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The Mathematical Vitality of Undergraduate Mathematics Student Teachers in Singapore

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Abstract: This paper explores the Mathematical Vitality (MV) of mathematics undergraduates who are being prepared to be teachers and compares the relative MV Test performance between student teachers who had completed a large proportion of their undergraduate mathematics courses and those who had just entered the degree programme. The results show that the senior students were not consistently strong in the various aspects of mathematical logic and techniques of proof although they were much stronger than the new students in overall performance on the MV Test.

Keywords: Mathematics Vitality; Mathematics Teacher Education; Undergraduate Mathematics Students

Introduction

Compared to decades ago, university mathematics in Singapore is now taught to an ever-widening spectrum of students who have diverse reasons and objectives for reading the subject. While only a small minority will go on to become research mathematicians, a large proportion of students who take mathematics courses do so as a service subject in order to complete their degrees in Engineering, Business, Science, Computer Science, Economics and so on. With such a purpose in mind, many students of undergraduate mathematics are only interested in the subject for its practical applications to their future profession. There is, however, one group of undergraduates for whom mathematics in general will remain extremely relevant, and these are students who will be teaching mathematics almost daily in their profession as future teachers.

While mathematics teachers may not need to understand or use deep level mathematics as is the case for mathematicians, research scientists, and research engineers, their knowledge of mathematics cannot be at the superficial level of merely carrying out algorithms or applying formulae at a level slightly above what they have to teach in the school curriculum. As mathematics educators, they have the all-important role of developing in their students an early conception of what the practice of mathematics is all about. The mathematical preparation of potential mathematics teachers thus occupies the significant intersection of mathematics teacher education and tertiary mathematics education, and this is exemplified by the *Preparation of Primary and Secondary Mathematics Teachers* being one of the working groups in the 1998 ICMI study conference on *The Teaching and Learning of Mathematics at University Level*.

The National Institute of Education (NIE) is the sole teacher preparation institution in Singapore, and almost all the thirty thousand teachers in Singapore schools have received their pre-service teacher education through one of its three pre-service programmes. While mathematics pedagogy courses are offered in all these programmes, it is only in the four-year Bachelor of Arts/Science (Education) (B.A./B.Sc.(Ed)) programme that the undergraduates being prepared to teach secondary mathematics take mathematics content courses within their teacher preparatory programme. These mathematics content courses, which are also available to those who are being prepared for teaching in primary schools, are similar to university mathematics courses elsewhere in depth and demands of rigour, although the coverage is substantially less than that taken by mathematics majors in faculties of science. This is because the education programme also includes two to three sets of subject pedagogy courses and several general education courses.

In the international comparative study on teacher education called the *Teacher Education and Development Study in Mathematics* (TEDS-M) (Tatto et al., 2012), Singapore's student teachers performed very well in mathematics content and mathematics pedagogy. In particular, future teachers in the B.A./B.Sc. (Ed) group were front runners among the primary student teachers, outperforming all the other groups in Singapore and participating countries (Wong et al., 2012).[1] In this context, a few teacher educators at the Mathematics department of NIE were interested to investigate further to find out if the mathematics undergraduates in the programme have sufficient mathematical facility in the fundamental processes of the mathematics discipline. While normal university mathematics examinations assess the candidates' content knowledge in terms of concepts, theorems, and applications, this study purposed to investigate the awareness and understanding of these undergraduates using the concept of *Mathematical Vitality* (MV), which originated some thirty years ago (Galbraith, 1982). In this study, the students' *Mathematical Vitality* was measured by an instrument called the *Mathematical Vitality* Test, modified from the Galbraith's instrument.

[1] At the time of the TEDS-M study, the B.A./B.Sc. (Ed) programme did not have a secondary track.

Since the undergraduates read mathematics courses over a three-and-a-half year period, the study investigated their MV at two points of their tertiary journey, at the beginning of the four-year program and as they approached the end of their mathematics courses. Findings on the performance of the senior group of student teachers had been presented in an earlier conference paper (Lim-Teo, Soon, & Ang, 2011). This paper presents snapshots of the MV Test performance of the two cohorts of undergraduates, with comparisons showing distinct differences particularly in the student teachers' understanding of proof techniques.

Theoretical Framework and Rationale for Study

Over the last three decades, teacher educators have studied the issue of different types of knowledge for teaching. The concept of subject-specific pedagogical content knowledge (PCK) as specialised knowledge just for teaching, which is different from subject matter knowledge, is a well-accepted concept among teacher educators. Within mathematics, there has been much research and discussion on teacher knowledge and its impact on their teaching and on student learning (Ball, Lubienski, & Mewborn, 2001; Mewborn, 2003). The issue of teacher knowledge has implications for teacher preparation and international comparative studies such as the TEDS-M study (Tatto et al., 2012), and reviews of research on the mathematical preparation of teachers (Tucker et al., 2001) have provided research information to teacher preparation institutions seeking to improve their programmes. In reviewing various research studies on teacher knowledge, Mewborn (2003) concluded that it is extremely difficult to establish links between teachers' mathematics subject matter knowledge and achievements of their students, particularly if the teachers' knowledge is measured by the number of mathematics courses taken at university level. Nevertheless, it is almost universally accepted that a teacher's mathematics subject matter knowledge is a necessary though not sufficient condition for being able to guide and facilitate their students to deeper mathematical thinking, either through deliberate planning or by spontaneously seizing teachable moments when they occur.

