MATHEMATICAL MODELLING AS PROBLEM SOLVING: PRIMARY 6 PUPILS' MODELLING ENDEAVOUR

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Abstract

Recent developments in the field of mathematics education are suggesting that mathematical modelling be the new focus of research in mathematical problem solving. In Singapore, "applications and modelling" has been included as part of a new process component in the recently revised mathematics curriculum in a move to be relevant to the changing mathematics education landscape. This paper draws on a larger research study in mathematical modelling carried out with Primary 6 pupils and addresses the way the pupils work within a problem-based learning instructional setting towards constructing models. The modelling task used in this study required pupils to go through four modelling stages towards successful completion of the task such that the pupils have to describe and interpret their conceptions in mathematical ways. This paper reports from selected excerpts that are used to exemplify the pupils' modelling aspects, detailing their generation of emerging models towards optimizing their solution.

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Introduction

Reformed efforts in mathematics education are calling for a shift towards engaging learners in more problem-based real-world problems (Hiebert et al., 1996; Erickson, 1999). Traditional word problems that children do in schools are increasingly seen as inadequate to prepare learners for a complex and dynamic world where problem solving is not a linear process with pre-determined steps that leads to only one correct answer but involves cycles of constructing, testing and revising (Lesh & Doerr, 2003). As well, mathematics education researchers in the US, Europe and Australia are focusing research efforts in mathematical modelling at the school level with the long term view of developing problem solvers who can apply their mathematics knowledge in real world situations (Mousoulides, Sriraman, & Christou, 2007). Locally, the recent revision to the Singapore Mathematics Curriculum saw new process components like "Reasoning, Communication, and Connections" and "Applications and Modelling" (MOE, 2007) included that underscore the importance of developing children towards expressing their mathematical ideas, making links between those ideas, and formulating and improving those mathematical ideas through the ways they represent data and the use of appropriate methods and tools in solving real-world problems. This paper discusses mathematical modelling as problem solving with respect to a group of Primary 6 pupils' endeavour in their attempt on the Floor-Covering Problem.

Mathematical Modeling as Problem Solving

Mathematical modelling has recently been advocated to be the new direction for research in mathematical problem-solving (Lesh & Zawojewski, 2007) and regarded as the most significant goals of mathematics education (Lesh & Sriraman, 2005). Mathematical modelling tends to be associated with higher school pure and applied mathematics (geometry, algebra, calculus, etc.) to solve real-world problems but this traditional sense of modeling has been claimed to be inflexible for want of trying to fit intact models into dynamic problematic situations (Yoon & Thompson, 2007). It is also claimed that traditional modelling involves

the direct mapping between the structure of the problem situation and the structure of a symbolic expression that leads to only one way of interpreting the problem (English, 2003) which does not depict problem solving in reality. The current view of mathematical modelling stemming from a modelling perspective (Lesh & Zawojewski, 2007) sees students' modelling process as going through multiple cycles in developing a mathematical model for the given problematic situation. The cycles of model construction, evaluation, and revision are valued as they are seen as befitting the professional practices of mathematicians and scientists as well as those of other disciplines such as biotechnology and aeronautical engineering (Lesh & Doerr, 2003). Such cycles convey a more realistic process in the light of generating models and conceptual tools towards problem resolution.

A modelling approach to mathematical problem solving focuses on the students' representational fluency through the flexible use of mathematical ideas where the students have to make mathematical descriptions of the problem context and data. When students paraphrase, explain, draw diagrams, categorize, find relationships, dimensionalize, quantify, or make predictions, they are generally developing their conceptual interpretations or models through the mathematizing. As they work with the rich contextual data, they would need to surface and communicate their mathematical ideas to clarify their thoughts and weigh the validity of their ideas. In other words, when students engage in model-eliciting activities, their "(internal) conceptual systems are continually being projected into the (external) world" (Lesh & Doerr, 2003, p. 11) thus making visible their sense-making systems of mathematical reasoning in the form of a variety of representational media such as spoken language, written symbols, graphs, diagrams, and experience-based metaphors. It is asserted that when students go through such cycles of expressing, testing, and revising, the full process of modelling as problem solving is seen as the process of "making practice mathematical" where mathematical practice is learned through experience of problem solving (Lesh & Zawojewski, 2007, p. 785). This contrasts with traditional notions of making mathematics practical. As interpretive cycles take place within the modelling process, multiple mathematical interpretations of students are elicited within each modelling stage. From this perspective, the modelling process is a non-trivial and thought-revealing problem-solving process.

