<table>
<thead>
<tr>
<th>Title</th>
<th>Relation between geometric phases of entangled bipartite systems and their subsystems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>D. M. Tong, E. Sjoqvist, L. C. Kwek, C. H. Oh and M. Ericsson</td>
</tr>
<tr>
<td>Published by</td>
<td>American Physical Society</td>
</tr>
</tbody>
</table>

This document may be used for private study or research purpose only. This document or any part of it may not be duplicated and/or distributed without permission of the copyright owner.

The Singapore Copyright Act applies to the use of this document.


The final publication is also available via [http://dx.doi.org/10.1103/PhysRevA.68.022106](http://dx.doi.org/10.1103/PhysRevA.68.022106)

© 2003 American Physical Society

 Archived with permission from the copyright owner.
I. INTRODUCTION

In 1984, Berry [1] proposed in a seminal paper that a quantum system in a pure state undergoing adiabatic cyclic evolution acquires a geometric phase. This discovery has prompted a myriad of activities on various aspects of the geometric phase in many areas of physics ranging from optical fibers to anyons. Simon [2] subsequently recast the mathematical formalism of Berry’s phase within the language of differential geometry and fiber bundles. While it is possible to consider Berry’s phase under adiabatic evolution, the extension to nonadiabatic evolution is usually nontrivial. The general formalism for the nonadiabatic extension was formulated by Aharonov and Anandan [3,4]. Samuel and Bhandari [5] further generalized the geometric phase by extending it to noncyclic evolution and sequential measurements. Further relaxations of the adiabatic, unitary, and cyclic properties of the evolution have since been carried out [6–8].

The concept of the geometric phase of mixed states has also been developed. Uhlmann [9] addressed this issue within the mathematical context of purification. Sjöqvist et al. [10] introduced a formalism that defines the mixed state geometric phase within the experimental context of quantum interferometry. As pointed out by Slater [11], these two approaches are not equivalent, and Ericsson et al. [12] have recently shown that the parallel transport conditions used in the two approaches lead to generically distinct phase holonomy effects for entangled systems undergoing certain local unitary transformations. Singh et al. [13] have given a kinematic formalism of the mixed state geometric phase with nondegenerate eigenvalues and generalized the analysis of Ref. [10] to degenerate states. Extensions of Ref. [10] to the off-diagonal case [14] and completely positive maps [15] have also been given. An experimental test of Ref. [10] in the qubit case has been reported [16], using nuclear magnetic resonance technique.

The geometric phase of entangled states should be another issue worthy of attention. It may have a potential application in holonomic quantum computation since the study of entangled spin systems effectively allows us to contemplate the design of a solid state quantum computer [17]. Moreover, it has been found that one could in principle devise more robust fault-tolerant quantum computation using the notion of geometric phase in designing a conditional phase shift gate [18–20]. Since the geometric phase depends solely on the geometry of the intrinsic spin space, it is deemed to be less susceptible to noise from the environment. As for the geometric phase of entangled states itself, Sjöqvist [21] considered the geometric phase for a pair of entangled spin-1/2 systems in a time-independent uniform magnetic field, and the relative phase for polarization-entangled two-photon systems was considered by Hessmo and Sjöqvist [22]. Tong et al. [23] calculated the geometric phase for a pair of entangled spin-1/2 systems in a rotating magnetic field. Another interesting question, which was mentioned in previous papers but has not been completely discussed, is the relation between the geometric phase of the entangled state and those of the subsystems.

In this paper, we consider a general entangled bipartite system under an arbitrary bilocal unitary evolution. We extend the conclusions obtained from the special mode in [23] to the general case. That is, we prove that the geometric phase for cyclic entangled states with nondegenerate eigenvalues under bilocal unitary evolution can always be decomposed into a sum of weighted nonmodular pure state phases pertaining to the separable components of the Schmidt decomposition, irrespective of the forms of the local evolution operators. However, we also see that this property does not manifest itself in the case of noncyclic evolutions. Moreover, we investigate the relation between the geometric phase of pure entangled states of the system and that of mixed states of the subsystem and conclude that the geometric phase of
mixed states of one subsystem is different from that of entangled states in general even if we do not act on the other subsystem. We also point out that the two phases are equal when the evolution operator acts only on the considered subsystem and satisfies conditions where each pure state component of the Schmidt decomposition is parallel transported.

