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# General formalism of Hamiltonians for realizing a prescribed evolution of a qubit

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We investigate the inverse problem concerning the evolution of a qubit system, specifically we consider how one can establish the Hamiltonians that account for the evolution of a qubit along a prescribed path in the projected Hilbert space. For a given path, there are infinite Hamiltonians which can realize the same evolution. A general form of the Hamiltonians is constructed in which one may select the desired one for implementing a prescribed evolution. This scheme can be generalized to higher dimensional systems.

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## I. INTRODUCTION

Evolution of a quantum system is completely determined by the given dynamics. For a closed system, the density matrix  $\rho(t)$  of the system satisfies the evolution equation,

$$\rho(t) = U(t)\rho(0)U(t)^\dagger, \quad (1)$$

where  $U(t) = T \exp[-i \int_0^t H(t) dt]$  is the unitary operator corresponding to the Hamiltonian  $H(t)$ . For an open system, its evolution is generally nonunitary and therefore it cannot be described by the above equation. However, an open system, denoted as  $S_a$ , can always be regarded as a part of a larger closed system, denoted as  $S_{ab}$ , comprising  $S_a$  and its environment, denoted as  $S_b$ . The evolution of combined system  $S_{ab}$  now obeys Eq. (1) whereas the evolution of the open system  $S_a$  is described by the Kraus representation [1]

$$\rho(t) = \sum_{\mu} M_{\mu} \rho(0) M_{\mu}^\dagger, \quad (2)$$

where Kraus operators  $M_{\mu}$  satisfy  $\sum_{\mu} M_{\mu}^\dagger M_{\mu} = I$ . The Kraus operators are usually expressed as  $M_{\mu} = \langle \mu | U_{ab}(t) | 0 \rangle$ , where  $U_{ab}(t) = T \exp[-i \int_0^t H_{ab}(t) dt]$  is determined by the Hamiltonian  $H_{ab}(t)$  of the combined system and  $|\mu\rangle$  are the orthogonal bases of the environment. Clearly, given the Hamiltonian  $H(t)$  for the closed system or the Hamiltonian  $H_{ab}(t)$  for the combined system, the evolution of the density matrix  $\rho(t)$  is completely determined through Eq. (1) or Eq. (2).

It is interesting to study the inverse problem. Supposing the density matrix  $\rho(t)$  of a quantum system, be it closed or open, is given as a time-dependent function, how can one obtain the Hamiltonian? This is an interesting and nontrivial issue because oftentimes in experiment, one needs to prepare a quantum system which is expected to evolve along a given path on the surface or inside the Bloch sphere. Certainly, one can always find this Hamiltonian through trial and error. To this end, one may look for some trial Hermitian operators and require them to satisfy the evolution equation with the

given  $\rho(t)$ , however, this is not easy. Indeed, it may happen that even if the form of the Hamiltonian is eventually found, it may still be too difficult to set it up experimentally. A suitable Hamiltonian must not only serve the given evolution mathematically, it should also be realizable physically. In particular, the problem becomes very difficult for nonunitary evolutions of open systems albeit its solvability for unitary evolution of closed systems.

The purpose of this paper is to put forward a general approach for finding the appropriate Hamiltonian that determines a given evolution of the systems. We restrict our discussion to two-level system (the qubit) which is generally the most prevalent system used in quantum computation and quantum information. We will provide a general formalism of the Hamiltonians for realizing an arbitrary prescribed evolution of the qubit system. The Hamiltonians are not unique. Our formalism gives a set of the equivalent Hamiltonians. Both unitary and nonunitary evolutions are investigated.

