offer a more functional and practical alternative to teachers for setting up the problem-solving environment for their students.

Foong (2006) examined the problem-solving processes in classrooms and found that successful problem-solving sessions have three phases, namely the "problem presentation" phase, the "solution effort" phase, and the "problem and solution discussion" phase. In the initial phase, students will seek to understand the problem through inquiry. Next, students will investigate and use the heuristics to search for the solution in groups where the teacher acts as the facilitator. Finally, in the last phase, after students have presented their solutions, teachers will direct the students to check and extend their work. Note that this is in fact the last stage of Polya's model of looking back, where problem solvers check and extend their solution. Although it may be hard to get our students to think about their own thinking, this is a vital and unavoidable step as it develops their thinking skills.

Students need to be exposed to a wide variety of problems and to be given the opportunity to experiment with different heuristics. As Posamentier and Krulik (1998) has mentioned, "the repetition of a "skill" is useful in attaining the skill", this also applies to the acquisition of problem-solving abilities. The more one practise, the better one becomes. Thus, teachers must be diligent to search and present sufficient questions for students to practise the application of different heuristics in varied areas. Teachers must be willing to factor in a substantial amount of time for their students to be involved in problem solving too.

Teachers must also reflect and be aware of their own attitude towards problem solving. If they themselves are motivated, passionate, and keen about solving problems, it will rub off their students. If they are frustrated, sceptical, and judgmental, their students will sense the negative emotions leading to detrimental effects.

CONCLUSION

One has to agree that teachers cannot, and should not be left alone to pilot the movement towards problem solving. Curriculum planners must come in to relook and amend the syllabus and assessment to make problem solving more visible and critical in school mathematics. Teacher training programmes should improve on the training that teachers received in problem solving, so that teachers will be more skilful, prepared, and confident. Perhaps, if teachers themselves immerse themselves in the problem-solving process substantively, they will have a better chance to transfer that knowledge, ability and appreciation to their students, making problem solving more meaningful.

In conclusion, one must acknowledge that problem solving is a long and tedious process and that it takes time for results to surface. It is essential to be mindful of the constraints on the schools, infrastructure, class organisation, examination boards, curriculum time, teachers and students' academic abilities. Nonetheless, it is necessary to embrace this change in order for our younger generation to be ready to meet the challenges of the future.

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