In clarifying the various concepts of teacher knowledge, Ball, Thames, and Phelps (2008) delineated subject matter knowledge (SMK) for teachers as a separate concept not to be confused with pedagogical content knowledge. Under mathematics SMK, they further separated the concept into three sub-concepts: common content knowledge (CCK), specialized (for teachers) content knowledge (SCK) and horizon content knowledge. The focus of this study, however, was not on mathematics SCK per se but an aspect of mathematics CCK, which could be pre-requisite for some parts of SCK. This is the concept of *Mathematical Vitality*, defined by Galbraith

(1982) as an attribute of a mathematically aware student of mathematics, who would have a deep understanding of mathematical processes, the ability to carry out a mathematical analysis, and the ability to construct a logical defence of mathematical statements.

In the context of the ability to defend mathematical statement, a most fundamental factor in any discipline is the determination of truth. In mathematics, the scarcity or even absence of “proofs” at secondary level mathematics in the Singapore curriculum has resulted in an erosion of this basic pillar of mathematics. There has been research evidence to show that proofs are difficult, not only for potential mathematics teachers but also for university mathematics undergraduates in general (Bell, 1976; Jones, 2000; Knuth, 2002; Selden & Selden, 2003; Weber, 2001, 2003). However, with the belief that the concept of mathematical proofs is fundamental to the discipline, the understanding of how proofs and logical arguments are used in justifying statements was given more prominence in this study’s concept of MV as compared with Galbraith (1982). It should be pointed out that MV does not focus on students’ conceptions of proof or their particular difficulties with proofs as discussed in the above papers, but on their understanding of when a proof is needed and when it is considered mathematically valid.

Mathematical Vitality (MV) for this study is thus conceptualized as the attribute of having a basic understanding of the fundamentals in the discipline of mathematics in terms of its language, structure, and in particular, the ways and processes of determining and accepting mathematical truths. While mathematics undergraduates can complete a university mathematics degree with the knowledge of much mathematics content, what lasts much longer is their MV because it is an attribute which becomes part of the learner’s mathematical disposition, having been developed through many mathematics courses taken by the learner.

As noted above, MV is an aspect or a part of common content knowledge (CCK) as defined by Ball, Thames, and Phelps (2008) rather than specialized content knowledge (SCK) for teachers. The word “common” in CCK means that the knowledge is to be achieved by students regardless of their future profession but it is clear that the knowledge varies depending on the mathematical level of people using the knowledge. For example, the knowledge of percentage or of area of a rectangle may be common to most people but the knowledge of how to express a relationship as an algebraic equation, especially a non-linear one, is not something many school leavers would have or need to use. In our context, MV is “common” in that it cuts across professions such as mathematicians, engineers, teachers, statisticians, or scientists but it need not be common knowledge to the general public.

Although there is a distinction between SCK and CCK, and MV is deemed part of CCK rather than SCK, MV could also be considered an essential ingredient of or a pre-requisite for SCK. In discussing SCK, Ball, Thames, and Phelps (2008) were "struck by the relatively uncharted arena of mathematical knowledge necessary for teaching the subject (SCK) that is not intertwined with knowledge of pedagogy, students, curriculum or other non-content domains" (p. 402). They looked into specific tasks which teachers are expected to engage in, tasks such as responding to students' "why" questions or recognizing what is involved in representing a particular mathematical idea, and noted that "these tasks require knowing how (mathematical) knowledge is generated and structured in the discipline and how such considerations matter in teaching" (ibid). So, although SCK refers to the ability to carry out these specific tasks, the knowledge and perhaps the disposition, required to carry them out depend much on the MV of the teachers. In addition, teachers need to have strong MV in order that they can in turn help their students to build up some understanding of the structures of the discipline beyond learning facts, concepts, and mathematical results. As role models to their students and initiators to the discipline, teachers should use proper mathematical language, and even if they are only going to introduce simple justification arguments in explaining results, they need to have strong MV to be immediately aware of fallacious logic used by students or to avoid such erroneous arguments themselves. Thus, while MV is not directly related to teaching and hence more properly classified under CCK as opposed to SCK, strong MV is nevertheless an important requisite for SCK.

Research Questions

The study was designed to answer the following research questions:

1. How strong is the *Mathematical Vitality* of the B.A./B.Sc. (Ed) students towards the end of their study of mathematics at the university?
2. Is the *Mathematical Vitality* developed during the study of mathematics at university level or do students who decide to do mathematics at university already have strong MV?

Answers to these questions will have implications for the review of the content and delivery of the mathematics curriculum for the potential mathematics teachers. Not only will the findings be of interest to mathematics teacher educators, they will also provide information on student learning to mathematicians teaching university level mathematics.

