Noisy quantum game

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In a recent paper [D. A. Meyer, Phys. Rev. Lett. **82**, 1052 (1999)], it has been shown that a classical zero-sum strategic game can become a winning quantum game for the player with a quantum device. Nevertheless, it is well known that quantum systems easily decohere in noisy environments. In this paper, we show that if the handicapped player with classical means can delay his action for a sufficiently long time, the quantum version reverts to the classical zero-sum game under decoherence.

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I. INTRODUCTION

Classical game theory [1,2] has always been applied successfully to economic and industrial decision models to resolve and determine the best possible strategy. Recent breakthroughs in quantum computations have shown that quantum mechanics continues to introduce surprising twists and novelties into our classical way of thinking. In particular, it has been shown that a quantum algorithm, such as the prime number factorization [3], can provide substantial improvement in speed and efficiency if one is equipped with an appropriate quantum device. However, it has also been shown that it is not always possible to perform better than quantum devices as in quantum bit commitment [4,5].

In quantum bit commitment, Alice (the sender) commits a bit to Bob (the receiver). At some later time, Alice must show Bob which bit she has committed and convince him that the revealed answer is the genuine bit that she has previously committed. It has been shown that if Alice is given a quantum computer, she can always cheat and therefore such commitments are never possible in the quantum case. This protocol serves to remind us that even if a quantum computer can be made, it may not always be possible to do better in the quantum situation.

Recently, there have been some attempts [6–9] to generalize the classical notion of game theory to an analogous quantum version. A natural question, therefore, is to find out whether it is possible in a two-party classically fair game for one party equipped with a quantum device to beat another party. In particular, it has been found instructive to consider two-party coin-tossing games [6]. Indeed, it has been found in this particular two-party classically fair game that Alice, equipped with a quantum computer, can always take full advantage of her resources to win. In fact for Alice to beat Bob in this classically fair game, it is necessary for Alice to "flip" the coin into a linearly superposed state of head and tail.

However, quantum systems are easily influenced by a noisy environment. Thus, neither Alice nor Bob can avoid the effects of decoherence since realistic quantum devices are especially prone to environment disturbances [10–13]. In this way, it is, therefore, useful to find out if Alice can continue to maintain her superiority and advantage in a noisy environment. In this paper, we will consider some simple decoherence models using Kraus operator representation and explore the issue of a noisy two-party classical fair game described in Ref. [6]. In Sec. II, we briefly describe the simple coin-flipping game. In Sec. III, we investigate the effects of decoherence channel, dephasing channel, amplitude-damping channel, and the two-Pauli channel in the game. Finally, in Sec. IV, we summarize our results and make some observations.

II. A QUANTUM COIN-FLIPPING GAME

In a recent paper [6], Meyer demonstrated that in a classical two-person zero-sum strategic game, if one person adopts a quantum strategy, then he has a better chance of winning the game. Meyer's strategy is as follows: two persons Alice and Bob take turns to flip a coin. Bob initially places the coin head up in a box. Thereafter, Alice, then Bob then Alice take turns to flip the coin. Alice wins if the coin is head up and loses otherwise.

In the quantum version, the initial state of the coin is represented by a density matrix, ρ_0 so that in the basis $\{|H\rangle,|T\rangle\}$ in which the symbols H and T denote head and tail, respectively, ρ_0 is given by

$$\rho_0=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}. \tag{1}$$

Suppose Alice adopts a quantum strategy, then Alice uses a unitary rather than a stochastic matrix to act on the coin. Let this unitary transformation be U_1, so that the state of the coin at the end of the transformation is given by $\rho_1=U_1\rho_0U_1^\dagger$. Bob however continues to play with a classical probabilistic strategy. Thus, Bob employs a convex sum of unitary (deterministic) transformation, namely, he either flips the coin using the transformation F with probability p or lets the coin rest in its original state (using the identity transformation, F_0) with probability $(1-p)$, where

$$F_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{2}$$

Thus, the operator F_0 leaves the coin in its original state, while the operator F_1 "flips" the original state to the other state. Thus, at the end of Bob's turn, the state of the coin should be described by the density matrix

$$\rho_2 = pF_1\rho_1F_1^\dagger + (1-p)\rho_1. \tag{3}$$

Finally, Alice "flips" the coin using the unitary transformation U_2, so that the final state of the coin is $\rho_3 = U_2\rho_2U_2^\dagger$. Meyer has shown that if Q selects the unitary matrices $U_1 = U_2 = H$, where H is the Hadamard transform given by

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{4}$$

then $\rho_3 = \rho_0$, independent of the probability p. Thus Alice wins the game every time.