Mathematical modelling as problem solving has been documented in a number of relevant works and the modelling processes have been postulated to comprise four stages known as *Description*, *Manipulation*, *Prediction*, and *Verification* (Lesh & Doerr, 2003; Lesh &

Zawojewski, 2007). The four stages are generic and they encompass most of what other mathematical modelling proponents essentially put forward as modelling processes. From a problem-solving lens, the four stages of mathematical modelling are reminiscent of Polya's (1973) four stages of problem solving but viewed from a fresh perspective. In many a classroom practice, Polya's problem-solving stages and heuristics are seen as intended strategies to help pupils function within current ways of thinking for they tend to prompt ways of selecting and carrying out procedures and rules in trivial problem-solving situations (Lesh & Zawojewski, 2007). From a modelling perspective, Polya's problem-solving process could have been viewed as using heuristics to help pupils go beyond current ways of thinking in complex problem-solving situations and as providing a language to describe the problem-solvers' processes to develop flexible prototypes of experiences to be drawn upon in future problem solving (Lesh & Zawojewski, 2007). This new perspective framed as a mathematical modelling process serves to address the emergence of pupils' engagement in descriptive processes for model development.

Conceptual Framework

As an instructional strategy, I situated mathematical modelling in a problem-based learning (PBL) platform. PBL has three main tenets, namely, an unstructured and complex task (in this case the modelling task), student collaboration, and teacher-scaffolding. The interaction between these three tenets would make visible the students' mathematical thinking during their modelling endeavour. Their mathematical thinking and reasoning is manifested as conceptual interpretations evidenced through the way they mathematize and operationalize data to create models.

Method

Participants

The participants were Primary 6 pupils from two classes of a neighbourhood schools. Each class comprised small groups of four or five pupils. Two groups from each class were selected by their respective mathematics teacher to be the target group for video-recording.

Modelling Task

The task, an adaptation of the modelling task from Gravemeijer, Pligge and Clarke (1998), was situated in the context of determining the most economic way of covering the floor of a study room given the different floor-covering materials that were priced differently. The details of the task given out as the task sheet are shown in Appendix 1. The pupils had to identify the key mathematics components from the task, conceive layout designs, establish key variable relationships and analyze their designs to determine which design was most economical.

Data Collection and Analysis

The target groups were video-recorded during their engagement of the modelling task. Other sources of data included my field notes, the pupils' written work and journals. The video data were transcribed and reviewed several times for evidence of conceptual interpretations that suggested the progression of the pupils' thinking for emerging models through modelling stages. Although a problem-solving coding scheme was used to code the pupils' protocols, this paper only addresses the various conceptual interpretations displayed by a group of pupils during the modelling process within each modelling stage.

Findings

Pupils' Mathematical Interpretations

Mathematical modelling has been variously defined in literature. In this study, the pupils' mathematical thinking and reasoning are reported as models of conceptual interpretations that involve the process of using mathematical language or terms to develop representational descriptions of specific situations whenever relationships between concepts of ideas or constructs are established. The pupils' conceptual interpretations are developed along the way as emerging models towards attaining the final solution and are assessed based on pre-determined models via a tasks analysis that had been carried out during the design of the modelling tasks. In this section, I describe using examples from one of the target groups' involvement in the said modelling activity.

Description Stage

The Description stage involves the breaking down of task information towards understanding and simplifying the problem task. This includes identifying mathematical components or variables essential to solving the problem.

Since from the task sheet, only one of the dimensions were given for a roll of floor-covering material, the pupils initially had problems with visualizing how the roll of carpet or met could be laid out.

2	S2	So the floor, right, you want to cover it, you are supposed to roll like that, roll like that, so we have 3 sheets. (*Gestured with both hands the unrolling of the floor material vertically and horizontally across the floor*)
3	S3	A bit greyish! If you want to (…) // You get what I mean or not?
4	S2	You can roll like that, then you can roll like that (*repeats the gesture of unrolling horizontally and vertically)*

The above excerpt suggests that pupil S2 conceived that the roll of carpet could be used in different orientations to cover the floor. However, there were uncertainties about what the given dimension meant as seen below.