Method

Instrument: Mathematical Vitality Test

The author sought and received permission to modify and use items from the *Mathematical Vitality Test* as designed by Galbraith (1982). The instrument, hereafter called the MV Test, comprised 12 items in multiple-choice format and space was provided for explanations if any. The items together with a short descriptor of the mathematical understanding being examined for each item are given in the Appendix. Besides testing for understanding of convention in the use of mathematical language, the majority of the items checked for understanding of different techniques of proof or whether the students could ascertain whether mathematical arguments are logically valid or otherwise. Performance on the MV Test would be used operationally as a measure of *Mathematical Vitality*. There was no time limit for completing the test.

Sample

The sample of this study is taken from undergraduates of the B.A./B.Sc.(Ed) programme. This programme has two separate tracks for the preparation of primary or secondary school teachers and, while the tracks had different curriculum structures and also different education and pedagogy courses, the Academic Subject (AS) courses are common to both tracks.

Students in the secondary track are trained to teach two subjects and their respective methods courses comprise 12 credits each, out of a total of 131 credits in the programme. They have to take two Academic Subjects (AS) that correspond to these two teaching subjects, but one of subjects will be taken at major level (13 courses totalling 39 credits) and the other at minor level (8 courses totalling 24 credits).

Students in the primary track study only one AS subject, such as Mathematics or English, at major level. However, they are trained to teach three school subjects, and for each of which they take a set of Curriculum Studies (teaching methods) courses (10 credits) with a corresponding set of Subject Knowledge (SK) courses (6 credits); the SK courses aim to extend understanding of topics in the primary school curriculum.

The sample for this study comprised undergraduates who were reading AS Mathematics[2] at major level. The first group was made up of students who had already completed at least 10 out of 13 AS mathematics courses and who were taking their eleventh and twelfth mathematics courses in the semester when they

[2] AS Mathematics courses are similar in content and depth to usual university mathematics courses, such as Calculus, Algebra, Statistics etc.

took the MV Test. Their performance on the test would answer the first research question about the level of MV towards the end of their training. This is referred to as the “senior” group in the following sections.

The second group, termed the “junior” group, comprised new students who had just begun their degree programme. None of them had mathematics levels beyond senior high school, although they had done well enough at the Cambridge ‘A’ Level Examination or NIE’s mathematics qualifying test to be admitted to read AS Mathematics. The purpose of administering the MV Test to this group was to obtain a benchmark of MV at entry level to university mathematics, hence providing baseline data to gauge the effectiveness of university mathematics in developing their MV. The performance of this junior group will be used as proxy for the entry level of MV in order to answer the second research question.

All the students in both cohorts were invited to participate in the study on a voluntary basis and were assured that the results would have no bearing on their course assessment. For the senior group, the test could be taken at any time during a two-day period at a fixed venue. Since the students knew that it was for research purposes, there was no concern that earlier participants would tell others about the items and this flexibility in test administration time was intended to encourage higher rates of participation. Nevertheless, of the 86 students in the cohort, only 20 (9 from the primary track and 11 from the secondary track) turned up to take part in the study. For the juniors group, the test was taken during their mathematics tutorial class time in the first week of their first semester. Out of the 56 students in this cohort, 41 (18 from the primary track and 23 from the secondary track) turned up to take the MV Test.

Among the 20 seniors, 14 of them had taken ten mathematics courses while the remaining six had completed eleven or twelve courses. Eight of these courses were compulsory core courses (Calculus 1 and 2, Algebra 1 and 2, Statistics 1, Finite Mathematics, Number Theory, and Computational Mathematics) while the other courses were electives with different combinations of choices. For the Junior group, they had just begun taking two basic courses, namely Calculus 1 and Algebra 1.

Findings

General results

Although there was no time limit, most students completed the test in less than 30 minutes and the maximum time taken was 45 to 50 minutes. The distribution is shown in Table 1 below.

Table 1
Time taken to complete MV Test

	Time in minutes			
	< 20	20-30	30-40	45-50
Seniors	3	15	0	2
Juniors	17	22	1	1

Since the samples were very small, in particular for the senior group, the results were not analyzed using statistical comparison tests but the overall performances of the two groups were compared relative to each other. The between-group differences in performance for each individual item, and especially the choices made, were also analyzed and will be discussed in the next section.

The test instrument had 12 items and the number of correctly answered items ranged from 1 to 11. The items were not given different marks based on difficulty and the each score on the test shows the number of correctly answered items. Table 2 presents the frequency distribution of students in percentage across the scores. From Table 2, it can be seen that 17 (85%) of the seniors answered at least 6 items out of the 12 items correctly while among the 41 juniors, only 7 (17%) of them were able to answer at least 6 items correctly. This distribution is also represented in Figure 1. It shows that the seniors' performance was stronger than that of the junior group.

In the senior group, the test results also showed a wide variation among the students. With the exception of one outlier student who only obtained two correct answers, the rest ranged from having 5 correct answers to 11 correct answers. The poor test results of the junior group were quite expected as they had yet to begin their mathematics courses at the university. The scores ranged from 1 to 7 with exception of one outlier who obtained 9 correct answers among the 12 items.

