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Quantum tunneling time

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We present a calculation of quantum tunneling time based on the transition duration of wave peak from one side of a barrier to the other. In our formulation, the tunneling time comprises a real and an imaginary part. The real part is an extension of the phase tunneling time with quantum corrections whereas the imaginary time is associated with energy derivatives of the probability amplitudes.

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Quantum tunneling, where a particle has a probability to penetrate a classically forbidden region, is one of the most important applications and the striking features of wave mechanics. It is well known that the total barrier penetration probability may be calculated directly from the stationary Schrödinger equation. However, its tunneling time remains an undefined and controversial topic for many years (for a recent and rather extensive review, see Ref. [1]). Indeed, an outstanding problem regarding quantum tunneling time concerns the issue of causality in the particle propagation.

The classical definition for the duration of a collision event is unambiguous and often straightforward. The time is measured by observing the movement of a particle from one point in space to another. In quantum mechanics, however, the uncertainty principle disallows such a simple measurement and there is, therefore, a need to redefine the notion of tunneling time.

Quantum tunneling processes and the definition of quantum tunneling times have been investigated extensively by using numerical [2,3], experimental [4–9], and analytic methods in a number of works. There have been several methods on how tunneling time can be estimated. The concept of phase time [10–12] was calculated from the temporal delay of the transmitted wave packet so that the tunneling time is associated with an energy derivative of phase shift. In another approach, quantum tunneling time was obtained from the Wigner function of the propagating wave packet [12,13]. It is also possible to investigate quantum tunneling time using the local Larmor time. The local Larmor time [14–18] was defined by using an averaged spin component $\langle s_y \rangle$ of the particles due to the Larmor precession arising from a homogeneous magnetic field confined to the barrier region. In yet another popular formalism, Büttiker and Landauer proposed that one should study quantum tunneling time using the transmission coefficient through a static barrier augmented by a small oscillation in the barrier height. An equally popular method concerns the computation of dwell time. The dwell time [19,20] was expressed as the total probability of the particle within the barrier divided by the incident probability current, which may average over all scattering channels. Other methods, such as Bohmian trajectories [21], Feynman path integral [22–26], Nelson's quantum mechanics [27,28], variational approach [29], and dispersion relations [30] have also been attempted. In most of these cases, the tunneling time was often defined as the time spent by path of particle within the barrier. Despite several attempts to unify these concepts, there is still no preferred notion of quantum tunneling time.

It has been known for sometime that different arrangements for the tunneling process could lead to different relevant time scales. The controversy surrounding this question is evident in the historical development of the problem. Therefore, a clear and an unambiguous interpretation of the temporal process of tunneling is important not only for its possible applications but also for a fundamental interest in physical problem. In this paper, we propose a definition of the quantum tunneling time using the invariant property of time-space translation and compare our results with some previous works on the subject.

We first begin by considering a scattering process under a static square-well potential localized in the interval $(0,d)$,

$$V(x) = \begin{cases} 0 & (x<0), \\ V_0 & (0<x<d), \\ 0 & (x<d). \end{cases} \tag{1}$$

The Hamiltonian for the system is $\mathcal{H}=\hat{k}^2/2m+V(x)$. It is obvious for a constant potential that $[\mathcal{H},\mathcal{H}]=[\mathcal{H},\hat{k}]=0$. This implies that the energy and momentum of the system are conserved. It is well known that the energy and momentum conservation correspond to an invariance of the time-space translation. In other words, the space and time are homogeneous. Specifically, one can consider the transformation,

$$\mathcal{T}: \quad x \to x+a, \quad t \to t+s, \tag{2}$$

with a corresponding unitary operator $u(\mathcal{T})$ acting on the Hilbert space as

$$\begin{aligned} |\psi'\rangle &= u(\mathcal{T})|\psi\rangle \\ &= u(\mathcal{T}) \int dxdt|x,t\rangle\langle x,t|\psi\rangle \\ &= \int dxdt|x+a,t+s\rangle\psi(x,t) \\ &= \int dx'dt'|x',t'\rangle\psi(x'-a,t'-s), \end{aligned} \tag{3}$$

where $|\psi'\rangle$ and $|\psi\rangle$ are state vectors in the Hilbert space and $\psi(x,t)=\langle x,t|\psi\rangle$ is the wave function. According to Eq. (3), the wave function possesses the following translational symmetry,

$$\psi'(x,t) = \langle x,t|\psi'\rangle = \int dx'\,dt'\langle x,t|x',t'\rangle\psi(x'-a,t'-s)$$
$$= \int dx'\,dt'\,\delta(x'-x)\,\delta(t'-t)\,\psi(x'-a,t'-s)$$
$$= \psi(x-a,t-s). \tag{4}$$

