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Teaching Mathematics:
Some Guidelines For Establishing A Problem-Solving Environment

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> *The primary aim of the mathematics curriculum is to enable pupils to develop their ability in mathematical problem solving.*
>
> *(Mathematics Syllabus, 1990, p. 6)*

Teaching students to be mathematical problem-solvers is a clearly stated aim of mathematics education in Singapore. It is NOT unique to the Singapore educational scene; similar suggestions appear in the mathematics education documents of virtually all countries. It is an aim with which few people would disagree. If you talk to parents, teachers, politicians or business people they say that they want people who are able to think and solve problems. For example, the Singapore Minister of Education, Mr. Lee Yock Suan, recently talked about the importance of thinking skills as a goal of education, and the ability to solve problems involves a high level of thinking.

However, agreeing with the aim and implementing a problem-solving approach in the classroom are very different. As Hodgson (1995) states, "We cannot simply transfer our perceptions and problem-solving expertise to students..." (p. 18). We need to develop relational thinking (Skemp, 1976).

Skills versus Problem Solving: A false contradiction

When we discuss how to teach problem solving we are often faced with statements that imply that there is insufficient time. "I have too much to cover" is a common response. Teachers feel that developing problem solving strategies means that they will not have the necessary time to develop algorithms. Furthermore, the development of algorithms is often seen as the "core" of mathematics education. Without them, it is felt that students will not be able to progress in their mathematics education. Consequently, teaching students to be problem solvers is somehow seen to be "contradictory" or at least not possible if we are to develop algorithms within the constraints of a mathematics curriculum.

Such views lead to problem solving not being given the necessary emphasis within a mathematics curriculum. Teaching problem solving needs to be viewed as an integral component of all mathematics education. In fact, problem-solving strategies and algorithmic skills are complementary.

Consider the following "question" in Figure 1. It can be answered by the mechanical application of an algorithm. Provided students can recall the definition of mean, median and mode and the appropriate formulae for their calculation, the "question" is easy to answer.

Calculate the mean, median and mode of the following data.

1.35, 1.48, 1.48, 1.49, 1.49, 1.50, 1.52
1.52, 1.53, 1.54, 1.66, 1.68, 1.68, 1.69
1.70, 1.70, 1.71, 1.72, 1.75, 1.75, 1.75

Figure 1. Calculation of Mean, Median and Mode

Now, consider the following problem (Figure 2). Let us assume that students have excellent skills and algorithms. They can compute efficiently, define terms, etc. Can they solve the problem? The answer is no; they need more than algorithmic knowledge. Consequently, this problem cannot be solved by a pure application of algorithms.

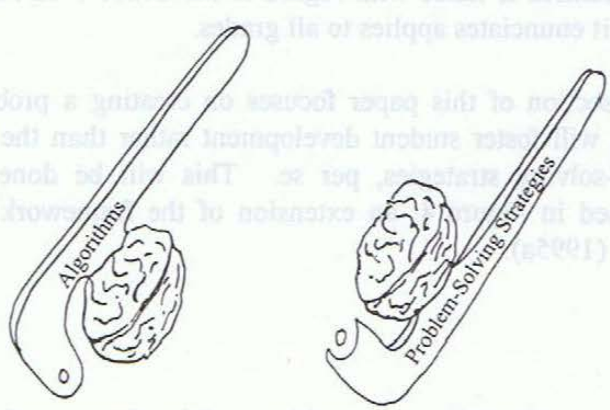
The following are the heights of various people attending a meeting. Calculate the mean, median and mode of the following data and indicate which you think would be the best to use to describe the data. Explain your choice.