Table 2
Frequency Distribution of MV Test Scores by Groups in Percentages

MV Test Score	Junior Group	Senior Group
0		
1	2.4	
2	12.2	5.0
3	26.8	
4	26.8	
5	14.6	10.0
6	7.3	10.0
7	7.3	25.0
8		10.0
9	2.4	15.0
10		15.0
11		10.0
12		

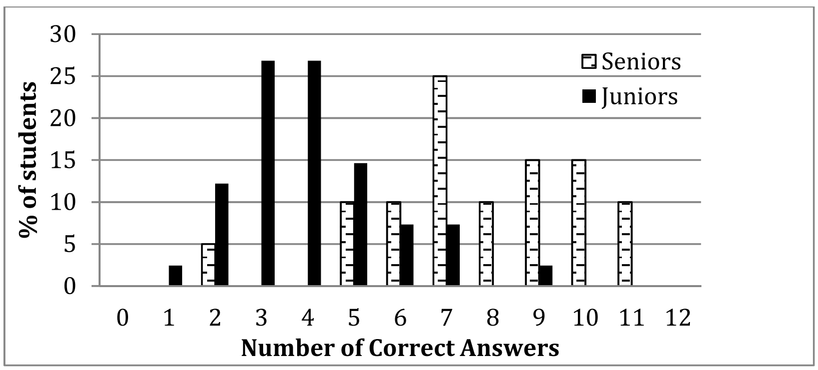

Figure 1. Distribution of Students across Scores.

Discussion of performance on individual items

Table 3 shows the frequency percentage distribution of answer choices made on each individual item for both groups. The items have been re-arranged to match the following discussion of clusters of items according to the concepts or processes being tested. The correct option for each item is underlined in bold.

Table 3
Percentages of Answer Choices by Clusters of Items

Item	Group	A	B	C	D	E	Blank
3	Seniors	15	15	15	25	**<u>30</u>**	
	Juniors	27	17	46	5	**<u>2</u>**	2
11	Seniors				**<u>100</u>**		
	Juniors	10	27	12	**<u>49</u>**	2	
12	Seniors	10	**<u>50</u>**	5	35	—	
	Juniors	10	**<u>7</u>**	17	66	—	
1	Seniors	**<u>95</u>**			5		
	Juniors	**<u>93</u>**			5	2	
5	Seniors	**<u>55</u>**	30		10	5	
	Juniors	**<u>22</u>**	39	5	20	12	2
6	Seniors		10	5	10	**<u>75</u>**	
	Juniors	17	22	5	17	**<u>39</u>**	
7	Seniors	**<u>70</u>**	5	5	15	5	
	Juniors	**<u>51</u>**	2	5	37	5	
8	Seniors		**<u>60</u>**	25		15	
	Juniors		**<u>37</u>**	41		22	
9	Seniors	25	5	**<u>50</u>**	20	—	
	Juniors	27	17	**<u>37</u>**	17	—	2
10	Seniors	**<u>80</u>**	5	10	5	—	
	Juniors	**<u>37</u>**	34	22	7	—	
2	Seniors	65	**<u>35</u>**				
	Juniors	95	**<u>5</u>**				
4	Seniors	10			**<u>70</u>**	20	
	Juniors	5		10	**<u>27</u>**	59	

The seniors found six of the items (1, 4, 6, 7, 10, and 11) relatively easy with at least 70% selecting the correct answer. In particular, items 1 and 11 had almost every senior selecting the correct option and the only one who chose a wrong option in item 1 was the student with the lowest score of 2 correct answers. On the other hand, the only item which could be considered easy for the juniors was item 1, where 93% of them selected the correct answer.

Using examples as proof: 3, 11, 12

Items 3, 11 and 12 were designed to test whether the students would accept a range of examples (whether a finite or infinite number of them) as proof of a result.

The results showed that item 3 had the worst performance among all the items. It was the only item where every distractor was selected by a substantial number of students, even among the seniors. Distractors (A), (B), and (C) were designed to test the misconception that an infinite number of objects satisfying a property lead to the conclusion that all such objects satisfying the property. Although it is not surprising that 90% of the juniors chose one of these three options, it is a matter of concern that a total of 45% of the seniors also selected these options. Very few students wrote additional explanations for their answers but one senior actually wrote "infinitely many means all". Five seniors chose option (D) and a closer study of the four explanations given showed that they did not have the same misconception: two wrote "infinitely many does <u>not</u> imply all" and the other two tried to give counter-examples (one used the fact that infinitely many right-angled triangles satisfied Pythagoras' Theorem but not all triangles do). However, a lack of understanding of the phrase "necessarily correct" and not taking into consideration the phrase "more likely" resulted in choosing Option (D) as the closest to their conclusion.

In item 12, a classroom situation and a specific well-known geometrical result were presented and the participants were to determine if the teaching activity was an acceptable "proof" of the result. Half of the seniors stated correctly that although many examples were used to verify the result, the proof was (mathematically) invalid as one must prove it for <u>ALL</u> triangles. Among the juniors, two-thirds of the participants, compared to 35% of the seniors, selected option (D), accepting the activity as valid because it covered different types of triangles. The researcher had expected that the students may select option (D) due to their pedagogical training since such a validating activity is normally used in primary classrooms. If this was so, the seniors would be more likely to choose this option compared to first year students who had not undergone any pedagogical training. However, the findings show the reverse; the high proportion of the junior group selecting option (D) suggests that they may have encountered such activities in their own school days. Furthermore, it was reassuring that the seniors could distinguish between what was suitable as a teaching activity and what was a mathematically acceptable proof. Even among the seven seniors who selected option (D), three of them explained that they felt the activity was valid in the context of teaching but more formal proofs were necessary for mathematics.

