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A Conceptual Framework for Investigating Pupils' Model Development During the Mathematical Modelling Process

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Abstract: From the perspective of modelling as problem solving, the nature of problem solving experiences is reassessed where students' interpretation of real-world contexts leading to sense-making and meaning construction becomes their solutions to the problem situations. This paper discusses the design of a conceptual framework for investigating pupils' mathematical modelling process in a problem-based learning setting and reports findings from a case study involving a group of Grade 6 pupils. The framework provides affordances for tracing the modelling process pupils undertake towards exemplifying the key areas of the framework. Through a protocol analysis method, the collaborative discourse of model development of one group of the pupils was summarized as a modelling path. The benefits of the framework are discussed.

Keywords: Mathematical modelling; Problem-based learning; Conceptual framework; Model development

Introduction

Mathematical modelling is gradually gaining prominence in the field of mathematics education. Since 1983, the International Community of Teachers of Mathematical Modelling and Applications (ICTMA) has been promoting the development of mathematical modelling and applications through its conferences. Recent publications such as the *Second Handbook of Research on Mathematics Teaching and Learning* (Lester, 2007), *Modeling Students' Mathematical Modeling Competencies* (Lesh, Galbraith, Haines & Hurford, 2010) and papers from the 14th International Commission on Mathematical Instruction Study (Blum, Galbraith, Henn, & Niss, 2007) testify to the growth and contribution that mathematical modelling has made towards understanding this field of study. As a forward looking future-oriented mathematics education curriculum, some researchers are proposing mathematical modelling as an approach for students to draw on knowledge within and outside the mathematics classroom to solve problems (English & Sriraman, 2010). Some educators have regarded mathematical modelling as the most significant goal of mathematics education (Lesh & Sriraman, 2005) and the new direction in mathematical problem solving research (Lesh & Zawojewski, 2007).

Mathematical modelling has been variously defined and it also takes on differing perspectives. According to Barbosa (2006), different theoretical frameworks and research agendas concerning mathematical modelling stem from different mathematical modelling perspectives. For example, Julie (2002) distinguished "modelling as content" (emphasis placed on the development of competencies to model real world situations) from "modelling as a vehicle" (modelling as a way to teach mathematical concepts). Kaiser and Sriraman (2006) saw modelling as having a pragmatic perspective (apply mathematics to solve practical problems), a scientific-humanistic perspective (for learners to create relations between mathematics and reality), an emancipatory perspective (to develop socio-critical attempts of mathematics teaching) and an integrative perspective (mathematical modelling having different aims to serve scientific, mathematical and pragmatic purposes harmoniously). Lesh and Doerr (2003) put forth a models-and-modelling perspective that sees the modelling endeavour as comprising cycles of expressing, testing, and revising of students' mathematical thinking through the mathematization. The models-and-modelling perspective also embraces the notion of mathematical modelling as problem-solving (Lesh & Zawojewski, 2007; Mousoulides, Christou & Sriraman, 2008). These perspectives show the diversity of research according to their central aims in relation to modelling (Kaiser & Sriraman, 2006). Despite the growth in research, researchers have also been lamenting the lack of research that focuses on students' activities in modelling (BorromeoFerri, 2006; Lehrer & Schauble, 2003).

The purpose of this paper is to describe an emergent conceptual framework for investigating Primary 6 pupils' mathematical modelling process in a problem-based learning (PBL) setting and reports the major findings with respect to the modelling stages and the models that the pupils have developed. The design of the conceptual framework was drawn from a body of research in the field of problem-based learning and mathematical modelling. The models-and-modelling perspective (Lesh & Doerr, 2003) was adopted to investigate pupils' mathematical thinking and thus their construction of models. An interpretive framework based on the protocol analysis method was subsumed under the conceptual framework to aid in the corroboration of a list of problem-solving behaviours and the interpretation of the students' model development.

Theoretical Background

In this paper, mathematical modelling is situated in a problem-based learning (PBL) setting context. It suffices, at this juncture, to deliberate on the theoretical

background with respect to these constructs in the design of a conceptual framework in guiding this research.

Mathematical modelling as problem solving and the models-and-modelling perspective

According to Lester and Kehle (2003), adopting a modelling perspective to problem solving seems a plausible way to address the inadequacies of past problem-solving efforts which were deemed as not producing the desired outcomes ever since mathematical problem solving first became central to school curricula from the 1980s. The new emphasis is to make mathematical modelling as a focus for undertaking rather than a new initiative. Thus, in viewing mathematical modelling as problem solving, it shifts the focus from the result to the solving process and from the calculations to the relationships among the problem variables (Doerr & English, 2003; Lesh & Doerr, 2003). It also shifts focus towards the flexible use of mathematical ideas and exercise of representational fluency (Chan, 2009).

In this study, the models-and-modelling perspective (MMP) is adopted. The MMP emphasizes the interpretation and communication aspects of understanding and procedural capabilities in problem solving (Lesh & Doerr, 2003). Based on this perspective, students develop models (internal conceptual models) that are powerful but are under-utilized unless they are expressed externally using spoken language, written symbols, concrete materials, diagrams, pictures, experience-based metaphors, or other representational media through solving the problem (hence the modelling activity is also commonly known as "model-eliciting activity").When students are solving non-trivial modelling problems, these representations (expressed models) are continually being projected into the external world signifying the *mathematizing of reality* as contrasted with *realizing mathematics*. Through these forms of expression, students develop ideas for the constructs and processes needed to solve the problem and the process comprises several express-test-revise cycles that help students' ways of thinking to evolve beyond current conceptions. Thus, solving problem results in the students making sense of the situation towards mathematizing it in ways that are meaningful to them.

Situating mathematical modelling in a problem-based learning instructional approach

Problem-based learning (PBL) as an instructional model had its origin in the medical discipline (Barrows & Tamblyn, 1980). It was gradually adopted for use in elementary and high schools (Delisle, 1997). PBL is seen as experiential learning organized around the investigation of messy, real-world problems (Torp & Sage, 2002). Three of the defining characteristics of PBL include:

(a) Learning being driven by confronting challenging, complex, open-ended problems that are considered authentic or simulate real-world situations. The problem-task, being ill-structured, suggests that not all the information is given. As the pedagogy is problem-driven, unlike traditional problem solving, the problem does not necessarily have a single correct answer (Hmelo-Silver, 2004). The openness of PBL suggests that students can manage the problem and yield various plausible solutions in the light of their decision making.

(b) Students work in small collaborative groups. This enables students to co-construct knowledge when they engage in a dialogic process of meaning making mediated through language (Vygotsky, 1978). The collaboration also enhances self and peer assessment of thinking as they work towards productive discussion, thereby leading to meaningful knowledge construction (Chin, Brown & Bruce, 2002).