II. NONCYCLIC GEOMETRIC PHASE OF THE ENTANGLED STATE

We begin by considering a quantum system \( S \) consisting of two subsystems \( S_a \) and \( S_b \) subject to the bilocal unitary evolution \( U(t) = U_a(t) \otimes U_b(t) \). The states of the system belong to the Hilbert space \( \mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \), where \( \mathcal{H}_a \) and \( \mathcal{H}_b \) are two complex Hilbert spaces with dimensions \( N_a \) and \( N_b \), respectively. Vectors in \( \mathcal{H}_a \otimes \mathcal{H}_b \) can be expanded as Schmidt decompositions. Thus, any normalized initial state of the system can be written as

\[
|\Psi(0)\rangle = \sum_{i=1}^{N} \sqrt{p_i} |\mu_i\rangle \otimes |\nu_i\rangle,
\]

where \( |\mu_i\rangle \) and \( |\nu_i\rangle \) are orthonormal bases of \( \mathcal{H}_a \) and \( \mathcal{H}_b \), respectively, \( N = \min[N_a, N_b] \), and the Schmidt coefficients \( \{p_i\} \) satisfy \( p_1 + \cdots + p_N = 1 \). When \( U_a(t) \otimes U_b(t) \) acts on \( |\Psi(0)\rangle \), we obtain

\[
|\Psi(t)\rangle = \sum_{i=1}^{N} \sqrt{p_i} |\mu_i(t)\rangle \otimes |\nu_i(t)\rangle,
\]

where \( |\mu_i(t)\rangle = U_a(t) |\mu_i\rangle \) and \( |\nu_i(t)\rangle = U_b(t) |\nu_i\rangle \).

Since \( |\Psi(t)\rangle \) is a pure state, its geometric phase can be obtained by removing the dynamical phase from the total phase. As we know, when a pure state evolves from \( t = 0 \) to \( t = \tau \) along a path \( C \) in projective Hilbert space, the nonadiabatic geometric phase can be obtained as \( \gamma(\tau) = \alpha(\tau) - \beta(\tau) \) with total phase \( \alpha(\tau) = \arg \langle \Psi(0) | \Psi(\tau) \rangle \) and dynamical phase \( \beta(\tau) = -i \int_{0}^{\tau} \langle \Psi(t) | \dot{\Psi}(t) \rangle dt \). Hereafter, we use \( \alpha, \beta, \) and \( \gamma \) to mark the total, dynamical, and geometric phases, respectively; and we use \( t, \tau, \) and \( T \) to represent instantaneous time, finite time, and the period of cyclic evolution, respectively. With Eqs. (1) and (2), we obtain the total, dynamical, and geometric phases of the entangled state \( |\Psi(t)\rangle \) under a bilocal unitary evolution \( U_a(t) \otimes U_b(t) \) as

\[
\alpha_{ab}(\tau) = \arg \sum_{i,j=1}^{N} \sqrt{p_i p_j} \langle \mu_i | U_a(\tau) | \mu_j \rangle \langle \nu_i | U_b(\tau) | \nu_j \rangle,
\]

\[
\beta_{ab}(\tau) = \sum_{i=1}^{N} p_i \left( -i \int_{0}^{\tau} \langle \mu_i | \dot{U}_a(t) | \mu_j \rangle dt \right) + \sum_{i=1}^{N} p_i \left( -i \int_{0}^{\tau} \langle \nu_i | \dot{U}_b(t) | \nu_j \rangle dt \right),
\]

\[
\gamma_{ab}(\tau) = \alpha_{ab}(\tau) - \beta_{ab}(\tau).
\]

Equation (4) entails that the dynamical phase can always be separated into two parts corresponding to the evolution of each of the subsystems \( S_a \) and \( S_b \). However, the total phase as well as the geometric phase cannot be separated into two parts in general. The latter observation arises primarily from the entanglement of the two subsystems.