1.35, 1.48, 1.48, 1.49, 1.49, 1.50, 1.52
1.52, 1.53, 1.54, 1.66, 1.68, 1.68, 1.69
1.70, 1.70, 1.71, 1.72, 1.75, 1.75, 1.75

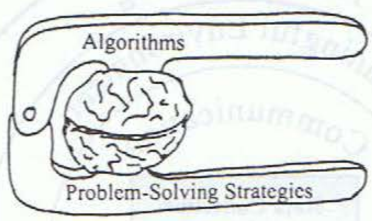
Figure 2. Calculation of Mean, Median and Mode in Context

If you tried to crack a nut (a problem) with a one-sided nutcracker you will not be successful. The problem will just 'fly' away (Figure 3a). Similarly, assume that students have excellent heuristics (problem-solving strategies). Again, if that is all the student has available, he or she is still using a one-sided nutcracker (Figure 3b). In order to crack the nut the nutcracker must have two sides (Figure 3c). It is through the combination of algorithms and problem-solving strategies that students can solve problems. If one accepts the analogy between the nutcracker and problem solving then the only logical conclusion is that both algorithmic skills and problem-solving strategies are necessary for the student. They are complementary; both need to be taught.

Figure 3: Cracking a Problem



3 (a) 3 (b)

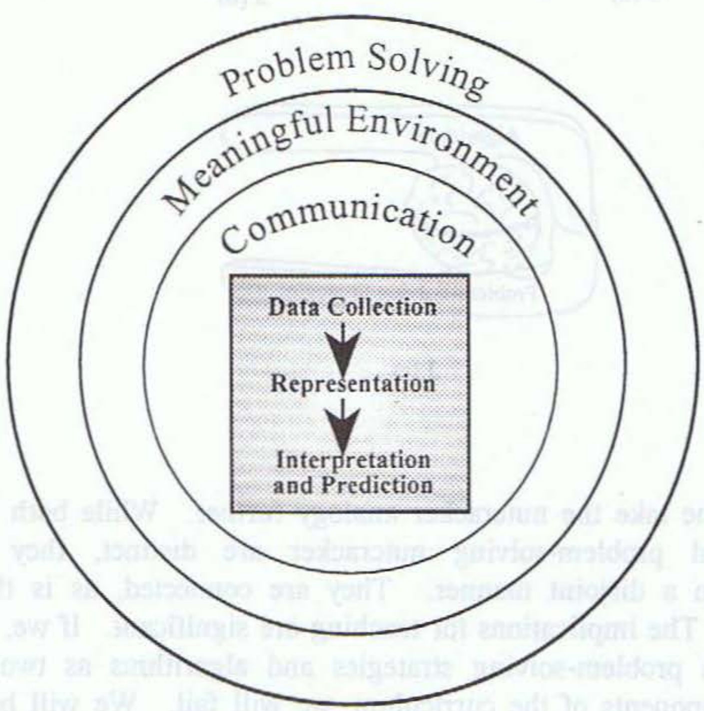


3 (c)

Let me take the nutcracker analogy further. While both sides of the mathematical problem-solving nutcracker are distinct, they cannot be developed in a disjoint manner. They are connected, as is the complete nutcracker. The implications for teaching are significant. If we, as teachers, try to teach problem-solving strategies and algorithms as two completely distinct components of the curriculum, we will fail. We will be constantly faced with the perceived pressure to trying to do both within an environment that already contains a crowded mathematics curriculum. We will see teaching algorithms and problem-solving strategies as two layers, and there will be insufficient time to "impose" problem-solving strategies on the curriculum. However, re-orienting our focus **within the present curriculum framework** is viable. We need to view mathematics and mathematical problem solving as indicated in the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards*, "Mathematical problem solving, in its broadest sense, is nearly synonymous with doing mathematics." (NCTM, 1989, p.137). Although this statement is made with regard to the Grade 9-12 standards, the general principle it enunciates applies to all grades.

The next section of this paper focuses on creating a problem-solving environment that will foster student development rather than the teaching of specific problem-solving strategies, per se. This will be done within the framework outlined in Figure 4, an extension of the framework outlined in Pereira-Mendoza (1995a).

Figure 4: Statistics in a problem-solving framework



This implies that the approach to teaching statistics (Data collection → representation → interpreting and predicting) is embedded within a structure in which the final aim is to develop problem solvers.

Let us start by examining some important components of a classroom environment structured to develop problem solvers.