(c) Teachers act as facilitators in the process of learning. Unlike a didactic instructional approach which is mainly based on teacher exposition and seen as the prescription of knowledge, teachers in the PBL setting helps to scaffold and support the students' learning. They help to ensure that key areas to be learnt are not overlooked and to promote reflective thinking, critical evaluation, and inventive thinking (Tan, 2003).

Research in PBL in the field of mathematics education internationally is scarce although the practice is encouraged as reform efforts in mathematics classrooms (Erickson, 1999; Hiebert et al., 1996). With the use of mathematical modelling tasks in a PBL setting, the mathematical modelling process exemplifies the features of what befits problem-based learning as the iterativeness of the process depicts the constant interaction between task, students, and teacher. Various representations of the modelling process found in literature (e.g., Ang, 2001; Galbraith, Goos, Renshaw, & Geiger, 2003; Penrose, 1978) have identified certain key stages in mathematical modelling. Despite the variations, the common stance is that mathematical modelling is a process that has reality (or a real-world situation) as the starting point for students to represent the problem or situation in mathematical terms. The mathematical solutions are then used to interpret the real-world problem. The process is often viewed as a cycle of stages where students need to understand and simplify the problem from the real-world situation, manipulate the problem and develop a mathematical model towards interpreting the problem solution, and verifying and validating the problem solution. The outcome at the end of the modelling process is a successful or improved solution, or the decision to revisit the model to achieve better outcome (Galbraith & Stillman, 2006). In a sense, mathematical modelling and the PBL instructional approach can be seen as

compatible pedagogies in surfacing students' mathematical thinking (Chamberlin & Moon, 2008; Hjalmarson & Diefes-Dux, 2008). Both the modelling perspective and PBL pedagogic approach are consistent with the theory of situated cognition which argues that knowledge building is linked to the conditions present in the environment and the knowledge that one brings to the modelling is part of an interacting system that is inclusive of individuals, relationships, tasks, tools, goals, teaching methods, and other contributing factors. In other words, knowledge is seen as a product of the activity, context, and culture in which it is developed and used (Brown, Collins, & Duguid, 1989). Situations, therefore, are deemed as co-producers of knowledge through activity, consequently suggesting that learning and cognition are fundamentally situated.

Problem-solving and mathematical modelling frameworks to understand students' problem-solving actions

To understand the cognitive activities that students are engaged in during problem solving, researchers (Artzt & Armour-Thomas, 1992; Calson & Bloom, 2005; Foong, 1993; Pape & Wang, 2003; Schoenfeld, 1985) have developed various taxonomies of problem-solving behaviours to give meanings to students' mathematical thinking during problem solving. These taxonomies are used as frameworks to analyze problem solving actions in managing the problem, solving the problem and/or in decision-making, and they include the students' manifestation of domain-specific knowledge, mathematical reasoning, critical thinking, and metacognitive thinking and even behaviours in the affective domain. How these behaviours are interpreted and analyzed depends on the research goal. Schoenfeld's (1985) problem-solving framework classified behaviours according to episodes such as read, analyze, explore, plan, implement, and verify to analyze cognitive-metacognitive behaviours. Through the analysis, he was able to infer students' application of a variety of problem-solving strategies, and that explicit heuristics instructions could make a difference to problem-solving performance. He also found that focusing on understanding and analysis could significantly impact perception and performance while emphasizing metacognitive behaviours could impact students' control behaviours. Artzt and Armour-Thomas (1992) expanded Schoenfeld's framework for small group problem solving to delineate the roles of cognitive and metacognitive processes. They reported that episodes need not occur in sequence. For example, students may bypass *planning* or *analysis* and become engaged in the *implementation* episode but hit an impasse which then would divert attention to *reading* and *understanding* and then later progress more sequentially through the episodes. Recently, Carlson and Bloom (2005) developed a multi-dimensional problem-solving framework that posited that students' problem solving comprises two cycles (plan-execute-check cycle, and conjecture-imagine-verify cycle), each of which comprising at least three of the four stages of *orientation*,

planning, *executing*, and *checking*. The findings through the use of the framework suggest that learning to become an effective problem solver requires the development and coordination of a vast spectrum of reasoning patterns, knowledge, and behaviours, as well as the effective management of resources and emotional responses during problem solving, while not discounting practice and experience.

Where mathematical modelling is concerned, Galbraith and Stillman's (2006) framework for identifying student blockages in transitions between modelling stages is applied to help the teacher know where they might help students overcome those blockages. It also identifies activities that modellers need to have competence in order to be successful in applying mathematics. They assert that an understanding of the blockages and competencies contributes to the planning of teaching towards the identification of the necessary pre-requisite knowledge and skills as preparation for intervention at significant junctures of the modelling process.

Towards a conceptual framework in investigating the Mathematical modelling process

Research has to be guided by a basic structure comprising ideas of abstractions and relationships that serve as the basis for the phenomenon that is to be investigated (Lester, 2010). Establishing a framework provides a structure for conceptualizing the way concepts, constructs, and processes of the research are defined. In investigating pupils' mathematical thinking and development of models as the research goal, a conceptual framework was developed by the authors to support the "argument that the concepts chosen for the investigation, and any anticipated relationships among them, will be appropriate and useful given the research problem under investigation" (Lester, 2010, p. 10). In other words, the conceptual framework would provide the notion of justification: justifying why we are doing what we are doing, and why our explanations and interpretations are reasonable.

Conceptual Framework

The conceptual framework for investigating pupils' mathematical modelling process in a PBL setting is shown in Figure 1 as a three-dimensional dynamic structure where the base is the foundation situated in the PBL instructional setting and the spiralling tiers as the modelling stages. The premise for the PBL instructional approach is considered a short-cycle approach as it does not require pupils to do a great amount of information mining or be involved in achieving multiple goals outside the classroom (Tan, 2003) since the participants are young children.

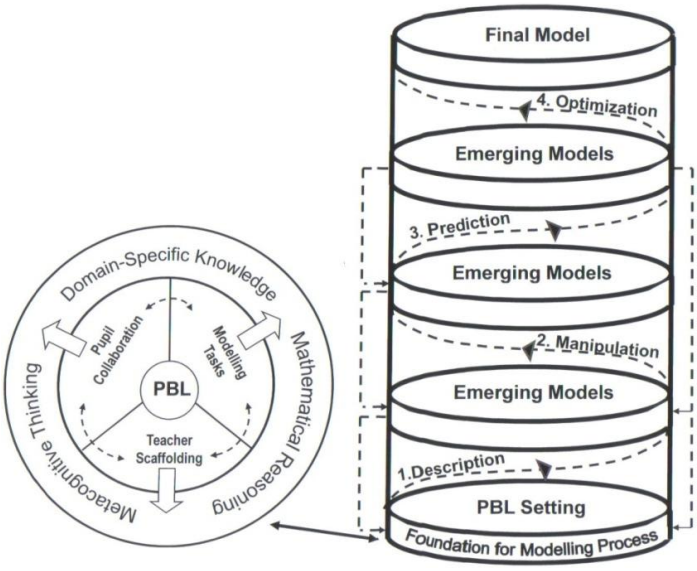


Figure 1. Conceptual Framework for Mathematical Modelling in a PBL Setting.