III. CYCLIC GEOMETRIC PHASE OF ENTANGLED STATES

In this section, we specialize the above discussion to cyclic states with nondegenerate Schmidt coefficients \( p_i \). Such states are characterized by the existence of a period \( T \) such that

\[
|\Psi(T)\rangle = e^{i\alpha_{ab}(T)} |\Psi(0)\rangle,
\]

that is,

\[
\sum_{i=1}^{N} \sqrt{p_i} |\mu_i(T)\rangle \otimes |\nu_i(T)\rangle = e^{i\alpha_{ab}(T)} \sum_{i=1}^{N} \sqrt{p_i} |\mu_i\rangle \otimes |\nu_i\rangle.
\]

As \( U_a(T) \) and \( U_b(T) \) are unitary, the vectors \( |\mu_i(T)\rangle \) and \( |\nu_i(T)\rangle \) are also orthonormal bases of \( \mathcal{H}_a \) and \( \mathcal{H}_b \), respectively. Moreover, under the consideration that the \( p_i \)'s are nondegenerate, i.e., that \( p_i \neq p_j \) for all pairs \( i, j \neq i \), the Schmidt decomposition is unique. So we have \( |\mu_i(T)\rangle \otimes |\nu_i(T)\rangle = e^{i\alpha_{ab}(T)} |\mu_i\rangle \otimes |\nu_i\rangle \), which implies

\[
\alpha_{ab}(T) = \arg \langle \langle \mu_i | U_a(T) | \mu_j \rangle | \nu_i | U_b(T) | \nu_j \rangle,
\]

where \( \arg \in [0, 2\pi] \). Since the left-hand side of Eq. (8) is independent of the summation index \( i \), \( \alpha_{ab}(T) \) can be written as

\[
\alpha_{ab}(T) = \sum_{i=1}^{N} p_i \left[ \arg \langle \langle \mu_i | U_a(T) | \mu_i \rangle | \nu_i | U_b(T) | \nu_i \rangle \right]
\]

\[
= \sum_{i=1}^{N} p_i \left[ \arg \langle \langle \mu_i | U_a(T) | \mu_i \rangle \right.
\]

\[
+ \arg \langle \langle \nu_i | U_b(T) | \nu_i \rangle | 2\pi \tilde{n}_i \rangle,
\]

where \( \tilde{n}_i \) are integers chosen as

\[
\tilde{n}_i = \begin{cases} 
0 & \text{if } \arg \langle \langle \mu_i | U_a(T) | \mu_i \rangle \rangle + \arg \langle \langle \nu_i | U_b(T) | \nu_i \rangle \rangle < 2\pi, \\
1 & \text{if } \arg \langle \langle \mu_i | U_a(T) | \mu_i \rangle \rangle + \arg \langle \langle \nu_i | U_b(T) | \nu_i \rangle \rangle \geq 2\pi.
\end{cases}
\]

Substituting Eqs. (4) and (9) into Eq. (5) yields

\[
\gamma_{ab}(T) = \sum_{i=1}^{N} p_i \left[ \gamma_{ai}(T) + 2\pi \tilde{n}_i \right],
\]

where
\[ \gamma_a(T) = \arg \langle \mu_i | U_a(t) | \mu_i \rangle e^{- \int_0^T \langle \mu_i | U_a'(t) | \mu_i \rangle dt}, \]
\[ \gamma_b(T) = \arg \langle \langle v_i | | U_b(T) | v_i \rangle e^{- \int_0^T \langle v_i | U_b'(t) | v_i \rangle dt} \]
are just the 2\pi-modular geometric phases of the pure states \(|\mu_i\rangle\) and \(|v_i\rangle\), respectively. The “winding numbers” \(n_i\) are integers and originate from the nonmodular nature of the pure state dynamical phases [6] and the integers \(\tilde{n}_i\). Equation (11) shows that the cyclic geometric phase for nondegenerate entangled states under a bilocal unitary evolution can always be decomposed into a sum of weighted pure state phases pertaining to the evolution of each Schmidt component. This is primarily a result of the uniqueness of the Schmidt decomposition on the time interval \([0,T]\). In such a procedure, each 2\pi \(n_i\) is the modulus 2\pi remainder of the corresponding dynamical phase.