## II. HAMILTONIANS FOR REALIZING UNITARY EVOLUTION OF A QUBIT

For a qubit system, the density matrix can be expressed as

$$\rho(t) = \frac{1}{2}(1 + \mathbf{r} \cdot \boldsymbol{\sigma}), \quad (3)$$

where  $\mathbf{r}$ , the Bloch vector, is a time-dependent vector function with  $|\mathbf{r}| \leq 1$  and  $\boldsymbol{\sigma}$  are the Pauli matrices. When we refer to a given evolution, it means that  $\rho(t)$ , as a matrix function of  $t$ , is explicitly given. In other words, the movement of the Bloch vector  $\mathbf{r}$  in Bloch sphere is known. The most general form of  $\rho(t)$  can be written as

$$\rho(t) = \frac{1}{2} \begin{pmatrix} 1 + r \cos \theta & r \sin \theta e^{-i\phi} \\ r \sin \theta e^{i\phi} & 1 - r \cos \theta \end{pmatrix}, \quad (4)$$

where  $r = r(t)$ ,  $\theta = \theta(t)$ , and  $\phi = \phi(t)$  are arbitrary time-dependent real functions with  $0 \leq r \leq 1$ ,  $0 \leq \theta \leq \pi$ ,  $0 \leq \phi \leq 2\pi$ . At  $t=0$ , we denote  $r(0)$ ,  $\theta(0)$ , and  $\phi(0)$  as  $r_0$ ,  $\theta_0$ , and  $\phi_0$ , respectively, and the general initial state reads

$$\rho(0) = \frac{1}{2} \begin{pmatrix} 1 + r_0 \cos \theta_0 & r_0 \sin \theta_0 e^{-i\phi_0} \\ r_0 \sin \theta_0 e^{i\phi_0} & 1 - r_0 \cos \theta_0 \end{pmatrix}. \quad (5)$$

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Our aim is to find the general form of Hamiltonians that realize the evolution defined by Eq. (4) with the initial state (5). For unitary evolution,  $r \equiv r_0$  is constant and there exist unitary operators  $U(t)$  satisfying

$$\rho(t) = U(t)\rho(0)U(t)^\dagger. \quad (6)$$

If we can find the operators  $U(t)$ , the Hamiltonians  $H(t)$  are easy to obtain as

$$H(t) = i\dot{U}(t)U(t)^\dagger. \quad (7)$$

Hence the problem reduces to the question of how to find the unitary operators  $U(t)$  satisfying Eq. (6). As mentioned before, the corresponding operators are not unique. There are infinitely many unitary operators  $U(t)$  that can realize the same evolution, all of which are equivalent for  $\rho(t)$ . We can find all the different but equivalent Hamiltonians by finding all the  $U(t)$ . To this end, we first construct one of the unitary operators. We find that the following operator

$$\tilde{U}(t) = \begin{pmatrix} \cos\frac{\theta-\theta_0}{2} e^{-i(\phi-\phi_0)/2} & -\sin\frac{\theta-\theta_0}{2} e^{-i(\phi+\phi_0)/2} \\ \sin\frac{\theta-\theta_0}{2} e^{i(\phi+\phi_0)/2} & \cos\frac{\theta-\theta_0}{2} e^{i(\phi-\phi_0)/2} \end{pmatrix} \quad (8)$$

satisfies Eq. (6) for an arbitrary evolution defined by Eqs. (4) and (5). With the operator  $\tilde{U}(t)$ , all the equivalent unitary operators can be constructed by

$$U(t) = \tilde{U}(t)V(t), \quad (9)$$

where  $V(t) \in U(2)$  and satisfies the following commutation relation,

$$[V(t), \rho(0)] = 0. \quad (10)$$

To provide an explicit form for matrix  $V(t)$ , we rewrite the initial density matrix  $\rho(0)$  in the form of the orthogonal decompositions,

$$\rho(0) = \sum_{k=1,2} w_k \rho_k(0), \quad (11)$$

where  $w_1 = (1+r_0)/2$ ,  $w_2 = (1-r_0)/2$ , and

$$\rho_k(0) = \frac{1}{2} \begin{pmatrix} 1 + \delta_k \cos \theta_0 & \delta_k \sin \theta_0 e^{-i\phi_0} \\ \delta_k \sin \theta_0 e^{i\phi_0} & 1 - \delta_k \cos \theta_0 \end{pmatrix}, \quad (12)$$