It is instructive to note that a classical coin has only two possible states, namely, head and tail. It is interesting to note that the explicit form of the operators F_0 and F_1 permits the definition of a density matrix G,

$$G \equiv \frac{1}{2}(F_0 + F_1) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathrm{Tr}\, G = 1. \tag{5}$$

One can easily verify that G commutes with F_0 and F_1, i.e.,

$$[G, F_j] = GF_j - F_jG = 0 \quad (j = 0,1). \tag{6}$$

Since F_j is unitary, the following identity holds

$$G = (1-p)F_0GF_0^\dagger + pF_1GF_1^\dagger, \tag{7}$$

so that G is independent upon the parameter p under a classical coin flip.

III. A NOISY QUANTUM COIN-TOSSING GAME

The standard procedure of understanding the behavior of one part of a bipartite quantum system is to extend the system to a larger one [in which the environment (E) is incorporated] so that the evolution of state becomes unitary. By assuming complete positivity of the superoperators, it is possible to study the nonunitarity evolution of the state of a subsystem using an operator sum representation. In terms of the operator sum or Kraus representation, we can then express this map, $\mathcal{S}$, more succinctly as

$$\rho_{\mathrm{out}} = \mathcal{S}(\rho_{\mathrm{in}}) = \sum_\mu M_\mu \rho_{\mathrm{in}} M_\mu^\dagger. \tag{8}$$

Unitarity of the evolution of the bipartite quantum system also requires that the Kraus operators satisfy the condition

$$\sum_\mu M_\mu^\dagger M_\mu = 1. \tag{9}$$

The explicit expressions for the Kraus operators depend on the type of channel chosen. Typically, one can consider the following quantum channels: (1) depolarizing channel, (2) phase-damping channel, (3) amplitude-damping channel, and (4) two-Pauli channel.

The coin-tossing game between Alice and Bob proceeds as described in Sec. II. However, Alice and Bob can now delay their decision to apply the unitary or classical flipping and allow the state of the coin to evolve nonunitarily under the noisy environment. Specifically, depending on the decoherence model, the game proceeds as follows. The initial state of the coin $|\chi_0\rangle\langle\chi_0| \equiv \rho_0$ is first allowed to evolved nonunitarily under the Kraus operators, $\{M_i^1\}$, so that the state becomes ρ_0'. Alice then applies her unitary transformation on the coin, so that the resulting state is ρ_1. Next, Bob can choose to delay his flip, thus allowing the state of the coin to evolve into the state, ρ_1' under the Kraus operators $\{M_i^2\}$. Bob then performs his flip and changes the state of the coin to ρ_2. This state is then allowed to evolve under Kraus operators, $\{M_i^3\}$. Finally, Alice performs her unitary transformation and reveals the coin.

A. Depolarizing channel

For the depolarizing channel, the Kraus operators are

$$M_0^j = \sqrt{1-r_j}\,\mathbf{1},$$

$$M_i^j = \sqrt{r_j/3}\,\sigma_i (j = 1,2,3),$$

where $0 \leqslant r_j \leqslant 3/4$, and $\sigma_i (i = 1,2,3)$ are Pauli matrices.

We next proceed to describe in some detail the steps in our calculation.

(i) We begin with the initial state of the coin as $|H\rangle$ so that its density matrix is given by $\rho_0 = |H\rangle\langle H|$.

(ii) After a time t_1, ρ_0 becomes $\rho_0' = \rho_0 - (2r_1/3)\sigma_3$, where r_1 is the parameter associated with the depolarizing channel.