While the seniors had problems with item 3 and, to a lesser extent item 12, they had no problems with item 11, where all of them realised that using specific numerical examples did not constitute a proof. This was, however, not the case for the juniors, where only 49% selected the correct option and another 49% thought that at least one of the two “proofs” using specific numerical examples was valid.

The large discrepancy between the MV Test performances of the two groups in this category of items provides evidence that students at school level often encounter numerical or other examples in illustrating mathematical results and hence take these illustrations as “proofs” of the results, a misconception to be corrected only through learning mathematics at university level. This is not to say that the senior group will be able to write mathematical proofs when they graduate but, at the very at least, a large majority of them will recognise that one cannot use examples to prove mathematical results.

Understanding of mathematical logic: 1, 5, 6, 7, 8

Items 1, 5 to 8 test students’ ability to apply mathematical logic to particular examples.

Item 1 tests the if-then logic applied to simple numbers. Both groups did well on this item since the numbers were simple and each option could be verified numerically. In general, the seniors performed well on items 6 and 7. For item 6, 75% of the seniors understood that a proven result does not imply the truth or otherwise of its converse. Among the juniors, however, only 39% realised this with another 39% believing that the converse was true, with or without further proof. For item 7, 70% of the seniors could identify the counterexample which could be used to disprove an if-then statement as did 51% of the juniors.

Item 5 checks on the participants’ understanding that a statement is equivalent to its contrapositive. The statement and its contrapositive were both stated in a general way but the word “contrapositive” was not used. A large majority (85%) of the senior students and 61% of the juniors felt that the contrapositive of the proven statement was true but only 55% of the seniors and 22% of the juniors realised that no further proof was necessary. This shows that the participants were able to link the statement with its contrapositive but the idea of logical equivalence was not that well established. Although items 5 and 6 were rather similar in phrasing, except that one dealt with the contrapositive while the other dealt with the converse, the better performance in item 6 as compared to item 5 indicates that it is easier to give the correct answer when it is of the “cannot tell if it is true” type rather than the more assertive “true and don’t need further proof” type of answer.

Item 8 gives a statement that is stated as true and participants were asked to select a true statement among five options, among which were the converse and contrapositive of the original. A large majority chose either the converse or the contrapositive but not the other three options with the seniors outperforming the juniors (60% versus 37%) in choosing the contrapositive. What was particularly interesting was that among the five seniors who selected the converse as also being true, four of them chose wrong options in item 5 which dealt with the contrapositive. Among them, three of them accepted that the contrapositive of a theorem was true but felt it still need to be proved.

Over these logic items, the seniors demonstrated a deeper understanding of logic compared with the juniors, and this is probably through their learning experiences in the various AS Mathematics courses. For both groups, the substantially stronger performance of items 1 and 7 compared with the performance of items 5 and 8, suggests that the students found abstract and general statements more difficult than specific results with given numbers which could be numerically checked and verified.

Specific proofs: 9, 10, (11)

Items 9, 10 and 11 present specific proofs of results and students were to determine whether the proofs are valid or where errors occurred. Item 11 was discussed above under using examples as proof.

Item 10 provides a correct proof by contradiction. The seniors found it rather easy, with 80% correct. Five of them explained that the given example was a "proof by contradiction". For the new students, however, only 37% thought that the proof was correct (option (A)) with 56% choosing either option (B) or (C). Option (B) stated that the proof was wrong because it began with a wrong assumption (the negation of the conclusion of the statement) and option (C) stated that the proof was wrong because it proved another result (the specific phrasing of the contra-positive of the given statement). It can be concluded that these students did not know that a proof by contradiction is mathematically acceptable since proving the contrapositive is logically equivalent to proving the original statement. This is again an expected result since "proof by contradiction" is not normally encountered at high school levels.

Item 9 proved to be difficult for both groups and the between group difference was low compared to other items, with 50% and 37% of the senior and junior groups respectively selecting the correct option (C), which stated that the proof was incorrect because it proved the converse instead of the result itself. Those who deemed the proof valid and selected Option (A) comprised 25% of the seniors and 27% of the juniors. As for the other distractors, the seniors were inclined towards option (D) that the proof was wrong for other reasons while the juniors were equally distributed between option (D) and option (B) which stated that squaring an inequality was wrong. For both groups, the students were unable to give valid explanations when they selected (D) with their explanations containing logical or algebraic errors.

While item 6 was categorized under Understanding of Mathematical Logic and item 9 under Specific Proofs, they both dealt with a statement and its converse. While three-quarters of the seniors understood that one cannot make conclusions about the converse of a proven statement in item 6, only half recognised that the given proof of item 9 is actually a proof of the converse instead of the proof of the given statement and is thus incorrect. Results from items 9 and 10 suggest that the seniors were better able to recognise a correct proof by contradiction than to distinguish a proof of a statement from a proof of its converse.