with  $\delta_1 = 1$ ,  $\delta_2 = -1$ . The above orthogonal decomposition is unique if  $r_0 \neq 0$  (we need not consider  $r_0 = 0$  because in this case the Bloch vector does not move at all during the unitary evolution). Thus, the unitary matrix  $V(t)$  satisfying Eq. (10) can be written as

$$V(t) = \sum_{k=1,2} e^{i\alpha_k} \rho_k(0), \quad (13)$$

where  $\alpha_k = \alpha_k(t)$  ( $k=1,2$ ) are arbitrary real parameters with  $\alpha_1|_{t=0} = \alpha_2|_{t=0} = 0$ . From Eqs. (9) and (13), the general form of the unitary operators  $U(t)$  is obtained as

$$U(t) = \tilde{U}(t) \sum_{k=1}^2 e^{i\alpha_k} \rho_k(0). \quad (14)$$

The corresponding general form of the Hamiltonians  $H(t)$  is

$$\begin{aligned} H(t) &= i\dot{\tilde{U}}(t)\tilde{U}(t)^\dagger - \sum_{k=1}^2 \dot{\alpha}_k \tilde{U}(t)\rho_k(0)\tilde{U}(t)^\dagger \\ &= \tilde{H}(t) - \sum_{k=1}^2 \dot{\alpha}_k \rho_k(t), \end{aligned} \quad (15)$$

where  $\tilde{H}(t) = i\dot{\tilde{U}}(t)\tilde{U}(t)^\dagger$  and  $\rho_k(t) = \tilde{U}(t)\rho_k(0)\tilde{U}(t)^\dagger$ . More explicitly, the Hamiltonians read

$$\hat{H}(t) = \frac{1}{2} \begin{pmatrix} \dot{\phi} - \dot{\alpha}_1(1 + \cos \theta) - \dot{\alpha}_2(1 - \cos \theta) & (-i\dot{\theta} - \dot{\alpha}_1 \sin \theta + \dot{\alpha}_2 \sin \theta) e^{-i\phi} \\ (i\dot{\theta} - \dot{\alpha}_1 \sin \theta + \dot{\alpha}_2 \sin \theta) e^{i\phi} & -\dot{\phi} - \dot{\alpha}_1(1 - \cos \theta) - \dot{\alpha}_2(1 + \cos \theta) \end{pmatrix}. \quad (16)$$

The formula expressed by Eq. (16) provides the recipe for finding all the Hamiltonians that can realize the prescribed evolution defined by Eq. (4) and (5). Since the evolution given by Eqs. (4) and (5) is arbitrary, the Hamiltonians given in the above equation are the most general ones. When a prescribed evolution  $\rho(t)$  is given, i.e.,  $r = r_0$  is a known constant, and  $\theta(t)$ ,  $\phi(t)$  are two known time-dependent functions, one can easily write down the corresponding Hamiltonians by directly substituting the variables into Eq. (16). Noting that there are two time-dependent parameters  $\alpha_k(t)$  ( $k=1,2$ ), which can be arbitrary real functions of  $t$  with  $\alpha_k(0) = 0$ , one can establish the suitable Hamiltonian for any special purpose by properly choosing the forms of  $\alpha_k(t)$ .

### III. HAMILTONIANS FOR REALIZING NONUNITARY EVOLUTION OF A QUBIT

We now turn to the nonunitary evolution of an open qubit system. For nonunitary evolution, the path traced out by the Bloch vector is a curve inside the Bloch sphere with variable  $r$ . In this case, Eq. (1) is no longer valid. To realize a prescribed evolution of the open system, we need to consider a larger closed system, comprising the open system and an ancillary system which can also be regarded as the environment of the open system. For simplicity, we take the ancillary system to be a qubit. We denote the qubit underprepared as  $a$  and the ancillary qubit as  $b$ . The two qubits constitute a combined system  $S_{ab}$ . The combined system is a closed one,

and its evolution is described by unitary operator  $U_{ab}(t)$ . The most general evolution of an open qubit system can still be described in terms of expressions (4) and (5) with variable  $r$ . Suppose the ancillary qubit is initially in the state  $|0\rangle\langle 0|$ , the initial state of the combined system can be written as  $\varrho(0) = \rho(0) \otimes |0\rangle\langle 0|$ , where  $\rho(0)$  is an arbitrary given initial state of the open system. Then, at time  $t$ , the density matrix  $\varrho(t)$  of the combined system reads

$$\varrho(t) = U_{ab}(t)\rho(0) \otimes |0\rangle\langle 0| U_{ab}(t)^\dagger. \quad (17)$$