(iii) Alice then applies the general unitary transformation (apart from an irrelevant phase factor)

$$U_1 = \begin{pmatrix} \cos\frac{\theta_1}{2} & -\sin\frac{\theta_1}{2}e^{i\omega_1} \\ \sin\frac{\theta_1}{2}e^{i\phi_1} & \cos\frac{\theta_1}{2}e^{i(\omega_1+\phi_1)} \end{pmatrix}$$

on the coin, where θ_1, ϕ_1, and ω_1 are real numbers. The state of the coin consequently becomes

$$\begin{aligned} \rho_1 &= U_1\rho_0'U_1^\dagger \\ &= \frac{1}{2}\mathbf{1} + \left(\frac{1}{2} - \frac{2r_1}{3}\right)[\sin\theta_1(\cos\phi_1\sigma_1 + \sin\phi_1\sigma_2) \\ &\quad + \cos\theta_1\sigma_3], \end{aligned}$$

independent of ω_1.

(iv) After a time t_2, ρ_1 decoheres (with parameter r_2) into $\rho_1' = \frac{1}{2}1 + \alpha_1[\sin\theta_1(\cos\phi_1\sigma_1 + \sin\phi_1\sigma_2) + \cos\theta_1\sigma_3]$, where $\alpha_1 = (1-4r_1/3)(1-4r_2/3)/2$.

(v) Bob continues to play classically by employing a convex sum of unitary (deterministic) transformation, namely,

he either flips the coin using the transformation F_1 with probability p or lets the coin rest in its original state (using the identity transformation F_0) with probability $(1-p)$. At the end of Bob's turn, the state of the coin is described by the density matrix $\rho_2=\frac{1}{2}1+\alpha_1[\sin\theta_1\cos\phi_1\sigma_1+(1-2p)\sin\theta_1\sin\phi_1\sigma_2+(1-2p)\cos\theta_1\sigma_3]$.

(vi) After a time t_3, ρ_2 under decoherence (with parameter r_3) becomes $\rho_2'\frac{1}{2}1+\alpha_2[\sin\theta_1\cos\phi_1\sigma_1+(1-2p)\sin\theta_1\sin\phi_1\sigma_2+(1-2p)\cos\theta_1\sigma_3]$, where $\alpha_2=(1-4r_1/3)(1-4r_2/3)(1-4r_3/3)/2$.

(vii) Finally, Alice implements the unitary transformation (apart from an irrelevant phase factor)

$$U_2=\begin{pmatrix} \cos\frac{\theta_2}{2} & -\sin\frac{\theta_2}{2}e^{i\omega_2} \\ \sin\frac{\theta_2}{2}e^{i\phi_2} & \cos\frac{\theta_2}{2}e^{i(\omega_2+\phi_2)} \end{pmatrix},$$

so that the density matrix of the final state of the coin is

$$\rho_3=U_2\rho_2'U_2^\dagger=\begin{pmatrix} 1/2+\alpha_2\xi & \alpha_2\eta \\ \alpha_2\eta^* & 1/2-\alpha_2\xi \end{pmatrix},$$

where

$$\xi=-\sin\theta_1\cos\phi_1\sin\theta_2\cos\omega_2+(1-2p)\times(\sin\theta_1\sin\phi_1\sin\theta_2\sin\omega_2+\cos\theta_1\cos\theta_2),$$

$$\eta=\{\sin\theta_1\cos\phi_1[\cos^2(\theta_2/2)e^{-i\omega_2}-\sin^2(\theta_2/2)e^{i\omega_2}]+(1-2p)[-i\sin\theta_1\sin\phi_1(\cos^2(\theta_2/2)e^{-i\omega_2}+\sin^2(\theta_2/2)e^{i\omega_2})+\cos\theta_1\sin\theta_2]\}e^{-i\phi_2}.$$