The PBL setting is situated in realistic and meaningful contexts where learning is in the doing when pupils interact with the environment and where knowledge is situated as being part of the activity and context. The two-dimensional concentric-circles diagram on the left unpacks the PBL setting. PBL, being the core instructional approach, is situated at the centre and is bounded by its three key tenets in the secondary ring: the modelling task, the pupil collaboration, and the teacher-scaffolding. The interaction of these tenets suggests that active mathematical problem-solving is going on. When pupils work on a modelling task, the interaction between the pupils (pupil-pupil interaction) in relation to the modelling task (pupil-task interaction) as well as the interaction with the teacher (teacher-pupil interaction) in relation to the task, generates discussion around problem interpretation, variables, and strategies towards solving the problem. The pupils' cognitive processing that occurs throughout the interaction is manifested as the pupils' mathematical problem-solving behaviours. These interactions are illustrated by the dotted arrows around pupil collaboration, modelling task, and teacher-scaffolding. The explicit interaction during the pupil collaboration should not be viewed as a mere aid to thinking but as almost tantamount to the thinking itself (Sfard, 2001).

While active problem solving is going on, the pupils' mathematical problem-solving behaviours are said to have become visible and are classed as *domain-specific knowledge* (K), *mathematical reasoning* (R), and *metacognitive thinking* (M) behaviours, as seen in the next outer ring, indicative of the resultant interaction of

the PBL tenets (shown as resultant outward pointing arrows from the PBL tenets). The framework highlights that the engagement of the modelling task would require the pupils to think deeply as they collaborate with one another and with the teacher to solve the problem. The situative context requires the pupils to communicate the mathematics that they know or have learned through some form of *mathematical reasoning* (interpreting, explaining mathematical steps, conjecturing, qualifying, etc.) or perform some arithmetic or procedural operations deemed as the exercise of *domain-specific knowledge*. *Metacognitive thinking* is exercised through the display of some form of awareness and monitoring of oneself, one another, the problem-solving situation, and even their personal knowledge within their personal experiences to guide the cognitive process. The interplay of these problem-solving behaviours results in the development of emerging conceptual models. These problem-solving behaviours can be captured via a problem-solving coding scheme.

The development of models during the modelling process takes place within modelling stages. Figure 1 captures this process as comprising the modelling stages *Description*, *Manipulation*, *Prediction*, and *Optimization*. *Description* refers to attempts at understanding the problem to simplifying it through explaining it. This includes drawing inferences from text, diagrams, formulas, or whatever data are given to make sense of the task details. Pupils also make assumptions from personal knowledge to simplify the problem. *Manipulation* refers to attempts at establishing relationships between variables and task details through constructing hypotheses, critically examining contextual information, retrieving or organizing information, mathematizing, or using strategies towards developing a mathematical model. Most of the mathematical computations and reasoning take place in this stage. *Prediction* refers to attempts at interpreting the models that they have conceived to ensure that they fit the parameters given or established. This includes affirming, verifying, or making decisions to justify their claim that they have attained a workable model. A workable (functional) model is usually attained after several revisions made to the emerging model as meeting the task requirements. *Optimization* refers to attempts at improving or extending their model solutions to achieve an ideal solution that is quantity efficient and yet is of maximizing value. Optimization could only be achieved when the preceding stage has been achieved, that is, a workable model has to be attained.

Although different in emphasis, the stages are characterized by the dynamic flows spiralling upwards where within each stage pupils are in the process of developing emerging models until the final model. It is posited that more prominent models surfaced while pupils are in the *Manipulation*, *Prediction*, and *Optimization* stages as pupils make mathematical relationships of the models. Models are usually under-developed in the *Description* stage. The spiralling concept implies a building up or

an expansion and evolution of ideas through the interaction of the PBL tenets. Each emerging model assumes the thinking becoming more complex as mathematical components are being worked upon thus increasing the refinement of the mathematical reasoning of specific situations. The dynamic nature allows the tracking back to previous stages, and whenever the need arises, to check back or stabilize previous models. The interplay of the problem-solving behaviours enables us to gain an understanding of the development and refinement of the models that the pupils construct within and across modelling stages. A more comprehensive account of the interpretive framework used to exemplify the conceptual framework is discussed in the Method section.

Method

This study adopted an exploratory, mixed methods design (Creswell, 2008). The purpose of this design was to gather qualitative data to explore a phenomenon and then collect quantitative data to support the qualitative findings. This research was strategically planned to be conducted over two phases. Phase 1 concerned the design of the conceptual framework leading to the development of tools such as the problem-solving coding scheme and the mathematical modelling interpretive framework. Phase 2 was the application of the tools towards answering the research questions. This paper focuses on Phase 1 of the study in the development of the conceptual and interpretative frameworks for use in Phase 2 of the study. In this paper, the model development of a group of pupils in Phase 1 is used to exemplify the conceptual and interpretive frameworks.

Participants, selection and data collection method

Two Primary 6 (grade 6) classes (one class of higher-ability pupils and one of mixed-ability pupils), one from each of two neighbourhood schools in Singapore were selected. As small group arrangements were central to PBL settings (Holen, 2000; Woods, 1994), the pupils were grouped (mixed gender) into four or five based on friendship to work collaboratively to solve a modelling task as there were positive benefits in such selection (Mahenthiran & Rouse, 2000; Webb & Palincsar, 1996). Two target groups were selected by their respective mathematics teachers for video recording for the purpose of providing rich data. All the other pupils were similarly involved in the modelling endeavour except that they were not video-recorded. The study took place in the school premise and all the pupils had not been involved in solving mathematical modelling problems prior to this study.

The modelling task

The Biggest Box problem (see Figure 2) was adapted and modified from Ang (2001) to be in line with the design principles of Lesh, Cramer, Doerr, Post, and Zawojewski, (2003). This modelling task was chosen as it was possible for pupils to develop models both descriptively and as an artifact. The pupils were given two 50 cm by 50 cm vanguard sheets, scissors, tapes, markers, rulers, and calculators.