The reduced density matrix of the open system is given by tracing out the state of the second qubit, i.e.,

$$\rho(t) = \text{tr}_b \varrho(t) = \text{tr}_b \{ U_{ab}(t)\rho(0) \otimes |0\rangle\langle 0| U_{ab}(t)^\dagger \}. \quad (18)$$

The problem of preparing an open qubit system for a given evolution reduces to that of finding the unitary operator

$U_{ab}(t)$  such that the reduced matrix  $\rho(t)$  remains the same as that assigned by Eqs. (4) and (5). To this end, we rewrite Eq. (18) by using the Kraus representation

$$\rho(t) = \sum_{\mu=0,1} M_\mu \rho(0) M_\mu^\dagger,$$

$$\sum_{\mu} M_\mu^\dagger M_\mu = I, \quad (19)$$

where the Kraus operators  $M_\mu = \langle \mu | U_{ab} | 0 \rangle$  and  $|\mu\rangle$  ( $\mu = 0, 1$ ) are the orthogonal bases of the ancillary qubit. If we know the Kraus operators, we may inversely deduce the unitary operators  $U_{ab}(t)$ . One readily verifies that the following operators  $\tilde{M}_\mu$  [3] satisfy Eq. (19),

$$\tilde{M}_0 = \begin{pmatrix} -\cos \frac{\theta}{2} \sin \frac{\theta_0}{2} - \sqrt{\frac{1-r}{1+r_0}} \sin \frac{\theta}{2} \cos \frac{\theta_0}{2} e^{i(\phi_0 - \phi)} & \cos \frac{\theta}{2} \cos \frac{\theta_0}{2} e^{-i\phi_0} - \sqrt{\frac{1-r}{1+r_0}} \sin \frac{\theta}{2} \sin \frac{\theta_0}{2} e^{-i\phi} \\ -\sin \frac{\theta}{2} \sin \frac{\theta_0}{2} e^{i\phi} + \sqrt{\frac{1-r}{1+r_0}} \cos \frac{\theta}{2} \cos \frac{\theta_0}{2} e^{i\phi_0} & \sin \frac{\theta}{2} \cos \frac{\theta_0}{2} e^{i(\phi - \phi_0)} + \sqrt{\frac{1-r}{1+r_0}} \cos \frac{\theta}{2} \sin \frac{\theta_0}{2} \end{pmatrix},$$

$$\tilde{M}_1 = \sqrt{\frac{r+r_0}{1+r_0}} \begin{pmatrix} \cos \frac{\theta}{2} \cos \frac{\theta_0}{2} e^{i\phi_0} & \cos \frac{\theta}{2} \sin \frac{\theta_0}{2} \\ \sin \frac{\theta}{2} \cos \frac{\theta_0}{2} e^{i(\phi + \phi_0)} & \sin \frac{\theta}{2} \sin \frac{\theta_0}{2} e^{i\phi} \end{pmatrix}. \quad (20)$$

The choice of the Kraus operators is not unique [2]. There are infinitely many (equivalent) operators  $M_\mu$  satisfying Eq. (19) for given  $\rho(t)$  and  $\rho(0)$ .  $\tilde{M}_\mu$  is only one special choice. Correspondingly, there are also infinitely many choices of  $U_{ab}(t)$  satisfying Eq. (18), so do  $H(t)$ . With the known Kraus operators  $\tilde{M}_\mu(t)$ , we can first construct one of the operator  $U_{ab}(t)$ , and denote it as  $\tilde{U}_{ab}(t)$ . In fact,  $\tilde{U}_{ab}(t)$  is a  $4 \times 4$  unitary matrix, where the elements in the first and the