It is easy to work out the probability of getting a head at the end of the game. This probability can be expressed as $P_{\text{head}}=1/2+\alpha_2\xi$. In general, r_1,r_2, and r_3 are different but in order to maintain her advantage, Alice would try to minimize any decoherence. Thus, we may set $r_1=r_3$ for convenience. In this case, let us redefine new variables x and y related to r_1,r_2, and r_3 as $x=2r_1/3=2r_3/3$, $y=2r_2/3$, we then have $P_{\text{head}}=[1+(1-2x)^2(1-2y)\xi]/2$. In order to establish a Nash equilibrium, Alice implements the quantum operations, U_1 and U_2 in her strategy while Bob can adopt unequal probabilities to his flip. It turns out that for a dominant strategy, Bob should play head or tail with equal probabilities. Since Bob continues to play "classically," a Nash equilibrium will be achieved when Alice maximize ξ with a proper choice of angles in her unitary transformations U_1 and U_2 [16]. Moreover, Alice does not know the parameter p, so she should set the coefficient of p to zero, i.e., $\sin\theta_1\sin\phi_1\sin\theta_2\sin\omega_2+\cos\theta_1\cos\theta_2=0$. It is not difficult to show that $\xi_{\max}=1$, when

$$\sin\theta_1\cos\phi_1\sin\theta_2\cos\omega_2=-1.$$

For convenience, Alice can choose $\theta_1=\theta_2=\pi/2$, $\phi_1=\phi_2=0$ and $\omega_1=\omega_2=\pi$. In this case, $U_1=U_2=H$, which implies that the Hadamard transformation is an optimal unitary transformation for Alice to optimize her strategy. We now consider the function $f(x,y)=P_{\text{head}}^{\max}-1/2$. It is straightforward to show that

$$f(x,y)=-4x^2y+2(x^2+2xy)-(2x+y)+\tfrac{1}{2}. \quad (10)$$

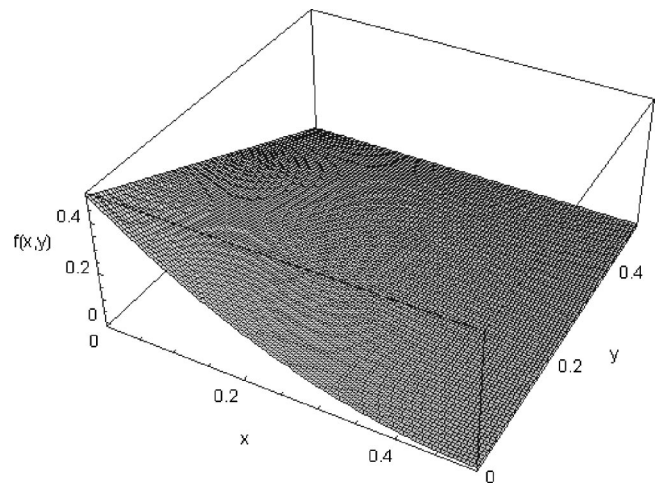

FIG. 1. A plot of the function $f(x,y)$ for $0<x,y<0.5$. Note that the function is identically zero only if x or y attains the maximum value of 0.5.

Figure 1 shows the variation of $f(x,y)$ with x and y and Fig. 2 shows the variation of cross-section plots $f(x,x)$ when $x=y$, $f(x,0.2)$ and $f(0,y)$ with respect to x or y. It is clear that even if Alice immediately performs her transformation, Bob can reduce her advantage drastically by introducing sufficient noise within the system.

B. Phase-damping channel

The Kraus operators are

$$M_0^j=\sqrt{1-r_j}\,\mathbf{1},$$

$$M_1^j=\sqrt{r_j}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$M_2^j=\sqrt{r_j}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (j=1,2,3).$$

Beginning with the initial state, ρ_0, after a time t_1 as before, ρ_0 becomes

$$\rho_0'=M_0^1\rho_0M_0^{1\dagger}+M_1^1\rho_0M_1^{1\dagger}+M_2^1\rho_0M_2^{1\dagger}=\rho_0.$$

In this case, an analysis similar to the previous case of the depolarizing channel shows that a Nash equilibrium is established when Alice uses a unitary transformation $U_1(=U_2)=H$ to act on the coin so that the state of the coin becomes $\rho_1=U_1\rho_0'U_1^\dagger=G$. Incidentally, such a strategy is always the best one possible for Alice to adopt in all the other channels discussed in this paper. Thus, henceforth, we need only consider Hadamard transformations for Alice's "flip."