> Your team is participating in a math project competition where you will need to present your findings in two days' time. In the competition, each team has been given only two square sheets made of vanguard. The team can decide if they want to use one vanguard sheet for trial. The team is supposed to make the biggest box (volume) using only ONE vanguard sheet. How would your team plan to solve the problem of making the biggest box? Show in detail how you reach a solution to convince the judge.
> (*Note that your box must have a base but need not have a cover*)

Figure 2. The Biggest Box Problem.

Development of problem-solving coding scheme

A preliminary taxonomy of problem-solving behaviours was developed as a coding scheme.

The main categories of behaviours were domain-specific knowledge (declarative and procedural knowledge), mathematical reasoning, and metacognitive thinking (showing awareness and critical thinking); see Figure 3. These were divided into more specific codes for the purpose of adapting the protocol analysis method (Ericsson & Simon, 1993) to capture the processes of solving a problem that corresponded to the sequence of problem states as well as for testing a theoretical model (Chi, 1997). The preliminary list of behaviours was applied to two groups of Primary 6 pupils who had worked on the modelling task.

A meet-in-the-middle approach (Meijer, Veenman, & van Houst-Wolters, 2006) was employed based on a top-down approach using the corroborated list of behaviours and a grounded approach to negotiate the codes towards changing them as well as adding new ones. The codes in Figure 3 show the revised list of problem-solving behaviours after several negotiations were carried out between coders based on their analysis of the two groups of pupils.

Domain-Specific Knowledge
K1 – Delineate mathematical components – state or mention mathematical components
K2 – Carry out computations – use routine mathematical procedures, algorithm, formulas
K3 – Incorrect use of procedures - incorrect use of mathematical procedures; attain incorrect answer

Mathematical Reasoning
R1 – Conceptualize representations of situations – formulate mathematical relationships in various representational forms
R2 – Make mathematical translations – use of mathematical reasoning or strategies
R3 – Conjecture / theorize – come up with a plausible theory
R4 – Generalize – reach a tentative conclusion or decision based on investigations
R5 – Flawed reasoning – show misconception in reasoning

Metacognitive Thinking
M1 – Situate the problem – attempt to understand or simplify parts or the problem
M2 – Monitor task facility – provide an assessment of the task as they perceive it
M3 – Check for understanding – find out about from others what one does not understand
M4 – Realize new information – use new information to revise own thinking
M5 – Realize one's own errors – discover own errors or realize when pointed out by others
M6 – Monitor / check general progress – desire to know the status or progress of the endeavour
M7 – Interpret and affirm / verify results - check and/or validate if problem has been answered
M8 – Impress personal knowledge – use personal knowledge / experience to contribute
C1 – Detect errors – raise an alert to errors or inconsistencies*
C2 – Challenge efficacy of ideas – show disagreement or question approach / method; offer counter arguments*
C3 – Propose unique conceptualizations – share novel ways to test ideas*

Others
RD – Read – read directly from task sheet
AK – Acknowledge – non-mathematical response to acknowledge / affirm what others say
PC – Promote collaboration – call for cooperation
EM – Emotional expression – emotional outburst or comments

* The C behaviours suggest critical thinking behaviours that are metacognitive as well.

Figure 3. Summarized Version of the Problem-solving Coding Scheme.

Development of the interpretive framework MMIF

The Mathematical Modelling Interpretive Framework (MMIF) takes the form of a horizontal protocol line representation, and it was developed to make sense of the coded protocols. The macro view is a time-line representation similar to

Schoenfeld's (1985) study of mathematical problem solving. The micro view displays the data that have been reduced into individual codes and as parsed episodes within modelling stages. The parsing of the protocols into the various stages of *Description*, *Manipulation*, *Prediction*, and *Optimization* provides an overview of the modelling pathways the pupils have gone through. The parsing of the protocols within stages into episodes provides a picture of the dominant problem-solving actions taking place within the particular stage. As the entire coded representation spans several pages, Figure 4 shows only a small part of it. The MMIF is then transformed diagrammatically into a Mathematical Modelling Pathway chart (see Figure 7) to depict and characterize the entire mathematical modelling process.

PL	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83
S1																				
S2		K2		M3			AK		C2 R1					C2 R1						
S3						C2				M6					C2 R1		R2	M1		
S4			K2					R1				K2							K2	
S5	R5				R1 R5						K2		R5			M5				M6
T																				
	PL				PL	DI								DI	DI	NL			TR	

Conceptualizing representations Carrying out computations

Figure 4. Mathematical Modelling Interpretive Framework (MMIF).

Figure 4 shows the protocol lines (PL) numbered sequentially as the first row. Subsequent rows show the codes for pupils S1 to S5 for these PLs. T is the teacher who has another set of codes. The codes in the last two rows are for another study and will not be discussed here. The mathematical modelling episodes are printed below the matrix.

Results

This section outlines the model development of only one group of five pupils. This group was selected as it provides a rich display of problem-solving behaviours for the purpose of exemplifying the conceptual and interpretive frameworks. The tracing of pupils' model development in the mathematical modelling process culminates in the production of the mathematical modelling pathway chart (see Figure 7) and the model development summary chart (see Figure 8).

The mathematical modelling process

The video transcriptions of the pupils solving the Biggest Box Problem were used alongside the viewing of the video to get a strong sense of the data. By adapting Schoenfeld's (1985) episode parsing method, the modelling stages were represented in a schematic timeline diagram as shown in Figure 5.

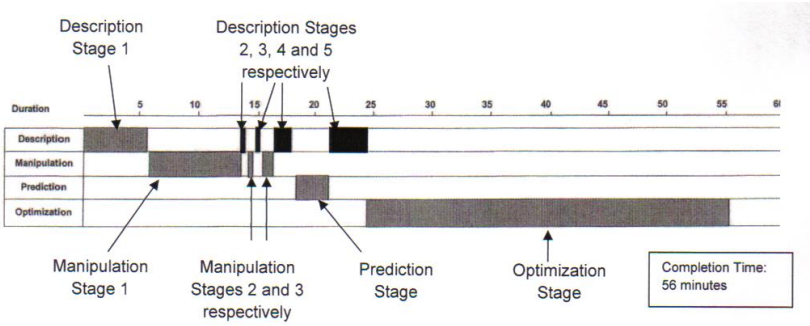


Figure 5. Time-line Representation of Modelling Stages.

The time-line representation enabled the group's modelling pathways to be traced. From beginning to end, the group took 56 minutes to complete the task. The corresponding mathematical modelling pathway is D(1) → M(1) → D(2) → M(2) → D(3) → M(3) → D(4) → P → D(5) → O where D, M, P, and O are the *Description*, *Manipulation*, *Prediction*, and *Optimization* stages respectively; the numbers in brackets are the successive numbers of the respective modelling stages. It can be seen that the modelling paths were not linear as the pupils went back and forth between *Description* and *Manipulation* on several occasions.