Allow sufficient time to think

Frank (1988) reported that a common belief among the 27 junior high school students he studied was that:

> *Mathematics problems should be quickly solvable in just a few steps.* These students believed, in general, that mathematics problems were supposed to be routine tasks in which known arithmetic or algebraic algorithms could be applied. . . . They believed that something was wrong either with themselves or with the problem itself if a problem took "too long" (more than five to ten minutes) to solve. (p. 33)

Given this belief it is important that students be given time to think about solving problems. A problem, as opposed to an exercise, involves analysing the situation and deciding on how to approach the problem. This takes time. Consequently, if we are to be effective it is necessary (but not sufficient) that students be allowed **time to think.**

Encourage alternative strategies

In the film, "Twice Five Plus the Wings of a Bird", there is an illustration of how a student solved 3 + 5 by "the other way of adding" (transforming the problem to 4 + 4 which he already knows). In the same film Arthur Benjamin discusses his own way of squaring numbers. He indicates how he could do it the way the teacher required, but his answers imply that he enjoyed his own way and found his approach more rewarding. By demonstrating "our" method of solving a problem students tend to accept that one method as correct. If they had an alternative procedure, most students will force it into the background and replace it with the teacher's method; the "correct" method. We need to encourage students to accept alternative procedures. I am not advocating that all methods of solving a problem are equally valid. However, during the learning process it is important that students be exposed to different methods. Through discussion they get a better understanding that there are alternative methods (and sometimes solutions) as well as alternative criteria by which "better" is judged in mathematics. That way they develop a greater feel for the nature of the problem-solving process.

Use group work

There is an extensive literature on group work which is not reviewed in this brief paper. This literature shows that interaction among students can play an important role in students learning effective problem solvers. Through an exchange of ideas one is "encouraged" to try alternatives, defend ones own ideas, etc.

Use different levels of questions

Often the questions students are asked are ***basic questions***, that is, questions that only require students to recall information or do simple calculations that are derived directly from the problem. In a problem solving environment it is necessary for students to be asked intermediate and higher-level questions. ***Intermediate questions*** involve such ideas as having students write their own questions, designing their own representations of data, etc. Finally, ***higher-level questions*** involve students having to make value judgements about a situation. The earlier example (Figure 2) in which students had to determine which statistic was most appropriate is a higher-level question. Examples of statistics activities involving different levels of question can be found in Pereira-Mendoza (1995b).

Create a meaningful environment

Students deal with mathematical situations every day. If they take the MRT or an SBS bus to school it involves money. Deciding on what to have for lunch at the cafeteria means deciding if they have sufficient funds. If they give directions to a friend on how to get to a shop (turn left, go straight down the road,...) they are using geometric ideas. In spite of this many students feel that the mathematics they meet in schools does not relate to their world. Furthermore, many of the questions that students find in textbooks or meet within the school are often of little interest to them. It is not possible or educationally sound to make all school mathematics related to everyday situations or to other school subjects. However, much can be done in this area, and, in particular, statistics provides an excellent opportunity to create a meaningful environment in which to "do" mathematics.

Foster communication

No one would argue that language is important. The ability to communicate is essential to living a full life. The ability to communicate mathematically is also essential in today's world. It is not sufficient to be able to solve a problem; it is necessary that you can communicate your process and answer to others. The obvious situation is on a test. Unless people can read and understand your answer you will not pass. However, this is only a small part of mathematical communication. The previous illustration of giving directions is a form of mathematical communication. Being able to describe the shape of something to a friend is communication. Explaining how you are going to share something equally is communication. There are many other examples.

A variety of considerations to develop a problem solving environment have been included. They have been selected to illustrate things that can be done within the context of the current school curriculum and with large classes. While large classes provide limitations on what can be done within the classroom, the Japanese experience shows that thinking and problem-solving can be developed effectively with large classes. As Nagasaki and Becker (1993) note,

> The activities and sequence of events in a lesson are commonly organised to draw out the *variety of students' thinking*. The teacher's "wait time" is crucial in this respect. The different ways students think about the mathematical topic or problem in a lesson are respected to a very significant degree by the teacher; in fact, the dynamics of a lesson centre on this, and *teachers rely on students as an "information source"* during the lesson. The *discussion* of ideas is also a prominent characteristic. (p. 41)

What it being suggested is a refocussing of teaching, not a re-invention of the wheel. The remaining part of this article provides an exemplar that illustrates how these ideas can be applied within the current curriculum and classroom.