10	S3	What do you mean by 4m?
11	S4	4m here *(using index finger to point at the side of the carpet icon)*
12	S3	This one can come up to what?
13	S4	This one is 4m. Really.
		…
31	S4	If this is the floor, and you have to cover, and the 4m does not change…

The pupils also tried to clarify what the given dimension meant - which length was fixed at 4m and which length could vary when laid across the floor as seen from (10) to (13). The negotiation of meaning took quite a while when up to protocol line 31, a pupil, S4, was still trying to convince others what she thought about the specificity of the dimensions. It was

important for the group to have a clear understanding as that would affect how they lay the carpet.

This modelling aspect saw the pupils delineating some key components from the task details such as length, dimensions, space (cover, area) and orientation layout which was an essential towards finding ways to make efficient the costing and the amount of materials used later.

Manipulation Stage

During Manipulation, pupils establish relationships between variables through constructing hypotheses, examining and organizing contextual information, mathematizing, or using strategies to develop a mathematical model.

Layout Design. On the development of a layout design as a modelling aspect, pupils were observed to reason how they economize on the use of materials towards a cost effective design. Only the carpet floor-covering material is used in the example below.

114	S2	I think we need 3.
115	S4	Why?
116	S2	Because you see, here the way you put it in, here 3m becomes here 1m, 1m ,1m (*gesturing the three parts in the A4 paper)*
…	…	...
126	S3	Why here 1m?
127	S2	No, here is 3m, but this carpet is 1m, 1m, 1m.
128	S3	Why here cannot be 1m, then here 0.5? *(pointing to the different portions in the A4 paper)*
129	S2	Because here is 0.3m. Because here you cover the carpet, (…)
130	S3	Why can't the 0.5 be here?
131	S2	Because if here 1m, then there'll be lots of wastage. You'll need more than 3 *(since S3's approach will require 6 pieces)*
132	S3	You need 3?
133	S2	Then you'll need 6.

By having the carpet layout to be 4m by 3m rolled vertically, the floor would have a gap of 0.3m by 3m. Since loose pieces of carpet measuring 0.5m by 1m would be needed to patch the gap, S2 conjectured that they needed 3 pieces of loose carpets (114) and then S4 generalized through reasoning that each 1m length of the loose carpet could be used to patch the gap (116). This conceptualized design could be depicted in Figure 2.

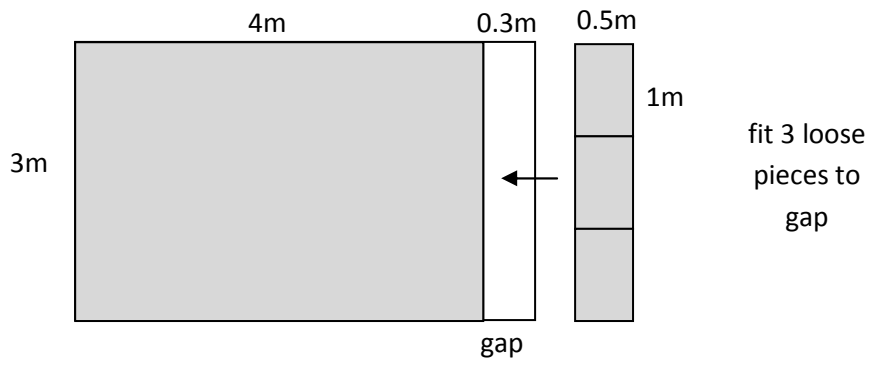

Figure 2. Conceptualized design using carpet from S2

S3 on the other hand had another design in mind. She wanted to know why the loose pieces of carpets could not be re-oriented to fit the gap (126). S3's conceptual design (128) could be presented as in Figure 3.