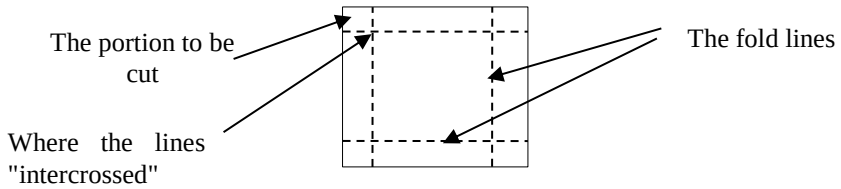
During *Description* stage 1, the pupils tried to understand and simplify the problem and describe what they know about the problem through assessing task properties by identifying or surfacing the mathematics components. They formulated their goals and provided a surface level account of how to manage the problem situation. Usually no obvious models were observed at this stage as they were deemed to be underdeveloped. The problem-solving behaviours were characterized mainly by K1 and some K2 and R1 behaviours when the pupils discussed length, possible shapes, volumes, and dimensions. Some of the metacognitive thinking behaviours include M1 and M3 in trying to situate the problem and check for understanding. An excerpt of *Description* stage 1 is shown below:

1	S2	What is the length of this? (K1)
2	S4	Find the length first. (K1)
3	S2	Who got a longer ruler? (*S2 measures the side of the vanguard sheet with his ruler*) (PC)
4	S4	All the rulers combined. All the rulers give you a bigger stretch. (R1)
		...
9	S3	OK. Our goal. What is our goal? (M3)
10	S4	Our goal. To make the box with the biggest volume. (M1, K1)
		...
32	S3	But what is the shape of the box? // Because 50, 50, 50. Would you like it to be shallow or…? (M3, R5)
33	T	That depends on your working and calculation. If it is a three dimensional thing, then what do you think? Three dimensional, what do you normally do? (El)
34	S1	You just fold it here, then you can fold up, it becomes a box already. (*talking about getting the net)* (R1)
35	S3	Depends on what shape you want. (K1)
36	T	Please be louder. (Su)
37	S5	Find breadth times length time height. (K2)

There are three iterative cycles of "*Description* → *Manipulation* → *Description*" from the start-up before the pupils entered the *Prediction* stage in the 18th minute. Revisiting the *Description* stages suggests that the pupils needed to understand or deliberate more deeply about the task details, and this could come about due to the teacher's interjection that required the pupils to go back. *Description* stages 2, 3, 4, and 5 are highlighted in Figure 5. The pupils' participation in these stages were *not* eventful: they were trying to think about whether there were other shapes apart from what they had been discussing (a square box). At *Description* stage 2, the teacher interjected and asked if the box had to be a square, and that led the pupils to abruptly leave their manipulation of data to think about boxes that could take other shapes. They talked about pentagonal-shaped boxes, triangular-shaped boxes, prism, and cylinder, but each time, the discussion did not move forward as they did not delve into how to get the net of these boxes or how to find their volumes. At every case, a pupil had to remind the group to return to working with a square box so that they could complete the task based on what they knew (*Manipulation* stages).

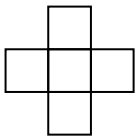
During *Manipulation* stage 1, the pupils manipulated the data more deeply. In the seven minutes of this stage, they used mathematical reasoning to produce a mental representation of the box to determine what could be done with the square paper. The excerpts below exemplify the emerging models that were continually revised in their discussion.

44	S5	If you fold, then some of the paper is not used yet. (R1)
45	S2	Then you don't fold, just cut *(to dispense with folding and resort to cutting).* (R2)
46	S5	If you fold, then it will overlap. (C2)
47	S2	You draw line from here, then you find the length, then when the thing inter-crossed, then you cut until there (*demonstrates with his fingers tracing the part to be cut on the butcher paper*). (R2)



The pupils deliberated about making folds to the vanguard sheet to construct a box. They cautioned that some parts of the paper would not be used or would overlap. They thought about cutting to enable the folds to be possible to avoid making overlaps. This was a tentative conceptual representation or an emerging model. This initial model led to further conceptualization of the net of the box with dimensions 50 cm by 50 cm by 50 cm as shown in the excerpt below.

54	S5	50, 50, 50 (R5)
55	S4	No. Cannot be 50, 50. Then the height is zero. 50 times 50 then the height is zero. (C2, R1)
…	…	…
58	S5	No, wait. You find the area of the vanguard sheet first. (K1, PC)
…	…	…
64	S5	Divide by 5. 2500 divide by 5, how much? 2500 divide by 5. (R5)
65	S2	2500 (*repeating aloud as he punches the calculator*)…500. (K2)
66	S5	500 square root *(to get the length of the square side)*. (K2)
67	S2	500 square root is what? (M3)
68	S5	Now we're finding area, right? So 5 parts. 5 pieces because the cover we don't need. // So the area is 500. One side is 500. (R1, R5)



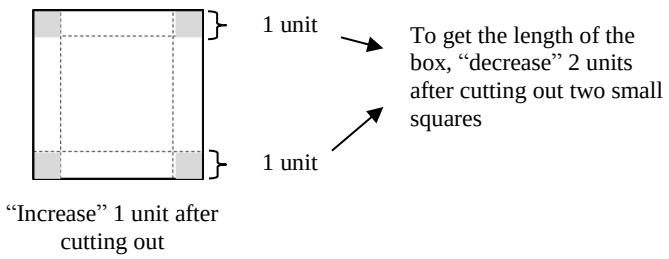
Net of a box with length, breadth and height 50 cm each

The conceptualizing of initial model was reframed by S5 but this emerging model was actually flawed because given the dimensions of the square vanguard sheet to be 50 cm by 50 cm, the box could never be 50 cm by 50 cm by 50 cm. It is interesting to note that the teammates were critically aware that the reasoning brought up was flawed and the representation was challenged by S4 in 55. S5 was quite persistent in wanting to bring his model across as he did not realize his reasoning was flawed when he suggested that the area of each face would be 500 square metres. In the excerpt below, the other teammates tried to convince S5 that they needed to cut out some parts and hence his reasoning of 2500 divided by 5 would not stand. S5 finally acknowledged his misconception (see line 79).