third columns are completely determined by the Kraus operators  $\tilde{M}_\mu$  with  $\tilde{U}_{11} = (\tilde{M}_0)_{11}$ ,  $\tilde{U}_{21} = (\tilde{M}_1)_{11}$ ,  $\tilde{U}_{31} = (\tilde{M}_0)_{21}$ ,  $\tilde{U}_{41} = (\tilde{M}_1)_{21}$ ,  $\tilde{U}_{13} = (\tilde{M}_0)_{12}$ ,  $\tilde{U}_{23} = (\tilde{M}_1)_{12}$ ,  $\tilde{U}_{33} = (\tilde{M}_0)_{22}$ , and  $\tilde{U}_{43} = (\tilde{M}_1)_{22}$  while the elements in the other two columns are yet to be determined. The second and the fourth columns' elements can be chosen by ensuring  $U_{ab}(t)$  to be unitary. Without loss of generality, we choose  $\tilde{U}_{ab}(t)$  as

$$\tilde{U}_{ab}(t) = \begin{pmatrix} -c_s - r_- s_c e^{i(\phi_0 - \phi)} & -r_+ s_s e^{-i\phi} & c_c e^{-i\phi_0} - r_- s_s e^{-i\phi} & r_+ s_c e^{-i(\phi + \phi_0)} \\ r_+ c_c e^{i\phi_0} & -s_c e^{i(\phi_0 - \phi)} - r_- c_s & r_+ c_s & -s_s e^{-i\phi} + r_- c_c e^{-i\phi_0} \\ -s_s e^{i\phi} + r_- c_c e^{i\phi_0} & r_+ c_s & s_c e^{i(\phi - \phi_0)} + r_- c_s & -r_+ c_c e^{-i\phi_0} \\ r_+ s_c e^{i(\phi + \phi_0)} & c_c e^{i\phi_0} - r_- s_s e^{i\phi} & r_+ s_s e^{i\phi} & c_s + r_- s_c e^{i(\phi - \phi_0)} \end{pmatrix}, \quad (21)$$

where  $s_s = \sin \theta / 2 \sin \theta_0 / 2$ ,  $s_c = \sin \theta / 2 \cos \theta_0 / 2$ ,  $c_s = \cos \theta / 2 \sin \theta_0 / 2$ ,  $c_c = \cos \theta / 2 \cos \theta_0 / 2$ , and  $r_+ = \sqrt{r+r_0}/(1+r_0)$ ,  $r_- = \sqrt{1-r}/(1+r_0)$ .

Expression (21) is only one of the many unitary operators

$U_{ab}(t)$  satisfying Eq. (18). To construct other equivalent unitary operators, we need to find other equivalent Kraus operators  $M_\mu$ . In terms of the known operators given by Eq. (20), the equivalent Kraus operators can be written as

$$M_\mu = \sum_{\nu=0,1} W_{\mu\nu} \tilde{M}_\nu V(t), \quad (22)$$

where  $W_{\mu\nu}$  are the elements of time-dependent matrix  $W(t) \in SU(2)$ , and  $V(t) \in U(2)$  satisfies Eq. (10). The explicit expression of  $V(t)$  is given by Eq. (13) if  $r_0 \neq 0$  and it can be any arbitrary  $2 \times 2$  unitary matrix if  $r_0 = 0$ . Equation (22) is the general expression of the Kraus operators realizing the prescribed evolution of the open system. With the general Kraus operators  $M_\mu$ , we can construct the general expression of unitary operators  $U_{ab}(t)$ . By using Eq. (22), the equivalent set of the unitary operators can be described by