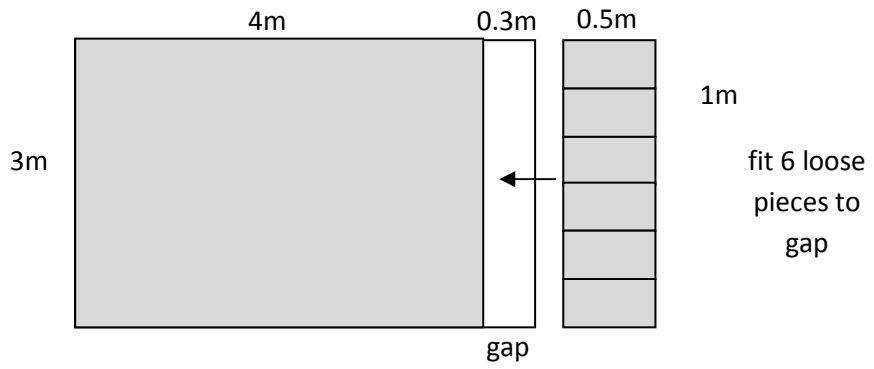

Figure 3. Conceptualized design using carpet from S3

With two designs conceptualized based on how the loose materials should be utilized to patch the gap, the first conceptualized design was chosen in favour of the second

because it prevented the "*wastage*" of materials and more loose carpets would be needed in the second design (131). S2 confirmed that they would need 6 loose carpets if the second design was chosen (133).

Manipulation Stage: Modelling Aspect on Area-and-Cost Relationship. The pupils were able to relate area and cost quite comfortably. Whenever a floor-covering area was determined, the pupils multiplied the area with the respective cost per square metre. The cost was then totalled taking into consideration the number of loose pieces required.

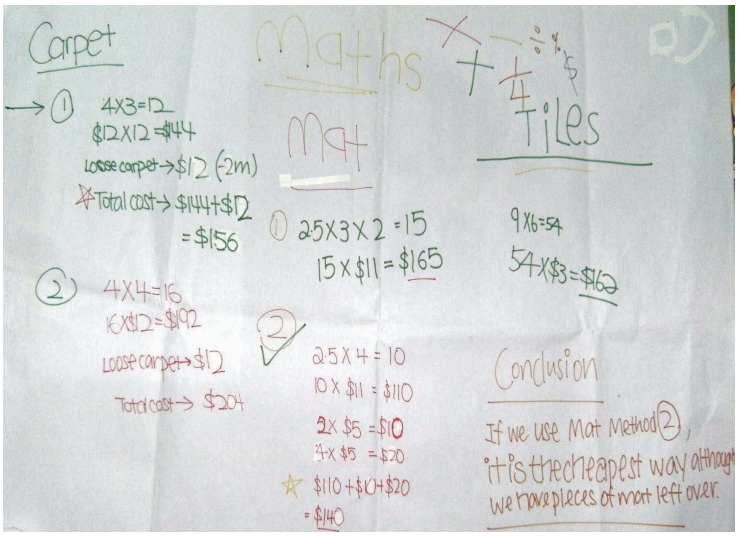

Figure 4. Area and cost relationships for the different floor-covering materials

Figure 4 shows that the circled numbers indicate two layout designs for carpet and mat each and their respective areas, inclusive of the areas of the loose materials used, were multiplied to the cost of the given material per square metre to obtain the cost of the respective floor coverings. For tiles, because there was only one way of tiling, there was no alternative design.

Prediction Stage

This modelling aspect involves analyzing and interpreting if the solutions obtained during the Manipulation stage fit the demands or expectations of the task. In this respect, pupils have to

discuss if their solutions options or layout designs are within parameters to ensure that they could cover the floor at a certain cost suggesting the attainment of a workable solution model. It was observed that the pupils did not quite fulfil this modelling aspect as they only briefly engaged to check if the way they penned the steps were correct.

224	S3	OK, so for the carpet we have $162, for the mat we have $145, and for the tiles, we have $162 again. So it is mat, correct? Mat, Method 2.
		(S2 records)
225	S2	Although for the loose pieces, it has lots of leftover, it is still the cheapest method.
226	S1	I agree
227	S3	Same here.
228	S2	Shall we double check?
229	S3	I don't think so. We should do the survey now.
		(S1 takes out their Group Journal sheet and prepares to write)
230	S3	Thus our conclusion is that (…) if we use mat, method 2, (…) eh, XXX, mat method 2 how to make?
		(S3 takes the roll of serviettes and tries to unroll and place it over the A4 paper in different orientations)
231	S2	Method 1 is like that right? And method 2 is like that, correct?
232	S3	Yeah, you want to write horizontal and vertical or not?
		…
234	S3	Thus our conclusion is that if we use mat, method 2, is the cheapest way although we have some pieces of mat left over.