69	S3	But it's not a cube. It's a box leh. (C2, R1)
70	S2	Yeah, it's a box. (AK)
71	S4	A box can be a square *{a box can have square faces}*. (R1)
72	S2	Some more we're folding it. We cut already then we're folding it *(trying to impress that S4's reasoning may not make sense).* (C2, R1)
…	…	…
76	S5	1, 2, 3, 4, 5. 2500 divide by 5. *(gestures with his hands again to show the net of five equal square faces)* (R5)
77	S2	No, cannot because we're cutting some away *(trying to impress that some parts had to be cut away)*. (C2, R1)
78	S3	No, it's the biggest box. It's not equal area *(trying to impress that the biggest box is not obtained from getting squares of equal areas)*. (C2, R1)
79	S5	(…) Oh yeah hor. (M5)

When the pupils were convinced that that was how the box could be constructed, they went about testing the various dimensions. They started off with 22 cm × 22 cm × 22 cm but soon came up with a firm conjecture instead of random ones. They realized that every time they cut away two small squares at the end of the vanguard sheet, the length of the cut-out was the height of the box (see excerpt below).

102	S2	So every time the length decrease by 2, the height will increase by what? (R3)
103	S4	Increase or decrease? (M3)
104	S2	Increase by 1. (R4)
105	S1	This one is 30, 30, (…), 10? (R1)



Across the various *Manipulation* stages, the pupils tested the dimension-combinations of using length × width × height but with a narrower range after their realization that the length of the square cut-out was the height of the box. During *Manipulation* stage 2, they even conjectured that the volume would peak before becoming smaller again when S2 said "*Once you reached a certain place, then it will stop, then it will start to decrease*". Eventually, in *Manipulation* stage 3, they concluded that 34 cm by 34 cm by 8 cm would give them the biggest volume. While the *Manipulation* stage involved the pupils working with the data that they had generated, there was active mathematization as evident from the dominant display of mathematical reasoning behaviours (R types). Misconceptions and conjectures surfaced as the pupils communicated their ideas mathematically, manifesting critical thinking (C types) and generic metacognitive behaviours (M types).

During the *Prediction* stage, the pupils affirmed their findings with one another and with the teacher when they were satisfied that they had obtained the biggest volume.

201	S3	Length 34, breadth 34, height 8. (K2)
202	S2	Length should be more, right? *(making a status check if the length quoted is correct)* (M6)
203	S3	34, 34, 8. (R1)
204	S2	For square, it is the most *(reaffirming that with the set of dimensions, they had obtained the biggest box)*. (R4, M7)
…	…	…
211	T	What is your result so far? (El)
212	S3	34, 34, 8. (M7)
213	S2	Highest for square base is 9248 cm^3. (M7)

The box with dimensions 34 cm by 34 cm by 7 cm was a workable model, but it might not be an "optimized" one as it was possible to increase the volume even further. In the *Prediction* stage, R4 (generalizing) behaviour was found to be dominant as the pupils reached a tentative solution and M7 (affirmation) was usually seen to be part of the verification of the solution.

The pupils entered the *Optimization* stage when they made improvements to their workable model to get even bigger volumes. The teacher played a part in scaffolding the pupils to realize that they could increase the volume further if they had considered numbers other than whole numbers; the pupils then realized they could consider decimals.

262	T	And what makes you think that in the number system, you have only whole numbers? In the number system, other than whole numbers, what do we have?
263	S5	Decimals.
264	T	Decimals. Are those numbers workable? Will those numbers offer you the bigger numbers (volume)?

The scaffolding enabled the pupils to scrutinize their solution further. As they used decimal fractions from the narrowed range of dimensions, they obtained marginally bigger volumes. They finally obtained the dimensions 33.3 cm by 33.3 cm by 8.35 cm as the optimal combination. As they were challenged further by the teacher again, they extended their model through re-using the square cut-outs so that there would be no wastage. The excerpt below shows the teacher trying to extend the pupils' thinking.

447	T	What if you don't sacrifice (the square cut-outs)? Everything counts?
484	T	By the way, the strategy is not to waste paper.
496	S2	Cut one square into four.
498	S2	Use the spare parts to patch it up…Hey, can...You can cut one square into four then you paste on top.

The attainment of the "optimized" model could be better explained by using the sequence of diagrams shown in Figure 6. Although there was a difference of 0.1 cm between the length of the strip and the box, this was not deliberated upon by the group. In building the box, the vanguard sheet was flimsy enough to make a box without much notice to the difference of 0.1cm. Even so, the 8.35 cm square cut-outs most probably were based on 8.3 cm or 8.4 cm since the ruler could not give exact markings up to two decimal places. The refining process did show a dominant display of making mathematical computations (K2) and reasoning (R1) behaviours through delineating the mathematical components. Checking with one another on the progress (M6) was a metacognitive process prevalent in the *Optimization* stage.

The mathematical modelling pathway chart

In the light of tracing the model development of the pupils through the interpretive framework, the entire modelling process can be summarized as a mathematical modelling pathway chart shown in Figure 7. This chart can be used as a working framework to report the overview of pupils' mathematical modelling endeavour.

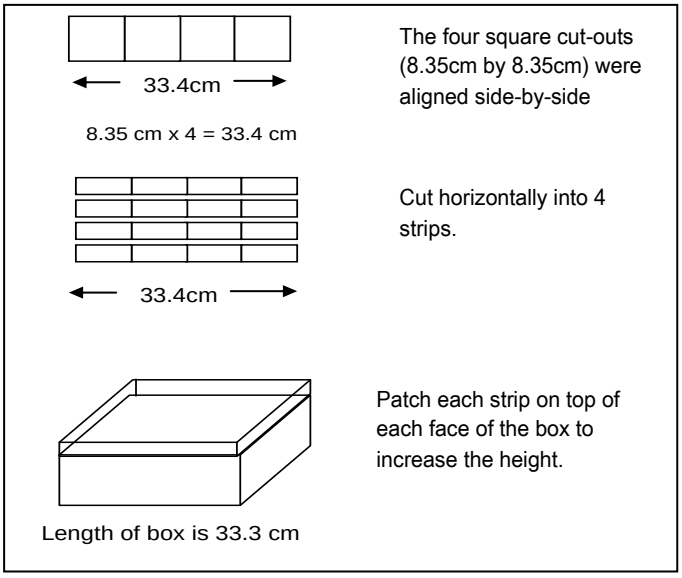


Figure 6. Enhancements made to get the Biggest Box.

The modelling pathway chart can be seen as an expanded version of the time-line representation of Figure 5. It affirms and enacts the conceptual framework in that when pupils solve modelling problems in a PBL setting, they go through different modelling stages and the paths are iterative in certain instances rather than the entire process being linear. The modelling pathway chart also shows in greater detail the following:

(a) Within each modelling stage, the dominant episodes were evident (seen in the left columns of the boxes) to suggest the modelling actions the pupils were involved in. In the right columns of the boxes, the problem-solving behaviours displayed by the pupils within the particular modelling stage are shown in parentheses beside the problem-solving behaviours. The codes and occurrences printed in bold suggest the most dominant behaviour exhibited for each category of domain-specific knowledge, mathematical reasoning, and metacognitive thinking. It also shows that when pupils were engaged in mathematical modelling, they exercised a repertoire of problem-solving behaviours. The dominant episodes are validated by interpreting the protocols and the codes.