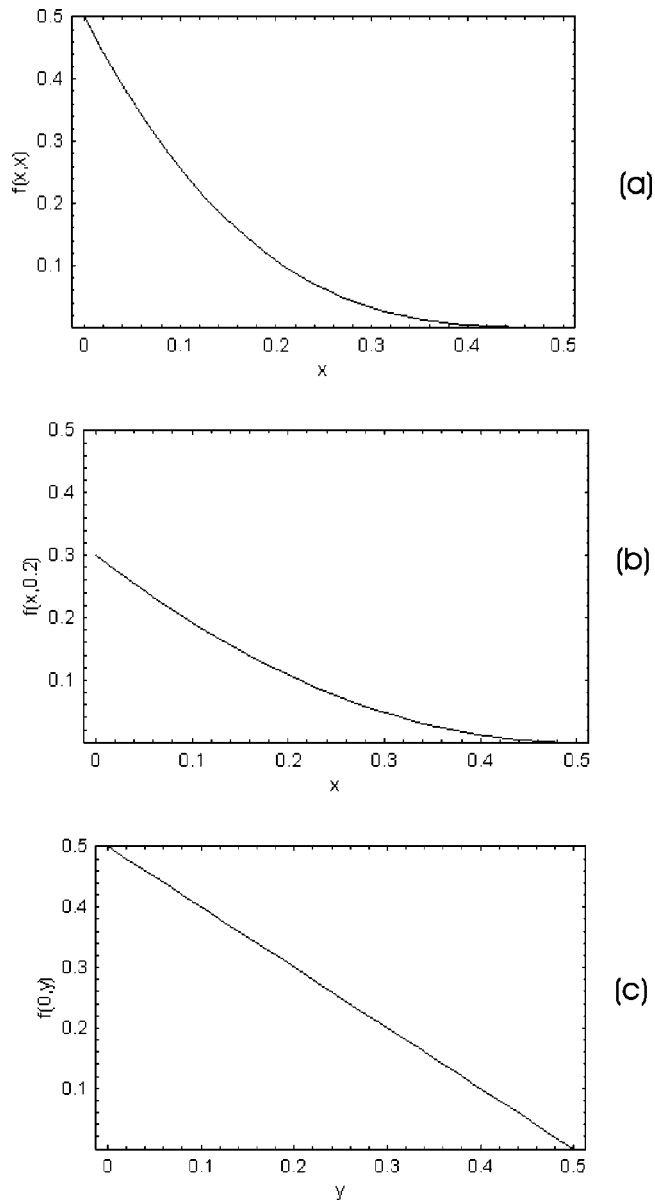

FIG. 2. Plots of the function $f(x,y)$ for (a) $x=y$, (b) $y=0.2$ and $x=0$.

Under decoherence, after a time t_2, the state ρ_1 becomes ρ_1'. After Bob's flip, the state of the coin can be described by the density matrix

$$\rho_2 = pF_1\rho_1'F_1^\dagger + (1-p)F_0\rho_1'F_0^\dagger = G - \frac{r_2}{2}\sigma_1 .$$

Finally, after a time t_3, ρ_2 becomes ρ_2' under decoherence with ρ_2' described by

$$\rho_2' = M_0^3\rho_2 M_0^{3\dagger} + M_1^3\rho_2 M_1^{3\dagger} + M_2^3\rho_2 M_2^{3\dagger} = G - \alpha_3\sigma_1 ,$$

so that the state after Alice's "flip" is

$$\rho_3 = U_2\rho_2' U_2^\dagger = \begin{pmatrix} 1-\alpha_3 & 0 \\ 0 & \alpha_3 \end{pmatrix},$$