To illustrate the significance of the winding numbers, consider a pair of qubits (two-level systems) initially in the entangled state
\[ |\Psi(0)\rangle = \sqrt{p_0} |0_a\rangle \otimes |0_b\rangle + \sqrt{p_1} |1_a\rangle \otimes |1_b\rangle \]
with \(p_0 \neq p_1\), and evolving under the influence of the time-independent Hamiltonian
\[ H = \hbar \omega \left| 1_a \right\rangle \left\langle 1_a \right| \otimes I_b \]
with \(\omega > 0\) and \(I_b\) the identity operator on \(\mathcal{H}_b\). As the Schmidt components \(|0_a\rangle \otimes |0_b\rangle\) and \(|1_a\rangle \otimes |1_b\rangle\) are eigenstates of \(H\), their corresponding 2\pi-modular geometric phases \(\{\gamma_{ab,0}, \gamma_{00}\}\) and \(\{\gamma_{11}, \gamma_{01}\}\) all vanish. However, they acquire different dynamical phases causing a nontrivial evolution of the entangled state. For one cycle \(T = 2\pi/\omega\) one obtains \(n_0 = 0\) and \(n_1 = 1\), yielding the total geometric phase
\[ \gamma_{ab}(T) = 2\pi p_1, \]
which is nontrivial for entangled states.

**IV. COMPARING THE PHASE OF ENTANGLED STATES WITH THAT OF MIXED STATES**

The above result concerning the phase of cyclic entangled states may provide some insight regarding the geometric phase of mixed states. Certainly, the state given by Eq. (2) is a pure state with density matrix \(\rho_{ab}(t) = |\Psi(t)\rangle \langle \Psi(t)|\). However, if we trace out the state of the subsystem \(S_b\), we obtain the reduced density matrix \(\rho_a\) corresponding to mixed states of the subsystem \(S_a\). From the reduced density matrix, we can deduce the geometric phase \(\gamma_a^M(T)\) of a mixed state. Here, we wish to examine, in the case where \(U_b(t)\) is the identity map \(I_b\), the relation between \(\gamma_{ab}(T)\) and \(\gamma_a^M(T)\).

From Eqs. (1) and (2), tracing out the state of the subsystem \(S_b\), we obtain the evolution of the reduced density matrix for subsystem \(S_a\) as
\[ \rho_a(t) = U_a(t) \rho_a(0) U_a^\dagger(t) \]
with
\[ \rho_a(0) = \sum_{i=1}^N p_i |\mu_i\rangle \langle \mu_i| . \]

For nondegenerate \(p_i\), the geometric phase of \(\rho_a(t)\) is found as [10,13]
\[ \gamma_a^M(\tau) = \arg \left( \sum_{i=1}^N p_i |\mu_i| U_a(\tau) |\mu_i\rangle e^{- \int_0^\tau \langle \mu_i | U_a'(t) | \mu_i \rangle dt} \right) . \]

This expression is valid for both cyclic and noncyclic states. In the cyclic case \(\tau = T\) we have \(\langle \mu_i | U_a(T) | \mu_i \rangle = 1\), and the above equation can be written as
\[ \gamma_a^M(T) = \arg \left( \sum_{i=1}^N p_i e^{i \gamma_a(\tau)} \right) , \]
where \(\gamma_a(T)\) is the geometric phase of the pure state \(|\mu_i\rangle\). Comparing Eq. (19) with Eq. (11) for \(U_b(t) = I_b\), we find \(\gamma_a^M(T) = \gamma_{ab}(T)\). Thus, the cyclic geometric phase of the whole system is in general different from that of the mixed state of the considered subsystem, basically because the former is a weighted sum of pure state phases while the latter is a weighted sum of pure state phase factors.