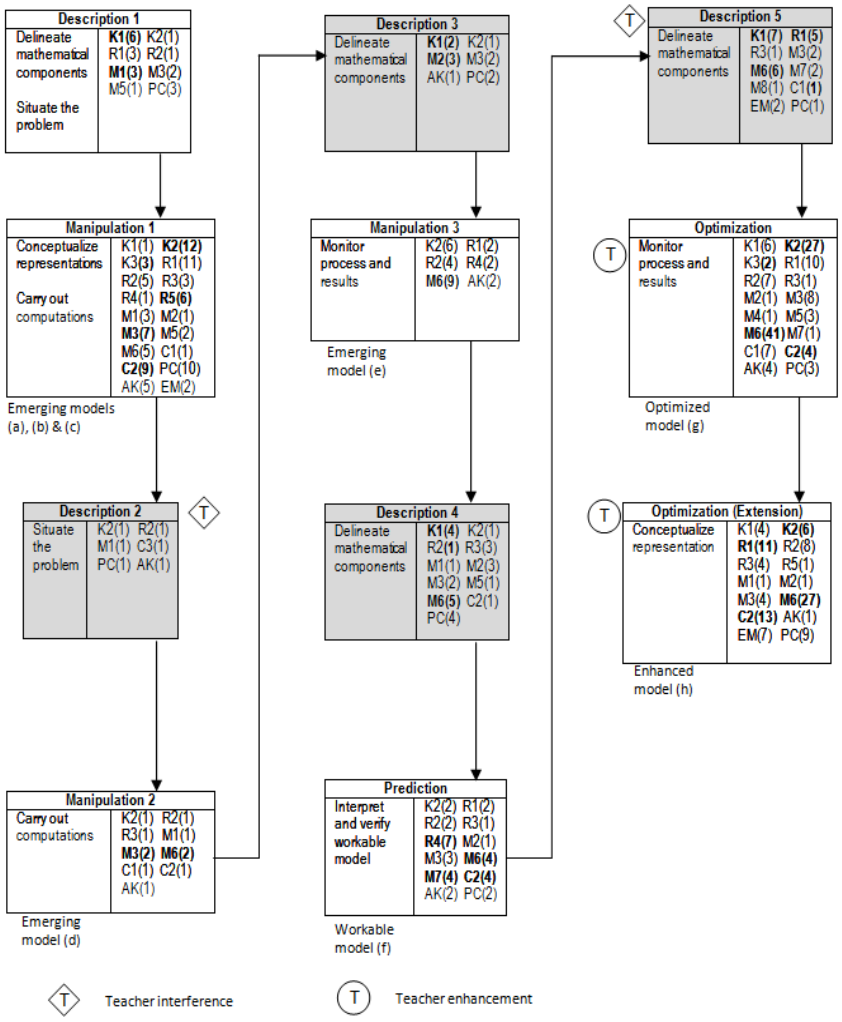


Figure 7. Mathematical Modelling Pathway Chart.

(b) As the pupils manipulated data, they began to mathematize, and this resulted in the development of models. The emerging models are represented beneath the boxes of the *Manipulation*, *Prediction*, and *Optimization* stages. Most of the models were emerging ones, that is, the conceptualized representations were

evolving towards having a more stable model. When the pupils reached the *Prediction* stage, they would have compared their models to affirm and decide that they had attained a workable model that satisfied the problem requirement. Pupils who made improvements to their workable models could enhance them further to get an improved or optimized one.

(c) Boxes that are shaded imply that the pupils' engagement during those stages is not eventful. While they tried to situate the problem or became involved in delineating mathematical components, they displayed several problem-solving behaviours, but the discussion may not advance towards acquiring new learning. For example, the pupils brought up the ideas of exploring cylinders, pentagonal-shaped boxes, triangular-shaped boxes, and prisms, but they did not work out the properties or volumes of these types of boxes. Sometimes, pupils' engagement in such discussion came from an abrupt interruption to their modelling process, for instance, when the teacher interjected and asked them questions that were unrelated to what they were currently engaged in. This led the pupils to pursue the teacher's agenda, and their current modelling endeavour had to be momentarily stopped. As shown in Figure 7, the rhombus shape labeled with a T is an instance of teacher interference that brought the pupils back into the *Description* stage towards discussing the shapes of boxes. The shaded boxes without the teacher's interjection show that the pupils continued with the discussion of boxes of other shapes, but they made no progress in that consideration.

(d) The teacher was found to advance pupils' thinking in the *Optimization* stage. It was through the teacher's assistance that the pupils managed to think further and improved their models.

Summary of model development

Figure 8 shows that the pupils' conceptualizations of emerging models grew in sophistication across the modelling stages. With the exception of the *Description* stages, the pupils' models developed from early notions of nets towards a list of dimension-combinations, and then refining the list to achieve their ideal dimension combination, the enhanced model. The pupils' development of models illustrates conceptualizing the models and making mathematical meanings through exercising mathematical reasoning and domain-specific knowledge. The evolution suggests that the pupils were involved in expressing, testing, and revising their models.

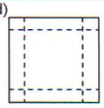
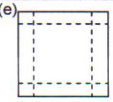
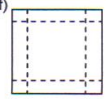
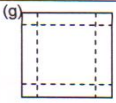
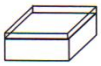
Modelling Stages	Model development (Developing conceptual representations)	Making mathematical relations through R and K behaviours
Manipulation Stage 1	(a)  Pupils conceptualized how a box could be made through folding the vanguard sheet	"You draw line from here...when the thing inter-crossed, then you cut until there" (R1)
	(b)  Pupils misconceived each face to be 50cm by 50cm.	"50, 50, 50" (R5) "1, 2, 3, 4, 5. 2500 divided by 5" (R5) "No. Cannot be 50,50. Then the height is zero". (R1)
	(c)   Pupils conceptualized dimension-combinations but not systematic.	Pupils conjectured that the length of the cut-out was the same as the height of the box. "30, 30, 10" (R1) "48, 48, 1" (R1)
Manipulation Stage 2	(d)  Pupils became more systematic by developing a dimension-combination tool to narrow the dimension range. Conjectured the volume would peak and then decrease.	"Once you reach a certain place, then it will stop, then it will start to decrease" (R3)
Manipulation Stage 3	(e)  Pupils made effective use the modelling tool to refine dimension-combinations.	Pupils worked on: 32cm x 32cm x 9cm (K2) 30cm x 30cm x 10cm (K2) 34cm x 34cm x 8cm (K2)
Prediction Stage	(f)  Pupils affirmed they had obtained their ideal dimension-combinations	"34, 34, 8" (R1) "For square, it is the most" (R4) "Highest for square base is 9248 cm^3" (R4)
Optimization Stage	(g)  Pupils used decimal-fractions to enhance the volume of the box	"33.9 x 33.9 x 8.1" (K2) "33.5 x 33.5 x 8.25. Decrease already". (K2) "33.3 x 33.3 x 8.35" (K2)
Optimization Stage (Extension)	(h)  Pupils increased the volume further by extending the height of the box.	"You cut one square into 4 and then you paste on top" (R2)

Figure 8. Pupils' Model Development Summary Chart.