We next suppose that the incoming particles are represented by a plane wave of unit amplitude and scattered by the potential $V(x)$ in Eq. (1). The incoming wave is partially reflected elastically and partially transmitted. Moreover, the wave function satisfies Schrödinger equation for a particle with energy $E=\hbar^2k^2/2m<V_0$ and assumes the following form:

$$\psi(x) = \begin{cases} e^{ikx-iEt/\hbar} + \sqrt{R(k)}e^{i\beta(k)-ikx-iEt/\hbar} & (x<0),\\ \sqrt{B(k)}e^{\kappa x-iEt/\hbar} + \sqrt{C(k)}e^{-\kappa x-iEt/\hbar} & (0<x<d),\\ \sqrt{T(k)}e^{i\alpha(k)+ikx-Et/\hbar} & (d<x), \end{cases} \tag{5}$$

where $V_0-E=\hbar^2(k_0^2-k^2)/2m=\hbar^2\kappa^2/2m$, $R(k)$ and $T(k)$ are the reflection and transmission probabilities, respectively; and $\beta(k)$ and $\alpha(k)$ are the corresponding phase shifts.

Since the values of a and s in Eq. (4) are arbitrary, we may set $a=s=0$ or $a=\pm\Delta x$, $s=0$ at points of $x=0$ and $x=d$ for arbitrary time t, respectively. Thus, one sees that

$$\psi'(0,t) = \psi(0,t), \quad \frac{\partial\psi'(0,t)}{\partial x} = \frac{\partial\psi(0,t)}{\partial x}, \quad \psi'(d,t) = \psi(d,t),$$
$$\frac{\partial\psi'(d,t)}{\partial x} = \frac{\partial\psi(d,t)}{\partial x}, \tag{6}$$

which are just some conditions for the continuity of the wave function and its space derivatives. Thus,

$$T(k) = 4k^2\kappa^2/D(k) = 1 - R(k), \tag{7}$$

$$\sqrt{B(k)} = \frac{k(\kappa+ik)}{\sqrt{D}}e^{-\kappa d+i\alpha(k)+ikd}, \tag{8}$$

$$\sqrt{C(k)} = \frac{k(\kappa-ik)}{\sqrt{D}}e^{\kappa d+i\alpha(k)+ikd}, \tag{9}$$

where

$$D(k) = 4k^2\kappa^2 + k_0^2\sinh^2(\kappa d), \tag{10}$$

and the scattering phases are

$$\alpha(k) + kd = \beta(k) + \frac{\pi}{2} = -\tan^{-1}\left[\frac{k^2-\kappa^2}{2\kappa k}\tanh(\kappa d)\right]. \tag{11}$$

Since $\psi(x_1,t_1)$ and $\psi'(x_2,t_2)$ in Eq. (4) can be regarded as the wave functions for the incoming (reflection) region and the transmitted region, respectively; we have

$$e^{i(kx_1-Et_1/\hbar)} + \sqrt{R(k)}e^{i[\beta(k)-kx_1-Et_1/\hbar]} = \sqrt{T(k)}e^{i[\alpha(k)+kx_2-Et_2/\hbar]}, \tag{12}$$

where the incoming plane wave and reflection wave packet may be combined as a single wave packet in the form of

$$e^{i(kx_1-Et_1/\hbar)} + \sqrt{R(k)}e^{i[\beta(k)-kx_1-Et_1/\hbar]}$$
$$= [\cos(kx_1) + \sqrt{R(k)}\cos(\beta(k)-kx_1)]e^{-iEt_1/\hbar}$$
$$+ i[\sin(kx_1) + \sqrt{R(k)}\sin(\beta(k)-kx_1)]e^{-iEt_1/\hbar}$$
$$= \sqrt{A_1(k)}e^{i[\gamma_1(k)-Et_1/\hbar]}, \tag{13}$$

with

$$A_1(k) = 1 + R(k) + 2\sqrt{R(k)}\cos(2kx_1-\beta), \tag{14}$$

and

$$\gamma_1(k) = \tan^{-1}\frac{\sin(kx_1) + \sqrt{R(k)}\sin[\beta(k)-kx_1]}{\cos(kx_1) + \sqrt{R(k)}\cos[\beta(k)-kx_1]}. \tag{15}$$