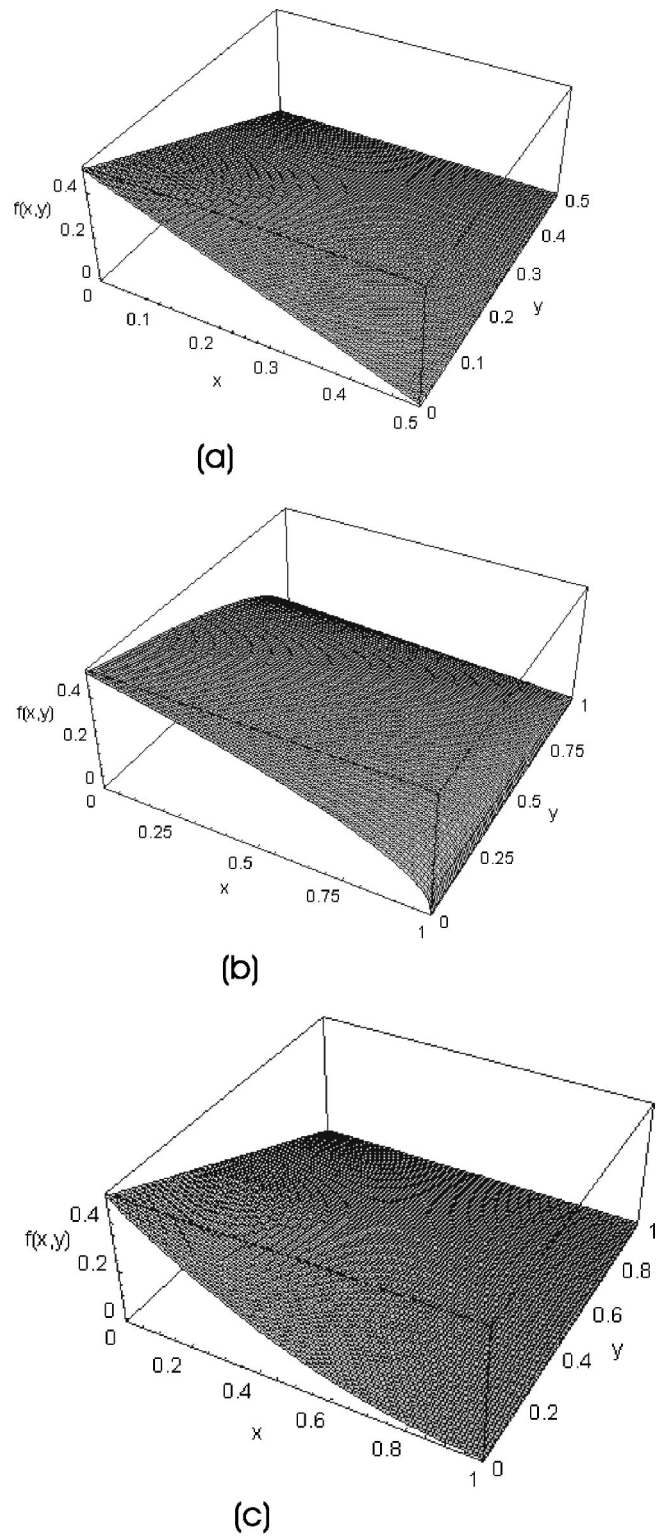

FIG. 3. Plots of the function $f(x,y)$ for (a) the phase-damping, (b) amplitude-damping, and (c) two-Pauli channels.

where $\alpha_3 = r_2/2 + r_3/2 - r_2 r_3/2$.