Similarly, for noncyclic evolution with \(U(t) = U_a(t) \otimes I_b\), we have in general
\[ \gamma_{ab}(\tau) = \gamma_a(\tau) \]
\[ = \arg \sum_{i=1}^N p_i \langle \mu_i | U_a(\tau) | \mu_i \rangle \]
\[ + i \int_0^\tau \sum_{i=1}^N p_i \langle \mu_i | U_a'(t) U_a(t) | \mu_i \rangle dt \]
and
\[ \gamma_a^M(\tau) = \arg \left( \sum_{i=1}^N p_i |\mu_i| U_a(\tau) |\mu_i\rangle e^{i \gamma_a(\tau)} \right) . \]

Thus, the geometric phase of the system \(S\) is dependent upon \(|\mu_i\rangle\) of the subsystem \(S_a\) but independent of \(|\nu_i\rangle\) of \(S_b\). Only the evolution of the subsystem \(S_a\) contributes to the geometric phase of the pure state system. Yet we find that, even in the case when the system’s geometric phase is completely determined by the evolution of the subsystem \(S_a\) and the Schmidt coefficients \(\{p_i\}\), \(\gamma_a(\tau)\) is generally different from \(\gamma_a^M(\tau)\) in both the cyclic and noncyclic case. It may seem unexpected, because only the subsystem \(S_a\) experiences a unitary evolution while the subsystem \(S_b\) is unaffected. The
geometric phase of the system \( S \) is attributed to \( U_a(t) \) only, and it seems natural to expect the phase obtained by the system to be same as that obtained by the subsystem \( S_a \), while regarding it as mixed state. However, they are different. This shows that the geometric phase of an entangled bipartite system is always affected by both subsystems.

V. PHASE RELATIONS UNDER PARALLEL TRANSPORT CONDITIONS

In this section, we again restrict our discussion to the case where \( U(t) = U_a(t) \otimes I_b \). As pointed out above, even in this case, the geometric phase of the system, which is determined only by the evolution of the subsystem \( S_a \) and the Schmidt coefficients \( \{ p_i \} \), is generally different from that of the corresponding mixed state. We now try to find the reason for the difference and give conditions under which the two phases are equal.

Under the evolution \( U(t) = U_a(t) \otimes I_b \), the state of the whole system is \( | \Psi(t) \rangle = \Sigma_{i=1}^{N} \sqrt{p_i} | U_a(t) | \mu_i \rangle \otimes | \nu_i \rangle \), which can also be expressed as the density matrix

\[
\rho(t) = \sum_{i=1}^{N} \sqrt{p_i} p_j [ U_a(t) | \mu_i \rangle \langle \mu_j | U_a^\dagger(t) ] \otimes | \nu_i \rangle \langle \nu_j | .
\]  

(22)

The corresponding mixed state of the subsystem \( S_a \) is

\[
\rho_a(t) = Tr_b \rho(t) = \sum_{i=1}^{N} p_i | U_a(t) | \mu_i \rangle \langle \mu_i | U_a^\dagger(t) .
\]  

(23)

When \( U_a(t) \) is given, the states \( \rho(t) \) and \( \rho_a(t) \) are definite, but when \( \rho(t) \) or \( \rho_a(t) \) is given, the evolution operator is not unique. That is, for a given path in state space, there are infinitely many unitary operators that realize the same path and so give the same geometric phase. All the operators form an equivalence set: two evolution operators are "equivalent" if and only if they realize the same path. If we know any one operator out of the equivalence set, say \( \bar{U}_a(t) \), we can write down all the operators of the set. For \( \rho(t) \), the equivalence set is

\[
\mathcal{S}_1 = \{ \bar{U}_a(t) e^{i\theta(t)} \},
\]  

(24)

where \( \theta(t) \) is an arbitrary real-valued gauge function of \( t \) with \( \theta(0) = 0 \). For the state \( \rho_a(t) \), the equivalence set is

\[
\mathcal{S}_2 = \{ \bar{U}_a(t) \sum_{i=1}^{N} e^{i\theta_i(t)} | \mu_i \rangle \langle \mu_i | \}
\]  

(25)

where \( \theta_i(t), i = 1, 2, \ldots, N \), are arbitrary real-valued gauge functions of \( t \) with \( \theta_i(0)=0 \). We see that the two sets are different in general and \( \mathcal{S}_1 \subset \mathcal{S}_2 \), which shows that the evolution operators that give the same path for \( \rho_a(t) \) may give different paths for \( \rho(t) \). So the two kinds of geometric phases \( \gamma_a(\tau) \) and \( \gamma_a^M(\tau) \) cannot be the same in general, otherwise they would be associated with the same equivalence sets of evolution operators. To see exactly the difference between the two phases, we substitute

\[
U_a(t) = \bar{U}_a(t) \sum_{i=1}^{N} e^{i\theta_i(t)} | \mu_i \rangle \langle \mu_i | ,
\]  