Limitations

The analysis of this study was mainly qualitative. Indeed, examining pupils' modelling process was very tedious and complex, and interpreting behaviours is still very much a human endeavour that will be biased based the lens the coder wears. Thus, the coding and interpretation of behaviours can never be absolutely accurate.

The analysis was carried out with only two classes of P6 pupils, and the findings are not meant to be generalizable. The conceptual and interpretive frameworks will be applied to more groups of pupils with different modelling tasks during Phase 2 of the study.

Time was a limiting factor. The pupils' modelling processes could have been different if more time had been given, for example, from one hour to one and a half hours. With this extra time, the pace could be more relaxed, and the teacher may feel less pressure of getting the pupils to complete the tasks by the hour.

Discussion Concluding Remarks

The conceptual framework proposed in this study is to provide "a skeletal structure of justification" (Eisenhart, 1991, p.210) for the research problem under investigation. The theories surrounding the models-and-modelling perspective, problem-based learning, and situated cognition serve as guides for which data are collected and analyzed towards addressing a concern in the field of mathematical modelling that little is known, in particular, about learners' modelling endeavours from a microscopic point of view (BorromeoFerri, 2006). The conceptual framework that is developed also reflects timeliness, that is, it is used to address a current state-of-affairs research problem for which data must be collected as empirical evidence (Eisenhart, 1991), which, in this case, is the growing recognition of mathematical modelling in the school curriculum at this point in time.

The conceptual framework has distinctive characteristics which differentiates it from other generic mathematical modelling frameworks. It is situated in a problem-based learning setting to highlight the importance of the pupil-task-teacher interaction with the emphasis towards making the mathematical thinking of the pupils visible. The focus is also on the mathematical modelling task that is used to drive the learning. While some of the modelling stages proposed may be similar to the generic ones found in literature (e.g., Lesh & Doerr, 2003; Mousoulides, Christou & Sriraman, 2008), this conceptual framework broadens the "verification" aspect as part of the *Prediction* stage and adds the *Optimization* stage to enhance the process to improve

and refine the models that the pupils have developed. It guides teachers and pupils towards the realization that problem solving does not end when an initial solution or answer is obtained. The spiralling and repetition of stages as the pupils build and refine models of the problem solutions reflect the real-world problem-solving practice of mathematicians and scientists (Lesh & Zawojewski, 2007; Romberg, Carpenter & Kwako, 2005). Moreover, the base of the model with domain-specific knowledge, mathematical reasoning, and metacognitive thinking built around pupil collaboration, modelling tasks, and teacher scaffolding incorporate critical aspects of the problem-solving process. In this respect, the conceptual framework promotes an advanced form of problem-solving pedagogy and experience that moves away from what the majority of pupils do as problem solving in the classroom, that is to solve structured word problems through mapping problem information onto appropriate operations or the application of previously taught rules. What has been demonstrated in this study is that problem solving is a complex human activity that involves managing information provided by others apart from the task. There is no guarantee that the problem will be solved, and sometimes there may be uneventful episodes compounded by task interpretation, group dynamics, or the teacher's untimely or inappropriate interruptions.

What has this conceptual framework to offer to researchers, teachers and pupils?

Researchers This conceptual framework can be used to analyze pupils' mathematical modelling process and their model development. Through a qualitative analysis of pupils' modelling endeavour, the different modelling stages can be identified. From a micro aspect, the interpretive framework can be used to relate the pupils' conceptual representations and mathematical translations towards making inferences of the models they are developing. The modelling pathway chart makes for easy interpretation at a glance and provides researchers and teachers with a valuable source of feedback with respect to what has taken place during mathematical modelling, what the pupils have done, what models they have constructed, how many iterations are involved, and even the ways the teachers have influenced the modelling process. Researchers can also capture how the models develop from naive ones to more sophisticated ones and what mathematical reasoning and decisions that pupils display that account for the success of their modelling endeavour.

Teachers It is not within the job scope of teachers to video-record and to transcribe pupils' work for analysis. But teachers who are convinced to carry out mathematical modelling activities in class can be aware of the modelling stages that pupils go through during mathematical modelling. They can observe that when pupils are unpacking task details and trying to situate the problem, they are in the *Description* stage. Or, when the pupils are involved in establishing and making sense of some

relationships between variables through manipulating data, they are likely to be in the *Manipulation* stage and some tentative models may emerge from their mathematizing. Teachers can play a part in scaffolding pupils at appropriate junctures of their modelling endeavours, especially if the pupils stop when they think they have obtained an answer or initial solution. Teachers can extend the pupils' thinking towards the *Optimization* stage so that they will refine and improve on their models.

By becoming involved in the learning community with the pupils, teachers know when pupils are highly engaged cognitively and are exercising a rich repertoire of problem-solving behaviours that are built around pupil collaboration, task engagement, and teacher engagement. Teachers can appreciate that such problem-solving endeavours will enable pupils to develop problem-solving skills and modelling competencies that traditional problem-solving experiences cannot offer.

Pupils In a literal sense, pupils do not rely on the conceptual framework to complete the task. They carry out the modelling activity as required in class. But what the pupils do during mathematical modelling in a PBL setting exemplify the important components that make up the modelling process as depicted in the conceptual framework. Operating within the framework, the pupils are seen to identify goals and variables, interpret the problem situation, interrogate the data, engage in inquiry, monitor oneself and others' thinking, and make improvements to their models (Chan, 2008). They may not know about modelling, but as they mathematize, they are involved in mathematical modelling.

From the models-and-modelling perspective, the conceptual framework is a tool to support research collaboration between researchers, teachers, and pupils. This tool is central towards the testing of its own design for a targeted purpose, and when tested, the important aspects of the conceptual system that it embodies are also tested (English, Lesh, & Fennewald, 2008).

In proposing the above conceptual framework to trace pupils' model development in the modelling process, we believe that we have brought additional clarity to the existing body of knowledge with respect to pupils' involvement in mathematical modelling. This framework needs to be further tested with more groups of pupils and more diverse modelling tasks across different levels in order to gain greater insights into pupils' model development.

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