The probability of getting a head can then be computed and is given by $P_{\text{head}} = 1-\alpha_3$. If we let $x = r_3/2$, $y = r_2/2$ and define the function $f(x,y) = P_{\text{head}} - 1/2$, we then have

$$f(x,y) = 2xy - x - y + \tfrac{1}{2}. \tag{11}$$

C. Amplitude-damping channel

The Kraus operators are

$$M_0^j=\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-r_j}\end{pmatrix},$$

$$M_1^j=\begin{pmatrix}0 & \sqrt{r_j}\\ 0 & 0\end{pmatrix}\quad (j=1,2,3).$$

The final density matrix ρ_3 is given by

$$\rho_3=\frac{1}{2}\begin{pmatrix}1+\sqrt{\alpha_4} & \zeta\\ \zeta & 1-\sqrt{\alpha_4}\end{pmatrix},$$

where $\zeta=[1-\alpha_4-4pr_2(1-r_3)]$ and $\alpha_4=1-r_2-r_3+r_2r_3$. Thus, the probability of getting a head is $P_{\text{head}}=(1+\sqrt{\alpha_4})/2$. Letting $x=r_3$, $y=r_2$ and defining the corresponding function $f(x,y)=P_{\text{head}}-1/2$, we have

$$f(x,y)=\frac{1}{2}\sqrt{xy-x-y+1}. \tag{12}$$

D. Two-Pauli channel

Finally, in the case of the two-Pauli channel, the Kraus operators are given by

$$M_0^j=\sqrt{1-r_j}1,$$

$$M_1^j=\sqrt{\frac{r_j}{2}}\sigma_1,$$

$$M_2^j=\sqrt{\frac{r_j}{2}}\sigma_3,\quad (j=1,2,3).$$

At the end of Alice's flip, the state of the coin is given by

$$\rho_3=\begin{pmatrix}1-\alpha_5 & 0\\ 0 & \alpha_5\end{pmatrix},$$

where $\alpha_5=\frac{1}{2}[r_1+r_2+r_3-(r_1r_2+r_2r_3+r_3r_1)+r_1r_2r_3]$. The corresponding probability of getting a head this time is $P_{\text{head}}=1-\alpha_6$. Defining $x=r_1=r_3$, $y=r_2$, we find that the function $f(x,y)$ is given by

$$f(x,y)=-\tfrac{1}{2}[x^2y-(x^2+2xy)+2x+y]+\tfrac{1}{2}. \tag{13}$$

For completeness, we have plotted the variation of $f(x,y)$ for the phase-damping, amplitude-damping, and the two-Pauli channels in Fig. 3.

IV. DISCUSSION

There has been ample evidence that a quantum system has tremendous advantages over a classical system. However, it is also well known that quantum devices are extremely sensitive to noise in the environment. Thus, any advantages due to quantum effects can be reduced or diminished through decoherence effects. Indeed, our computations reemphasize this fact.

In Meyer's quantum game, it is possible for one player (Alice) who possesses a quantum device to play a non-zero-sum game with another player (Bob) who continues to use classical devices. However, it is clear from our computations that if Bob suspects that Alice is using a quantum device, he can easily eradicate any advantages due to the quantum device by simply delaying his response in his classical flip. Such delay has no significant effect if Alice is using classical means. However, in a quantum system, such a time delay inevitably introduces noise which can easily decohere the state of the coin.

Since Bob can control the channel by introducing sufficient noise, it is interesting to study the behavior of the function $f(0,y)$ in each case. In our computations, it is not hard to see that the behavior of $f(0,y)$ for the depolarizing, phase-damping, and two-Pauli channels are essentially the same, namely, $f(0,y)$ varies linearly with the parameter y whereas for the amplitude-damping channel, $f(0,y)=1/2\sqrt{1-y}$. By delaying his flip for a sufficiently long period, Bob can be assured of a zero-sum game. In particular, under the current technology, this time is less than 0.1 s for the ion trap and 10^{-5} for the optical cavity [15]. Thus, Alice's employment of quantum device poses no significant danger or difficulty in this quantum game. In fact, the same analysis holds for many other quantum games, for example, the prisoner dilemma [7] or coin-tossing experiments.

In summary, we have shown that unless Alice can control the noise in the system completely, she stands to lose her advantages through the utilization of quantum devices. Furthermore, our analysis can be easily applied to the N-state generalization of Meyer's quantum game [9]. It is also interesting to explore its behavior under a corrupted source [14].

ACKNOWLEDGMENT

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