The wave packets between the two sides of the barrier can be assumed to distribute sharply at a given energy without much loss of generality. Using the peak positions of the two wave packets, Eq. (12) and its energy derivative are given by

$$\frac{\partial\ln\sqrt{A_1(k)}}{\partial E} + i\frac{\partial\gamma_1(k)}{\partial E} = \frac{\partial\ln\sqrt{T(k)}}{\partial E} + i\frac{\partial\alpha(k)}{\partial E}$$
$$+ \frac{\partial k}{\partial E}x_2 - i\frac{t_2-t_1}{\hbar}. \tag{16}$$

Thus, a temporal delay caused by the tunneling process can be expressed as

$$\Delta t_T = -i\frac{1}{v(k)}\left[\frac{\partial\ln\sqrt{T(k)}}{\partial k} - \frac{\partial\ln\sqrt{A_1(k)}}{\partial k}\right]$$
$$+ \frac{1}{v(k)}\left[x_2 - x_1 + \frac{\partial\alpha(k)}{\partial k} - \frac{\partial\gamma_1'(k)}{\partial k}\right], \tag{17}$$

where $v(k)=\hbar^{-1}dE/dk=\hbar k/m$ is the group velocity, which may be defined by assuming that $\psi(x_1)$ and $\psi'(x_2)$ are two plane waves in Eq. (12). Moreover,

$$\frac{d\gamma_1'(k)}{dk} = \frac{\sqrt{R}}{A_1(k)}\left[\frac{\partial\ln\sqrt{R(k)}}{\partial k}\sin(\beta-2kx_1)\right.$$
$$\left.+ [\sqrt{R(k)} + \cos(\beta-2kx_1)]\left(\frac{\partial\beta(k)}{\partial k} - 2x_1\right)\right]. \tag{18}$$

It is interesting to note that any corrections to the wave packets can be obtained through the term $\partial \ln\sqrt{R(k)}/\partial k$ in the temporal delay (17).

The tunneling time in Eq. (17) has real and the imaginary parts. The imaginary part also appeared in the path-integral approach [22–26], the Larmor procession in two perpendicular directions [16,17], and second-order expansion of Wigner distribution [31,32]. The imaginary time associated with the transmitted amplitude was first discovered under the calculation of the flux-flux correlation function [33,34]. The real time associated with a momentum derivative of phase shifts is similar to the results of phase time [10–13]. If the terms in Eq. (17), $\partial\ln\sqrt{A_1(k)}/\partial k$ and $\partial\gamma_1'(k)/\partial k$, are neglected, our imaginary time is exactly the same as the calculation of the flux-flux correlation function and our real time equals to the transmission time obtained by the Wigner function of wave-packet distribution [12]. The transmission phase time is in a complete agreement with classical tunneling process [10,21]. Therefore, the correction terms, $\partial\ln\sqrt{A_1(k)}/\partial k$ and $\partial\gamma_1'(k)/\partial k$, are the quantum-mechanical effects.

In fact, the phase time with the imaginary part may be obtained from our calculation by neglecting the reflection wave packet on the left side of Eq. (12). It is obvious from invariant property of the space-time translation (4) that this solution is merely an approximation. Therefore, our quantum tunneling time could be regarded as a more general formalism of phase time.

In the reflection region, we set

$$e^{i(kx_1-Et_1/\hbar)} + \sqrt{R(k)}e^{i[\beta(k)-kx_1-Et_1/\hbar]} = e^{i(kx_2-Et_2/\hbar)} + \sqrt{R(k)}e^{i[\beta(k)-kx_2-Et_2/\hbar]}, \tag{19}$$

which can be written as

$$\sqrt{A_1(k)}e^{i[\gamma_1(k)-Et_1/\hbar]} = \sqrt{A_2(k)}e^{i[\gamma_2(k)-Et_2/\hbar]}, \tag{20}$$

where $A_2(k)$ and $\gamma_2(k)$ may be obtained by replacing x_1 by x_2 in Eqs. (14) and (15), respectively.