For the juniors, only about a third gave the correct answers to items 6, 9 and 10, showing a lack of understanding of the relationships between a statement, its converse and its contrapositive. This may be because at high school level, most of the results are of the if-and-only-if type and there is little emphasis on formal proofs. In particular, proofs-by-contradiction is not normally encountered prior to university mathematics.

Miscellaneous items: 2, 4

Item 2 tests the convention involving the square root ($\surd$) sign. It was the single item where the wrong option (A) was chosen by a sizeable majority of the seniors and by almost all the juniors. In their earlier learning of mathematics, many students had been so conditioned to write down both the positive and negative square roots when solving $x^2 = a$ (where $a > 0$) that it was quite natural for many of them to think that $\sqrt{9}$ is ± 3. This result is not surprising as similar results were reported for the same item by Galbraith (1982) while testing prospective teachers in Australia and by Wong (1990) for postgraduate diploma in education (PGDE) students in Singapore. This error is not too worrying since the misconception can be corrected through explicitly making the notations clear and emphasising the distinction between $y^2 = x$, $y = \pm\sqrt{x}$, $y = \sqrt{x}$ and $y = -\sqrt{x}$.

Item 4 checks the basic knowledge that π is irrational and the awareness of the nature of product of rational and irrational numbers. While the senior group performed fairly well, a majority of the juniors did not make the deduction that the product of an integer and an irrational number is necessarily irrational. This may be because they regard π as ${}^{22}/_{7}$ and hence rational or because they are not used to dealing with a generalised but unknown quantity for the radius.

Discussion and Implications

As stated in the research questions, the study sought to find out the Mathematical Vitality of the senior students in the degree programme and whether it was developed during their undergraduate studies in Mathematics. The MV test results of the two groups in the MV test indicate an affirmative answer to the second research question as the senior group performed substantially better than the new students. The answer to the first research question is more complex. Despite having far stronger MV than the new students, the seniors did not perform entirely satisfactorily. There was a wide variance among performance of individual students and different performance levels in different items or types of items.

As a group, the senior students still had gaps in their understanding of fundamental mathematical logic, mathematical language, or proof techniques. Considering that they had already completed at least ten of their thirteen courses for AS mathematics, only half of them managed to get 8 or more items correct out of the 12 items. The difficulty could be due to two features of the instrument: (i) many of the questions were not contextualised with explicit numerical examples or specific geometrical properties, and (ii) the rather unfamiliar mathematical language used. This is also a cause for concern because it means that these students lacked fluency in mathematical language and were unable to draw general conclusions.

While the results cannot be generalised due to the small sample and low response rates, the fact that the students voluntarily participated indicated that it is likely that weaker students would decline to participate and it was indeed found from the names of the senior participants that a substantial number of them were high scorers in AS Mathematics.

The findings also indicate that mathematical logic could be better understood by the senior students but they were not able to argue for the correct responses particularly where the examples were not specific enough. The relationship between a statement and its contrapositive, for example, as tested in items 5 and 8, was only well understood by slightly more than half of the seniors who participated, although the group as a whole performed well in item 10 concerning the validity of a proof-by-contradiction. It thus appears that the ability to recognize the steps for carrying out a process (proof-by-contradiction) is easier to acquire than to relate statements with their contrapositives.

The department has debated whether various aspects of mathematical logic coupled with methods of proofs should be covered in a separate course or taught implicitly through the various courses covering different topics. The current curriculum does not have a separate course since it would be difficult to teach proofs without content topics to provide the context. From the fact that some senior students actually used terms like "contrapositive" and "proof by contradiction", it can be deduced that these terms had been taught. Furthermore, compared to juniors who had just begun university mathematics courses, the stronger performance of the seniors shows that the AS courses had certainly made substantial impact in their understanding of what constitutes valid mathematical proofs.

However, it is clear from the above results that the understanding of fundamental mathematical ideas and processes of proof had not been well internalised by all the senior students. From their study on how undergraduates validate proofs, Selden and Selden (2003) concluded that few university teachers teach the validation of proofs explicitly and that students "appear to be substituting their feeling of overall understanding, plus a focus on surface features, for validation" and while "such a feeling of overall understanding is useful and reliable for mathematicians who will have included it in their notion of correctness", it is "often unreliable and misleading for students" (p. 28). The findings in this study support the need for explicit teaching of proof techniques and mathematical logic even if it is not done in any separate "bridging" or "introductory" courses in the current curriculum. Under this circumstance, faculty could and should make a concerted effort to continually emphasise the various methods of proof and the understanding of mathematical logic in their teaching, to the extent of explicitly naming and explaining the methods or terms repeatedly across as many courses as possible.

One specific limitation of this set of data is the small sample size of the senior group. The participation rate was much better with the first-year students with more than two-thirds taking part. It is difficult to encourage greater participation since the MV Test was not part of course assessment. Furthermore, the submission of the test papers within 20 minutes by 17 of the 43 first-year students shows a lack of effort in trying to figure out the correct answers because they knew the test was for research purposes only. Nevertheless, although the findings cannot be generalised, they are useful for informing faculty of their students' mathematical understanding.