$$U_{ab}(t) = [I \otimes W(t)] \tilde{U}_{ab}(t) [V(t) \otimes I]. \quad (23)$$

Substituting Eq. (23) into Eq. (7), one obtains the Hamiltonians

$$H_{ab}(t) = i \dot{U}_{ab}(t) U_{ab}(t)^\dagger. \quad (24)$$

For the prescribed evolution of the open system, there are many possible choices of Hamiltonians, one of which is given by  $\tilde{H}_{ab}(t) = i \dot{\tilde{U}}_{ab}(t) \tilde{U}_{ab}(t)^\dagger$  while the others are obtained by Eq. (24). For a given prescribed evolution  $\rho(t)$ , we can immediately write down the Hamiltonians by substituting the corresponding  $r = r(t)$ ,  $\theta = \theta(t)$ ,  $\phi = \phi(t)$  into Eqs. (21), (23), and (24). All the Hamiltonians fulfil the prescribed evolution of the open system. As the matrices  $W \in SU(2)$  and  $V(t) = \sum_{k=1,2} e^{i\alpha_k} \rho_k(0)$  if  $r_0 \neq 0$ , or  $V(t) \in U(2)$  if  $r_0 = 0$ , there are five or seven pending parameters to serve any other special requisitions. One can obtain any required Hamiltonian by choosing properly the pending parameters.

#### IV. EXAMPLE FOR APPLICATIONS

In the preceding sections, we develop a general procedure of Hamiltonians for realizing a prescribed evolution of a qubit system, be it closed or open. In spite of the complicated details, it is still a useful device due to its generality. As an illustration of its applications, we consider the measurement of geometric phase as an example.

As we know, when a system evolves along a given path in the project Hilbert space, the system will acquire the dynamic phase as well as the geometric phase. Geometric phase can be regarded as the total phase minus dynamic phase. For the mixed state of a qubit system, if the evolution is defined by Eqs. (4) and (5) with  $r = r_0$ , the geometric phase acquired by the system is [4,5]

$$\gamma(\tau) = \text{arg} \left\{ \sum_{k=1,2} \text{tr} \left[ w_k \rho_k(0) U(\tau) \right] \times \exp \left( - \int_0^\tau \text{tr} [\rho_k(0) U(t)^\dagger \dot{U}(t)] dt \right) \right\}, \quad (25)$$

where  $w_1 = (1 + r_0)/2$ ,  $w_2 = (1 - r_0)/2$ , and  $\rho_k(0)$  are defined by Eq. (12). Mathematically, we may use any one of

the unitary operators  $U(t)$  given by Eq. (14) to evaluate the geometric phase. All of them result in the same value of  $\gamma(\tau)$ . However, the experiment situation for phase measurement is quite different. The phase measured in the experiment is usually the total phase of the system, not the geometric phase. The different choices of  $U(t)$  will lead to different values of the measurement. If one wishes to measure the geometric phase, one must choose a special unitary operator satisfying the parallel transport conditions

$$\text{tr} [\rho_k(0) U(t)^\dagger \dot{U}(t)] = 0 \quad (k=1,2). \quad (26)$$

The special unitary operator will render the dynamical phase zero, hence the total phase equals the geometric phase. Substituting Eqs. (12) and (14) into Eq. (26), one will find that the parallel transport conditions constrain the functions  $\alpha_1(t)$  and  $\alpha_2(t)$  to be [6]

$$\alpha_1 = -\alpha_2 = \frac{1}{2} \int_0^t \cos \theta \dot{\phi} dt. \quad (27)$$

Substituting Eq. (27) into Eq. (16), the Hamiltonian is obtained as

$$H(t) = i \dot{U}(t) U(t)^\dagger = \frac{1}{2} \begin{pmatrix} \dot{\phi} \sin^2 \theta & (-i \dot{\theta} - \dot{\phi} \sin \theta \cos \theta) e^{-i\phi} \\ (i \dot{\theta} - \dot{\phi} \sin \theta \cos \theta) e^{i\phi} & -\dot{\phi} \sin^2 \theta \end{pmatrix}, \quad (28)$$

with which the measurement for the total phase results in the geometric phase. Let us be more specific. Suppose a qubit system, say an electron, with initial state  $\rho(0)$  ( $r_0 \neq 1$ ,  $\phi_0 = 0$ ) given by Eq. (5) is expected to evolve along a path with  $\theta = \theta_0$ ,  $\phi = \omega t$ ,  $t \in [0, 2\pi/\omega]$ . The evolution is a closed circle inside Bloch sphere. By using expression (28) the special Hamiltonian can be immediately written down as

$$H(t) = \frac{\omega \sin \theta_0}{2} \begin{pmatrix} \sin \theta_0 & -\cos \theta_0 e^{-i\omega t} \\ -\cos \theta_0 e^{i\omega t} & -\sin \theta_0 \end{pmatrix}. \quad (29)$$