Similarly, for the two peaks of the wave packets between two sides of Eq. (20), we have

$$-i\frac{\partial \ln\sqrt{A_1(k)/A_2(k)}}{\partial E} + \frac{\partial\,\gamma_1(k)}{\partial E} - \frac{\partial\,\gamma_2(k)}{\partial E} + \frac{t_2-t_1}{\hbar} = 0. \tag{21}$$

Thus, a time delay in the reflection region can be written as

$$\Delta t_R = -i\frac{1}{v(k)}\frac{\partial \ln\sqrt{A_2(k)/A_1(k)}}{\partial k} + \frac{1}{v(k)}\left(x_2 - x_1 + \frac{\partial\,\gamma_2'(k)}{\partial k} - \frac{\partial\,\gamma_1'(k)}{\partial k}\right), \tag{22}$$

where $d\gamma_2'/dk$ may be obtained by inserting x_2 on x_1 in Eq. (18). If the left side of Eq. (20) can be considered as the incoming plane wave, the real part of Eq. (22) is the same as the reflection delay calculated by using Wigner distribution [12,13]. It is obvious that our results involve quantum-mechanical corrections.

It is known that the transformation relation for the wave function (4) determines the continuity condition (6). However, Eq. (4) does not include spin effects. If the interaction between the spin and the magnetic field is considered within the potential barrier, Eqs. (4) and (6) should also be modified to take the spin effects into account under the space-time transformation.

Over the last few years, several numerical simulations and experimental papers presenting various indirect measurements of barrier traversal times have appeared. Some of these works seem to agree with the semiclassical time of

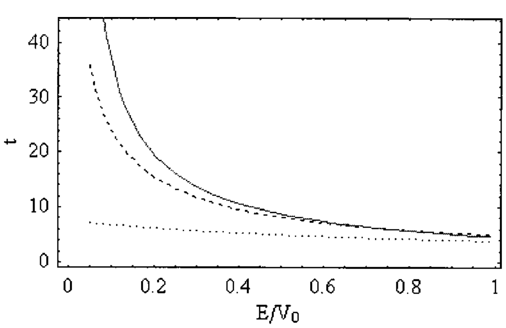

FIG. 1. A comparison of numerical traversal times for a thin barrier (width 25 Å, height 0.1 eV, effective mass $m^*=0.063m_e$), where t is in unit of femtosecond. The solid line is our result, the dotted line denotes Büttiker's expression, and the dashed line is the result for phase time.

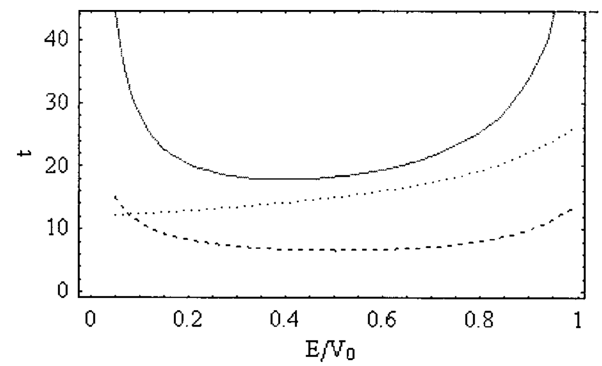

FIG. 2. Tunneling time as Fig. 1 for a thick barrier (width 110 Å, height 0.2 eV, effective mass $m^*=0.063m_e$).

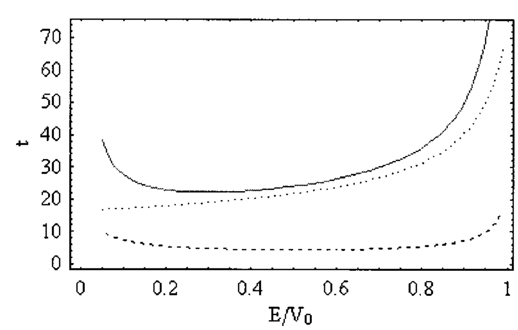

FIG. 3. Same as Fig. 1 for a thick barrier (width 200 Å, height 0.3 eV, effective mass $m^*=0.063m_e$).