As seen in (224) and (225), both S3 and S2 were generalizing which floor-material was the cheapest but did not re-analyze to justify if their solutions fit the task details and expectations. It was noted that S2 suggested checking their work in (228) but S3 did not want to (in 229) and they came to their conclusion merely based on the costs compared (234). The pupils' modelling endeavour in this stage is therefore not adequate because based on the design of

the task, there could only be one layout design for the carpet floor-covering instead of two. This breach of task parameter had escaped the pupils' attention but it did not adversely affect the overall problem solving because the other layout designs were acceptable and within task parameters.

Optimization Stage

In this stage, pupils improve or extend the working model. The various floor-covering layouts and their associated costs in the Prediction stage suggested only the ways that the materials could have been laid at a certain costs. Thus the various layouts were workable designs (save for the one which was not within task parameter) but not optimized designs. Because the pupils did not adequately discuss nor reflected about their solutions, they easily concluded their choice of floor material for the particular cost computed (in 234 above). However, in a PBL instructional setting, the teacher-facilitator plays an important part in extending the pupils' thinking towards optimizing their solutions. In this instance, the teacher-facilitator intervened through scaffolding the pupils' thinking towards improving their solution.

258	S3	Because here is 3m, so we must buy 3 pieces.
259	T	Why 3 pieces?
260	S3	Because here 1m, 1m, 1m.
261	T	So if here is 0.5m, here will be…
		(conversation becomes muffled)
262	S3	0.2,
263	T	OK, so here is 0.2, right? So you'll have these pieces. Can you use these remaining pieces?
264	S2	You mean you can cut along the loose pieces?
265	T	It's a carpet. What did they say? You can "further cut". (*T points to the task sheet)* Think about it.
266	S3	We can cut it as small as possible.
267	T	Further cut. Think about it.
		(Pause as pupils try to figure out as they look at the task sheet)
268	S2	OK, let's go back to this one. Carpet. 4 times 3. So how do we do it? *(S1 begins to unroll the serviette again model the situation)*

269	S3	If she is saying that we can use the remaining loose carpet to fix instead of buying another loose carpet to fix, that means instead of buying another loose carpet, we can use the remaining to fix.
270	S2	0.3 here. 0.3 times 3. Here is 0.2, 0.2. Then here is 0.4.
271	S2	Then we only need 2.
272	S3	Yeah, we only need 2.
273	S2	OK, then 2 lor. Then change lor (*S1 uses the correction fluid to make changes)*
274	S3	So instead of 18 and instead of 3 pieces, now we have 2 pieces of loose carpet.

As seen in (261) and (263), the teacher-facilitator got the pupils to focus on the three pieces of loose carpet that the pupils had found that would be needed to patch the floor gap. Her intent was to get pupils to think about how to reuse the wasted loose materials as seen in (263), (265) and (267). It prompted the pupils to reanalyze their designs for cutting the loose materials by working on the dimensions of the materials that would be wasted to enhance the patchwork (268, 269). As they reasoned about the dimensions of the loose pieces in relation to the area of the gap (270), they found they only needed 2 pieces instead of 3 (271, 274). With this idea of saving materials, the floor-covering design would enable the pupils to adopt value-for-money methods and this knowledge was transferable to optimizing the material-cost layouts for the carpet and the mat.

Discussion and Concluding Points

When pupils are given a modelling task such as the Floor-Covering Problem, they have to undergo the four modelling stages, Description, Manipulation, Prediction, and Optimization towards successful completion of the task. During the design of the task, the emerging models have been pre-determined. This helps in assessing if the pupils have been able to adequately model specific situations like in this case, construct appropriate and economical layout designs, and relate the respective areas and costs. Since the goal is to promote material and cost savings, the pupils will have to work towards optimizing those modelling aspects. The modelling endeavour in working towards that goal is seen as problem-solving.