The above Hamiltonian ensures the given state to evolve along the prescribed path with zero dynamic phase. The measurement result of the geometric phase should be  $\gamma = -\tan^{-1} \{ r_0 \tan[\pi(1 - \cos \theta_0)] \}$ . This Hamiltonian can be experimentally realized in physics by applying a time-dependent magnetic field,  $\mathbf{B}(t) = (B_1, B_2, B_3) = \omega \sin \theta_0 (-\cos \theta_0 \cos \omega t, -\cos \theta_0 \sin \omega t, \sin \theta_0)$ , to the electron system, and  $H(t) = \boldsymbol{\sigma} \cdot \mathbf{B}(t)/2$ . As the geometric phase has been used as quantum gate, the above discussion may be useful for quantum computation. As is well known, if an evolution causes nonzero dynamic phase, one must eliminate it by some means [7–9]. However, this cancellation is not easy. In our case, given the generic expression for the Hamiltonian, one can try to select those Hamiltonians for which the dynamic phase is zero. Recall that Eqs. (28) and (29) give no dynamical phase for any path of the state, be it closed or open. It is also interesting to consider noncyclic evolutions instead of cyclic closed circuit. In this context, we would like to mention a recent paper [10], where it is argued that some

systems may obey the same fault-tolerance properties like those of the geometric phase also for nonvanishing dynamical phase.

The above example provides a concrete application of our formalism in the form of the geometric phase of mixed state under unitary evolution. Similarly, we can also consider examples of mixed state under nonunitary evolutions. For instance, suppose we need to prepare a qubit system whose Bloch vector  $\mathbf{r}$  is expected to evolve along a prescribed path inside the Bloch sphere, say, an ellipse with  $r_x^2 + 4r_y^2 = 1$ ,  $r_z = 0$ . We can realize this evolution through the combined system of the qubit and an ancillary qubit. The unitary operator  $\tilde{U}_{ab}(t)$  can be easily calculated by substituting  $r = (\cos^2 \omega t + 4 \sin^2 \omega t)^{-1/2}$ ,  $\theta = \pi/2$ ,  $\phi = \omega t$ , and  $r_0 = 1$ ,  $\theta_0 = \pi/2$ ,  $\phi_0 = 0$  into Eq. (21). With  $\tilde{U}_{ab}(t)$ , the general unitary operators  $U_{ab}(t)$  and the general Hamiltonian  $H_{ab}(t)$  can be obtained by Eq. (23) and Eq. (24), respectively. Each Hamiltonian  $H(t)$  obtained in this way satisfies the same evolution with the appropriate reduced matrix  $\rho(t)$  along the given ellipse. If there are other physical constraints, one can then choose a preferred Hamiltonian by determining the suitable parameters in the Hamiltonians. As the Hamiltonians involve complex  $4 \times 4$  matrices again, we will not describe the example in detail in the present paper. Instead, we prefer to show an explicit form of the Hamiltonians for a simple example, the pure shrinking of the Bloch vector along the polar axis. In this case, by putting  $\theta = \theta_0 = 0$ ,  $r_0 = 1$  in Eq. (21), the Hamiltonian  $\tilde{H}_{ab}(t)$  is obtained as

$$\tilde{H}_{ab}(t) = i \tilde{U}_{ab}(t) \tilde{U}_{ab}(t)^\dagger = \frac{\dot{r}}{4\sqrt{1-r^2}} [\sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_x], \quad (30)$$

where  $\sigma_x$ ,  $\sigma_y$  are Pauli matrixes, and  $r(0 \leq r \leq 1)$  is an arbitrary prescribed time-dependent function with  $r \leq 0$ .  $\tilde{H}_{ab}(t)$  will render the Bloch vector of the state shrink from the pole (1,0,0) towards the center of the Bloch sphere with the prescribed speed  $|\dot{r}|$ . Other Hamiltonians equivalent to  $\tilde{H}_{ab}(t)$  can be obtained by using Eqs. (23) and (24).