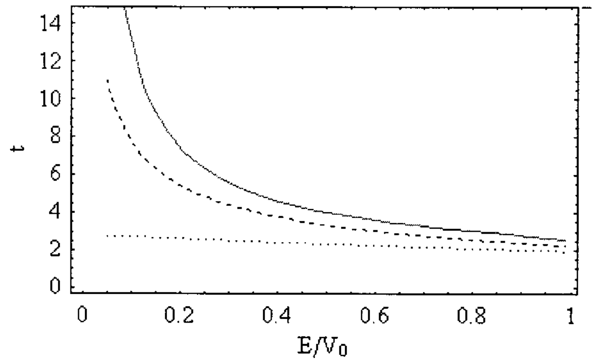

FIG. 4. A comparison of numerical traversal times for a very small effective mass and a thick barrier (width 200 Å, height 0.3 eV, effective mass $m^* = 0.00063 m_e$), where t is in units of femtoseconds. The solid line is our result, the dotted line is Büttiker's expression, and the dashed line is phase time.

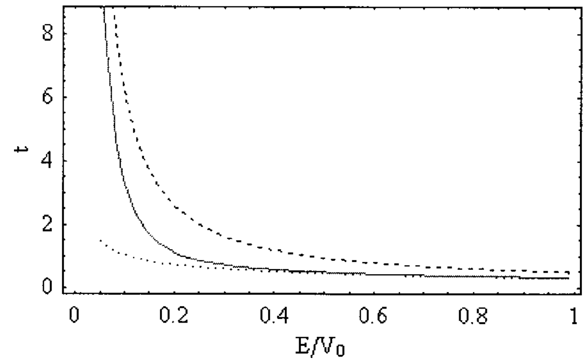

FIG. 5. Same as Fig. 4 for a thin barrier (width 25 Å, height 0.1 eV, effective mass $m^* = 0.00063 m_e$).

Büttiker and Landauer (BL time), while others seem to agree with phase time [2,5–7,35–37]. The typical results of such a sequence of experiments are usually illustrated using a typical thin or thick barrier. We have therefore chosen some physically plausible values appropriate for tunneling barriers in small GaAs structure (i.e., effective mass of $0.063 m_e$ was used—m_e being the bare electron mass). With appropriate units and setting the thin barrier as 25 Å and the barrier height as 0.1 eV, the tunneling time for our model is illustrated in Fig. 1. For a thick barrier corresponding to a barrier width of 110 Å and 200 Å with height of 0.2 eV and 0.3 eV, it is shown in Figures 2 and 3, respectively.

Figs. 1–3 also show a comparison of tunneling times for phase time and BL time with our result. In case of the thin barrier, our result is a good agreement with the phase time (see Fig. 1). For the thick barrier, our results tend to agree with the BL time for $E > 0.2 V_0$ (see Figs. 2 and 3). However, our curve appears to have a similar shape to the phase time. Therefore, it is not surprising why some experiments seem to agree with the BL time, while others seem to agree with the phase time for the different barrier.

The results are found to depend on the effective mass. When the effective mass decreases to a smaller value, our results tend to agree with the phase time for both thin and the thick barriers (see Figs. 4 and 5). Therefore, our tunneling process using photon as incoming beam agrees well with the phase time. This conclusion was also confirmed by the experimental results in Ref. [4].

It is noted in Eq. (17) that the phase time is an approximation of our quantum tunneling time. In addition, our tunneling time is directly defined by the symmetry property of the wave function. Therefore, Eq. (17) is a self-consistent tunneling time based on the homogeneity of the space-time.

The imaginary time may be regarded as the signal velocity of a truncated wave packet [38]. It is therefore important for dynamic tunneling events. Over the past few years, the imaginary time has been used to study baryon and lepton number violation processes in collision experiments in the TeV range [39,40]. This process is associated with the tunneling time between topologically different vacua in the standard electroweak model based on the baryon and lepton number anomaly. Finally, we reiterate that by using property of space-time transformation, we can define the quantum tunneling time as the transition time of wave peak between two sides of a barrier.

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