The findings in this study could not be directly compared to the studies by Galbraith (1982) and Wong (1990) but for seven similar items (items 1-4, 6-8), a greater proportion of the senior students in the current study were able to select the correct answer as compared to those in Wong (1990). In fact, for items 1, 4, 7 and 8, the junior group did better than or equal to the Wong's sample. One reason could be the currency of their tertiary mathematics courses because the sample in Wong's study comprised students who had graduated a few months or even years prior to their postgraduate diploma in education program, while the students in this study were still enrolled in undergraduate mathematics courses.

Galbraith (1982) has argued that pre-service teachers seemed to approach mathematics with an instrumental view and he also conjectured that simply taking more mathematics courses would not enhance *Mathematical Vitality*. He suggested more research into establishing a link between student performance and *Mathematical Vitality* as a measure of relational mathematics, both of the teachers themselves and their approach to teaching mathematics. In the current study, ethics requirements prevented the researcher from requiring the participants to indicate their names on the test scripts unless they wished to do so. Many of the senior students did indicate their names and with the small sample, it was possible to match the remaining scripts with names on the consent forms to a high degree of accuracy. The MV test scores were then matched with their cumulative grade point average for their 13 AS Mathematics courses which they had completed by the time the results were analysed and the correlation was found to be 0.59. This must be taken as only indicative due to the small sample size, but there seems to be positive correlation between MV and mathematics performance in university courses. Further study would need to be undertaken after the senior students in this study had graduated in order to (a) know if they have retained their *Mathematical Vitality* and (b) establish links, if any, between *Mathematical Vitality* and teaching approaches.

Conclusion

As in most tertiary institutions, mathematics courses are taught by experts in the respective fields. It is quite a norm for mathematicians to cover a systematic collection of definitions, theorems, and applications with the assumption that their students will somehow, through doing mathematics, learn to be fluent in mathematical language and develop a general understanding of the logical processes of arriving at mathematical truths. This study has shown that there are students who fail to acquire such *Mathematical Vitality* after taking most of their university mathematics courses, although they performed substantially better than first year students.

Continual curriculum review and investigation into the teaching approaches of faculty and the learning of their students are necessary for mathematics teacher education to remain relevant and effective. In addition to international issues raised by the ICMI Centenary Symposium (Menghini et al., 2008), in particular, working groups 1 (Disciplinary mathematics and school mathematics) and 2 (The professional formation of mathematics teachers), local findings on mathematical understanding of NIE students will contribute towards targeted and specifically designed enhancements which are based on professional evidence-based and informed judgement. The author conjectures that enhancing the *Mathematical Vitality* of undergraduates is possible but requires a concerted, conscious, and consistent endeavour on the part of faculty members to include MV notions in a large majority of mathematics courses through agreed approaches. The results of this study form the starting point for just such an endeavour, and the fact that mathematicians and mathematics educators work together in the same department at the NIE holds promise for successful collaboration.

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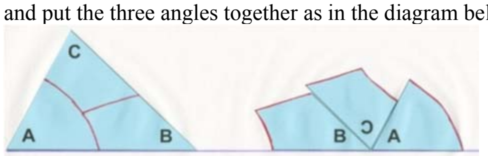
Appendix: Mathematical Vitality Test Items

This test comprises 12 multiple choice questions. For each question, select only one answer (A, B, C, D or E). You are encouraged to explain your answer in the space provided.

No	Question	What is being tested
1	Which of the following statements is correct? (A) If $x = 4$, then $x^2 = 16$. (B) If $x^2 = 16$, then $x = 4$. (C) $x = 4$ if and only if $x^2 = 16$. (D) If $x \neq 4$, then $x^2 \neq 16$. (E) None of the above.	Understanding of the if-then logic in a simple example
2.	Which of the following statements is correct? (A) $\sqrt{9} = \pm 3$ (B) $\sqrt{9} = +3$ (C) $\sqrt{9} = -3$ (D) $\sqrt{9} = \|9\|$ (E) None of the above	Knowledge of basic symbol/terminology of square root sign
3	It has been proven that infinitely many triangles possess property H. *Statement S*: All triangles possess property H. Which of the following statements is *necessarily* correct? (A) *Statement S* is true but further proof is needed. (B) *Statement S* is true and no further proof is needed. (C) *Statement S* is more likely to be true than false. (D) *Statement S* is more likely to be false than true. (E) None of the above.	Having an infinite number of the object satisfying the statement has no bearing on the truth of the statement applied to all such objects.
4	The radius of a circle is k cm where k is a positive integer. Select whichever of the following you believe to be correct. (A) The circumference could be exactly 88 cm. (B) The circumference could be exactly 9 cm. (C) Either (A) or (B) could be correct. (D) Neither (A) nor (B) could be correct. (E) It is not possible to decide in favour of any of the above alternatives.	Knowledge that π is irrational and applying the fact that the product of a rational with an irrational must be irrational.