(26)

into Eqs. (18) and (20), respectively, and get

\[
\gamma_a^M(\tau) = \arg \left( \sum_{i=1}^{N} p_i | \mu_i \rangle \langle \mu_i | \bar{U}_a(t) | \mu_i \rangle e^{- \int_0^\tau \langle \mu_i | \bar{U}_a(t) | \mu_i \rangle dt} \right).
\]  

(27)

\[
\gamma_a(\tau) = \arg \sum_{i=1}^{N} p_i | \mu_i \rangle \langle \mu_i | \bar{U}_a(t) | \mu_i \rangle e^{i\theta_i(t)} + i \int_0^\tau \sum_{i=1}^{N} p_i | \mu_i \rangle \bar{U}_a(t) | \mu_i \rangle dt - \sum_{i=1}^{N} p_i \theta_i(\tau).
\]  

(28)

We see that \( \gamma_a^M(\tau) \) is invariant under the choice of the member in \( \mathcal{S}_2 \), while \( \gamma_a(\tau) \) is not as it depends upon \( \theta_i(t) \).

With the above analysis, we can conclude that, for a given local evolution operator \( U_a(t) \), the two phases \( \gamma_a(\tau) \) and \( \gamma_a^M(\tau) \) are different in general. But when can they be the same, that is, for what kinds of \( U_a(t) \) can we consider the two phases to be the same? We prove that when the evolution operator \( U_a(t) \) satisfies "the stronger parallel transport conditions" [10]

\[
\langle \mu_i | \bar{U}_a(t) | \mu_i \rangle = 0, \quad i = 1, 2, \ldots, N ,
\]  

(29)

the two phases are the same. Substituting Eq. (29) into Eqs. (18) and (20), we find

\[
\gamma_a(\tau) = \gamma_a^M(\tau) = \arg \left( \sum_{i=1}^{N} p_i | \mu_i \rangle \langle \mu_i | \bar{U}_a(t) | \mu_i \rangle \right)
\]  

(30)

and

\[
\gamma_b(\tau) = \gamma_b^M(\tau) = 0.
\]  

(31)

Equations (30) and (31) show that the two kinds of geometric phase for unitaries of the form \( U_a(t) \otimes I_b \) are always the same when the evolution operator \( U_a(t) \) satisfies the stronger parallel transport conditions Eq. (29).

Substituting \( U_a(t) = V_a(t) \sum_{i=1}^{N} e^{i\theta_i(t)} | \mu_i \rangle \langle \mu_i | \) into Eq. (29), we get the general form of the evolution operators satisfying the stronger parallel transport conditions

\[
U_a(t) = V_a(t) \sum_{i=1}^{N} | \mu_i \rangle \langle \mu_i | e^{- \int_0^\tau | \mu_i \rangle \langle \mu_i | V_a^\dagger(t) V_a(t) | \mu_i \rangle dt},
\]  

(32)

where \( V_a(t) \) is an arbitrary unitary operator. So we see that when \( U_a(t) \) has the form of Eq. (32), the geometric phase of the pure state of the system under the evolution \( U(t) = U_a(t) \otimes I_b \) is the same as that of the mixed state of subsystem \( S_a \) under the evolution \( U_a(t) \). The phase relations are shown as Eqs. (30) and (31).

It might also be useful to consider the problem from an operational point of view. Let us first consider the two-particle Franson type interferometry setup shown in the up-
Note that $\rho_a(\tau) = \text{Tr}_b |\Psi(0)\rangle \langle \Psi(0)|$ is measured as a shift at $D_a$ by ignoring the subsystem $b$.