Students are often interested in characteristics that can be used to describe themselves or the other students in the class. You could start by asking the students the following question:

> You are going to have a visitor to the class. How would you describe yourself so that the visitor would recognise you?

Split the class *into groups* and *allow time* for them to discuss possible characteristics that they might use. Such characteristics might involve height, hair colour, type of clothing, etc. Although the teacher could pre-select the characteristics that could be used, it is important that the teacher allow the class to make the initial selection. This means that the students have a greater degree of ownership of the problem, a greater involvement is discussion and it *creates a more meaningful environment* for them. They feel it is their problem they are exploring.

The question can now be broadened to ask how these characteristics apply to the whole class. Each group can collect the appropriate data for their group. This can then be pooled into a data set that the class can then discuss. Part of this discussion can be on which characteristics would be the best to focus on and how might these characteristics be represented. This is a *higher-level question* that gives an opportunity to discuss *alternatives*. The level of discussion and sophistication of choice will depend on the class. A primary class might focus on hair colour and sex, say, and then they might decide how to represent this data (chart or graph). A secondary class might select different characteristics.

For the purpose of continued discussion height will be selected. While height will be used to illustrate the approach, any appropriate data that interests your class could be used. If height were discussed at the primary level the discussion might centre on doing the measuring and on different possible representations. At the secondary level the data could be discussed in terms of measures of central tendency and dispersion. As in Figure 2, students might be asked to think about which measure of central tendency is better.

This activity has been described very briefly. It is just meant to give you a starting point for reflection to create a problem-solving environment. Such an approach appears to take more time. However, it involves the application of important statistical skills (graphing, calculation of the mean), other mathematical concepts (measurement) and problem-solving strategies. It integrates the use of these skills into a problem-solving environment. It provides practice in their application. It allows students to practice algorithms and problem-solving strategies in a different way. Through such an activity students develop a better understanding of mathematics and are more likely to be better problem solvers. Consequently, the apparent additional time is really just a refocussing of time towards the primary objective of the programme; enabling pupils to be mathematical problem solvers and think mathematically. This is the basis on which they can continue to learn in the future. The ability to solve problems is essential if students are to be able to adapt to the future needs of society.

> What is crucial, therefore, for the education system in preparing our young people for the future is to impart to them the ability and understanding of how to learn. This is not a luxury reserved for the gifted nor an alternative method for schools inclined to go off on the far side. It is an absolute necessity because no one today can predict what needs to be learnt in five to ten years from now. (Koh, 1995)

Teaching students algorithms and problem-solving strategies is a significant step towards educating students who can think mathematically and continue to learn.

References

Frank, M. L. (1988). Problem solving and mathematical beliefs. *Arithmetic Teacher*, 35(5), 32-34

Hodgson, T. P. (1995). Connection as problem-solving tools. In P. House & A. F. Coxford (Eds.). *Connecting mathematics across the curriculum: 1995 yearbook of the National Council of Teachers of Mathematics* (pp. 13-21). Reston, VA: National Council of Teachers of Mathematics.

Koh, B. H. (1995). The challenges for our education system. *Speech given to the Principals' Conference in Singapore, September, 1995.*

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston: VA: NCTM.

Mathematics Syllabus (Primary) (1990). Curriculum Planning Division, Ministry of Education, Singapore.

Mathematics Syllabus (Lower Secondary) (1990). Curriculum Planning Division, Ministry of Education, Singapore.

Nagasaki, E, and Becker, J. (1993). Classroom assessment in Japanese mathematics education. In N. Webb and A. F. Coxford (Eds.) *Assessment in the mathematics classroom: 1993 yearbook of the National Council of Teachers of Mathematics* (pp. 40-53). Reston, VA: National Council of Teachers of Mathematics.

Pereira-Mendoza, L. (1995a). Statistics is communication. *Teaching and Learning*, **15(1)**, 52-62.

Pereira-Mendoza, L. (1995b). Using newspapers and other documents to teach statistics in the primary school. Mathematics Newsletter of the Curriculum Planning Division of the Ministry of Education, **1(12)**, 8-12.

Skemp, R. R. (1976). Relational Understanding and Instrumental Understanding. *Mathematics Teaching*, **77 (December, 1976)**, 20-26.