Mathematical modelling as problem solving can be achieved through a PBL setting, in this case, a short-cycle PBL instructional approach. The problem solving becomes more authentic because the pupils are working in groups and solving a real world problem that needed them to produce and present their findings. As they do so, the student-student interaction, the student-task interaction, as well as with the student-teacher interaction elicits the pupils' problem-solving behaviours. Unlike relying on a standard or memorized procedure to solve problems as in a word-problem situation, pupils are developing powerful reasoning capabilities such as the assessing of properties, conceptualizing layouts, strategizing through identifying relationships, revising, improving and extending their models which are manifested as representational conceptual interpretations by way of the pupils' mathematical descriptions and explanations. These processes show that the pupils are engaged in mathematizing – where evidence of their application of mathematical reasoning aspects such as quantifying, dimensionalizing, coordinating, categorizing, algebratizing, and systematizing relevant objects, relationships, actions, patterns, and regularities in realistic situations are seen.

By working on mathematical modelling tasks in a PBL setting, not only do the pupils' mathematical thinking becomes explicit, they develop and heighten their metacognitive thinking as well through the interaction as there is greater need to check, query, validate, and monitor the problem-solving situation. In fact, the modelling activity also saw the pupils working independently of the teacher most of the time. The pupils therefore are more self-directed and the teacher offered scaffolding only on a few short occasions to elicit, support, and extend their thinking. The pupils' modelling endeavour suggests that the pupils could enter the Description and Manipulations stages well but did not quite adequately analyze if their solution models fit the task parameters in the Prediction stage. Perhaps when pupils are made more aware about the importance of assessing for functionality would they be able to interpret their solution more adequately. Also, where the Optimization stage is concerned, it was through the teacher's scaffolding that has enabled the pupils to stretch their thinking and thus extend their model.

This study has shown that pupils can develop conceptual interpretations of specific situations through constructing variable relationships between idea, concepts or constructs. Mathematical modelling as problem solving as seen in this study holds promise for pupils to think mathematically in everyday situations that are associated with design specifications,

minimizing costs, maximizing values, and predicting which designs are workable or are cost and material effective. Such mathematical modelling and reasoning processes would provide valuable experiences in children's mathematical development that can hardly be obtained through working on traditional word problems.

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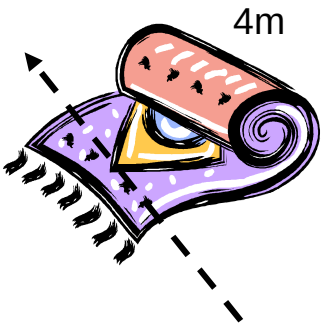
Problem-Based Learning in Mathematics

Problem Scenario – The Floor Covering Problem

You have been asked by your mother to suggest a covering for the floor of your study-room. The room is rectangular and measures 4.3 m by 3 m. There are three ways to cover the floor. You can use the mat, carpet or tiles but they are of different costs. Explain clearly and mathematically your best choice and how you arrive at your decision. Drawing diagrams may make your explanation clearer.

Carpet

4m



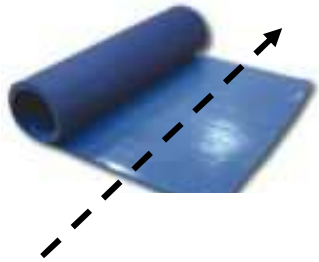
$12 per m^2

Can be cut in only one direction as indicated by dotted arrow.

Loose carpet of 0.5m by 1m for patchwork at $6 per piece. Each piece can be further cut to fit size.

Mat

2.5 m

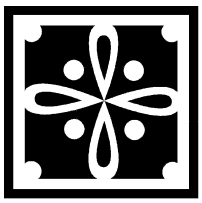


$11 per m^2

Can be cut in only one direction as indicated by dotted arrow.

Loose mat of 0.5m by 1m for patchwork at $5 per piece. Each piece can be further cut to fit size.

Square Tile



0.5 m

$5.00 each

If a tile cannot completely fit part of the floor space, it will require professional help to cut the tile for fitting. Service charge will be at $2.00 per tile plus cost of the tile.