The difference between the geometric phases of bipartite systems and their parts has its primary cause in entanglement and thus vanishes in the limit of separable states. We hope that the present analysis may trigger multiparticle experiments to test the difference between the phases of entangled systems and of their subsystems.


c\) 
Suppose now that we first arrange these experiments so that $\hat{U}_a(t)$ yields the pure state parallel transport condition $\langle \Psi(0)| \hat{U}_a(t) \hat{U}_a(t) \otimes I_b|\Psi(0)\rangle = 0$. In this case, the interference pattern obtained by measuring on both subsystems in coincidence would be shifted by the pure state geometric phase $\gamma_a(\tau)$. On the other hand, measuring only on subsystem $a$ would yield the same shift, but the interpretation is different: it is the total phase $\alpha_a^M(\tau)$ acquired by $\rho_a(\tau)$, which is in general at variance with $\gamma_a^M(\tau)$ as this latter phase is based upon the stronger parallel transport conditions. The situation is different when setting up $U_a(t)$ so as to parallel transport $\rho_a(\tau)$, i.e., by implementing $U_a(t)$ according to Eq. (32).

Here, the coincidence and marginal interference patterns are shifted by $\gamma_a(\tau)$ and $\gamma_a^M(\tau)$, respectively, in accordance with the above analysis.

\section{VI. Conclusion and Remarks}

We have discussed geometric phases of entangled states of bipartite systems under bilocal unitary evolution and of the mixed states of their subsystems. We draw the following conclusions.

(1) The cyclic geometric phase for entangled states with nondegenerate eigenvalues under bilocal unitary evolution can always be decomposed into a sum of weighted nonmodular pure state phases pertaining to the separable components of the Schmidt decomposition, irrespective of the forms of the local evolution operators, although the same cannot be said for the noncyclic geometric phase.

(2) The mixed state geometric phase of one subsystem is generally different from that of the entangled state even if the other subsystem is kept fixed, although it seems as if the two phases might be the same. However, when the evolution operator satisfies the stronger parallel transport conditions for mixed states, the two phases are the same, and the general forms of the operators are given.

\section{Acknowledgments}

The work of D.M.T. was supported by NUS Research Grant No. R-144-000-054-112. This work is also supported in part by the Agency for Science, Technology and Research, Singapore under the ASTAR Grant No. 012-104-0040 (WBS: R-144-000-071-305). E.S. acknowledges financial support from the Swedish Research Council. M.E. acknowledges financial support from the Foundation BLANCEFLOR Boncompagni-Ludovisi, née Bildt.

FIG. 1. Two-particle interferometry setups to measure phases for internal degrees of freedom in the entangled pure state $|\Psi(0)\rangle$. In the upper panel, the pure state phase $\arg\langle \Psi(0)| U_a(\tau) \otimes I_b |\Psi(0)\rangle$ is measured as a shift in the coincidence interference oscillations obtained at detectors $D_a$ and $D_b$ by varying the $U(1)$ phase $\chi$. In the lower panel, the mixed state phase $\arg\text{Tr}[\rho_a(0) U_a(\tau)]$, with $\rho_a(0) = \text{Tr}_b |\Psi(0)\rangle \langle \Psi(0)|$ the mixed input state of subsystem $a$, is similarly measured as a shift at $D_a$ by ignoring the subsystem $b$.

\[ \mathcal{I}_{ab} \sim \mathcal{L}(\tau) + \rho_a(\tau) \otimes I_b |\Psi(0)\rangle |\Psi(0)\rangle \]

Thus, the relative phase $\arg\langle \Psi(0)| U_a(\tau) \otimes I_b |\Psi(0)\rangle$ shifts the interference oscillations when $\chi$ varies. To observe the phase of $\rho_a(\tau) = \text{Tr}_b |\Psi(\tau)\rangle \langle \Psi(\tau)|$ acquired by subsystem $a$ alone, we instead use the setup shown in the lower panel of Fig. 1 with the same source but where the subsystem $b$ is ignored. Here, the output intensity at $D_a$ reads [10]

\[ \mathcal{I}_a = 2 |\text{Tr}[\rho_a(0) U_a(\tau)]| \cos \chi \arg \text{Tr}[\rho_a(0) U_a(\tau)] \]

Note that $\text{Tr}[\rho_a(0) U_a(\tau)] = \langle \Psi(0)| U_a(\tau) \otimes I_b |\Psi(0)\rangle$ due to the triviality of the evolution of the $b$ subsystem.