## V. DISCUSSION AND CONCLUSIONS

The inverse problem for the evolution of a qubit is investigated in this paper. We proposed a general formalism to establish the Hamiltonian for a qubit system that is required to evolve along a particular path. Both unitary and nonuni-

tary evolutions of mixed states of a qubit system are discussed.

For a closed qubit system, its evolution is unitary. The general form of Hamiltonians for realizing a prescribed evolution is explicitly given by Eq. (16). One can directly write down the Hamiltonians by substituting  $\theta = \theta(t)$  and  $\phi = \phi(t)$  into Eq. (16). The explicit expression for the Hamiltonian contains two arbitrary time-dependent parameters. Regardless of the values of the parameters, all Hamiltonians give rise to the same evolution. One may choose appropriate parameters to satisfy the physical constraints.

For an open qubit system, its evolution is generally nonunitary. However, the system with its environment continues to evolve as a closed system. We therefore investigate the open system by considering the closed system constructed by adding an ancillary qubit to the open system. The evolution of the open system is expressed by the reduced density matrix given by Eq. (18). If the desired evolution is as given by Eqs. (4) and (5), the unitary operators  $U_{ab}(t)$  and the Hamiltonians  $H_{ab}(t)$  can be calculated by Eqs. (23) and (24). The expression of the Hamiltonians contain five or seven arbitrary parameters. All Hamiltonians of the given form cause the qubit to evolve along the desired path while they really act on the combined system. One may choose the suitable parameters for the Hamiltonians to meet any physical constraints, just like the unitary case.

As an example for illustrating the applications of our result, we discuss the parallel transport conditions on geometric phase measurement. A general expression of the Hamiltonians satisfying the conditions is given by Eq. (28). With the kind of Hamiltonian, the value of the measurement on total phase results in the geometric phase.

The study in present paper focuses on the qubit system. However, the approach can be generalized to higher dimensional systems. Some main formulas are similar. In fact, for unitary evolution of  $N$ -dimensional system, Eqs. (13)–(15) are still valid with  $k = 1, 2, \dots, N$ . For nonunitary evolution of  $N$ -dimensional open system, Eqs. (23) and (24) still hold. The difference for an  $N$ -dimensional system from the qubit system appears in  $\tilde{U}(t)$  given by Eq. (8) or  $\tilde{U}_{ab}(t)$  given by Eq. (21), which will be replaced by some new matrices corresponding to an  $N$ -dimensional system.

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- [1] K. Kraus, *States, Effects and Operations* (Spring-Verlag, Berlin, 1983).  
 [2] J. Preskill, Lecture notes: *Information for Physics 219/Computer Science 219, Quantum Computation*, www.theory.caltech.edu/people/preskill/ph229.  
 [3] D.M. Tong, Jing-Ling Chen, L.C. Kwek, and C.H. Oh, e-print quant-ph/0311091 (2003).  
 [4] E. Sjöqvist, A.K. Pati, A. Ekert, J.S. Anandan, M. Ericsson, D.K.L. Oi, and V. Vedral, Phys. Rev. Lett. **85**, 2845 (2000).

- [5] K. Singh, D.M. Tong, K. Basu, J.L. Chen, and J.F. Du, Phys. Rev. A **67**, 032106 (2003).  
 [6] The discussion is for mixed state  $r \neq 1$ . For pure state  $r = 1$  the constraint is that  $\alpha_1 = \frac{1}{2} \int_0^1 \cos \theta \dot{\phi} dt$  while  $\alpha_2$  is arbitrary.  
 [7] Givseppe Falci *et al.*, Nature (London) **407**, 355 (2000).  
 [8] I. Fuentes-Guridi *et al.*, Phys. Rev. Lett. **85**, 5018 (2000).  
 [9] Artur Ekert *et al.*, J. Mod. Opt. **47**, 2501 (2000).  
 [10] Shi-Liang Zhu and Z.D. Wang, Phys. Rev. Lett. **91**, 187902 (2003).