No	Question	What is being tested
5.	*Theorem X*: If three positive integers a, b and c satisfy condition P, then they satisfy condition Q. *Statement Y*: If three positive integers a, b and c do not satisfy condition Q, then they do not satisfy condition P. *Theorem X* has been proved. Which of the following statements is <u>*necessarily*</u> correct concerning *Statement Y*? (A) *Statement Y* is true and does not need further proof. (B) *Statement Y* is true but needs to be proved. (C) *Statement Y* is false and does not need disproof. (D) *Statement Y* is false but need disproof. (E) None of the above.	Understanding the relation of a statement's contrapositive to the original statement.
6	The following are two statements: *Statement X*: If a polygon has property P, then it has property Q. *Statement Y*: If a polygon has property Q, then it has property P. *Statement X* is a theorem in geometry which has been proved. Which of the following statements is <u>*necessarily*</u> correct concerning *Statement Y*? (A) *Statement Y* is true and does not need further proof (B) *Statement Y* is true but needs to be proved. (C) *Statement Y* is false and does not need disproof. (D) *Statement Y* is false but needs disproof. (E) It is not possible to determine whether *Statement Y* is true or false from the information given.	Understanding of the truth of a statement has no bearing on that of its converse.
7.	A statement *S* reads as follows: *S*: A whole number n is divisible by 6 if the sum of its digits is divisible by 6. Select whichever of the following you believe to be true. (A) The number 39 shows that *S* is false. (B) The number 42 shows that *S* is false. (C) The numbers 39 and 42 both show that *S* is false. (D) *S* is false but neither 39 nor 42 is adequate to disprove it. (E) *S* is true.	Ability to use counter examples appropriately to disprove an if-then statement.
8	We know that a given statement *S* is true if $y < 0$. Which of the following statements must be true? (A) If *S* is true, then $y \geq 0$. (B) If *S* is false, then $y \geq 0$. (C) If *S* is true, then $y < 0$. (D) If *S* is false, then $y < 0$. (E) None of the above.	Ability to apply the relationship between a statement and its contrapositive.

No	Question	What is being tested
9	Result: For $a > 0$ and $b > 0$, $\frac{1}{2}(a+b) \geq \sqrt{ab}$. Proof: Step 1: If $\frac{1}{2}(a+b) \geq \sqrt{ab}$, then $(a + b)^2 \geq 4ab$. Step 2: Hence, $a^2 - 2ab + b^2 \geq 0$. i.e. $(a - b)^2 \geq 0$. Step 3: Since $(a - b)^2 \geq 0$ is always true, $\frac{1}{2}(a+b) \geq \sqrt{ab}$. Do you think the proof is valid? Why? (A) Yes, the proof is valid. (B) No, Step 1 is wrong because we cannot square both sides of an inequality. (C) No, the proof should not begin with what is to be proved and lead to a correct result. (D) No, for other reasons. (Please give/explain your reason(s).)	Ability to recognise the error of proving the converse instead.
10	Result: If $x + 4$ is an odd integer, then x is an odd integer. Proof: Given that $x + 4$ is an odd integer, assume that x is even. Let $x = 2k$ where k is an integer. Then $x + 4 = 2k + 4 = 2(k + 2)$ which is even, since $(k + 2)$ is an integer. Since this contradicts the fact that $x + 4$ is odd, the assumption is wrong and hence x is odd. Which of the following statements is correct? (A) The proof is valid. (B) The proof is wrong because it begins with a wrong assumption. (C) The proof is wrong because we are actually proving that if x is even, then $x + 4$ is even. (D) The proof is wrong but for other reasons. (Please give/explain your reason(s).)	Recognising a proof by contradiction and its validity.

<table>
<tr><th>No</th><th>Question</th><th>What is being tested</th></tr>
<tr><td>11</td><td><u>Result</u>:
If m is a factor of n and n is a factor of k, then m is a factor of k.
<u>Proof 1:</u>
Consider the numbers: 4, 8 and 24.
4 is a factor of 8 since 4 x 2 = 8.
8 is a factor of 24 since 8 x 3 = 24.
Hence 24 = 4 x 2 x 3. So 4 is a factor of 24.
<u>Proof 2</u>:
3 is a factor of 6 and 6 is a factor of 24. We see that 3 is a factor of 24.
3 is not a factor of 5 and 5 is not a factor of 7. We see that 3 is not a factor of 7.
Which of the following statements is correct?
(A) Proof 1 only shows one case. Proof 2 is valid because it shows both factors and non-factors.
(B) Proof 2 is wrong because the result is not about non-factors. Proof 1 is correct because it deals with numbers which meet the conditions.
(C) Both proofs are valid, they just use different numbers.
(D) Both proofs are not valid because we need to show that the result is true for all numbers m, n and k.
(E) None of the above.</td><td>Recognising that numerical examples are not proofs.</td></tr>
<tr><td>12.</td><td><u>Result</u>: The sum of the measures of the interior angles of a triangle is 180º.
<u>Proof</u>:
Forty pupils of a class each drew a triangle. Some drew acute-angled triangles, some drew obtuse triangles and others drew right-angled triangles. They cut out their triangles, tore them up and put the three angles together as in the diagram below.


Since the three angles lay on a straight line, they add up to 180º.
Is the above a valid proof for the result?
(A) No, diagrams cannot be used in proofs.
(B) No, because only 40 triangles were used and we must prove for all triangles.
(C) No, because there are no mathematical statements in the proof.
(D) Yes, because different types of triangles were used to verify the result.</td><td>Recognising that a good range of examples does not constitute a valid mathematical proof.</td></tr>
</table>