Asymmetric multipartite Greenberger-Horne-Zeilinger states and Bell inequalities

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We study the multiparticle generalized GHZ states. It has been shown that for an odd number of qubits and for a specific range of parameters, they do not violate any Bell inequality for correlation functions. We show here both analytically and numerically that, nevertheless, such states violate local realism, once a more detailed analysis of the correlations is made than the one allowed by correlation functions. The results imply that multiparticle Clauser-Horne-type inequalities involving probabilities are stronger tools for analyzing violations of local realism in multiparticle systems than inequalities involving the correlation functions.

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I. INTRODUCTION

Quantum entanglement is not only a fascinating phenomenon contradicting common sense (see, for instance, [1]) but also a valuable resource in quantum computation and quantum communication [2–5]. Therefore, it is crucial to understand the nature of entanglement from the perspective of fundamental research on quantum theory, as well as from the perspective of possible future applications.

Historically, the earliest tool for investigating quantum entanglement is the Bell inequality [6,7]. It states that correlations generated by some entangled states cannot be simulated in a classical way (by classical simulation we understand the existence of a local realistic model for the correlations). This means that there exist some quantum predictions (probabilities), given by the usual formula

$$p(A_1,A_2,\ldots,A_n) = \mathrm{Tr}\ \varrho \otimes_{i=1}^{n} \hat{P}_i(A_i)$$

[where ϱ is the density matrix of the (compound) system and $\hat{P}_i(A_i)$ is a projector on a one-dimensional subspace of the Hilbert space of the ith subsystem, dependent on some parameters A_i], which do not admit a (local realistic) model of the form

$$\int d\lambda \rho(\lambda) \prod_{i=1}^{n} P_i(A_i,\lambda),$$

where λ is a set of ("hidden") parameters over which we integrate or sum, $\rho(\lambda)$ is their classical probabilistic distribution, and $P_i(A_i|\lambda)$ is a (classical) probability, conditioned on λ.

Thanks to the work of Gisin [8], Gisin and Peres [9], and Popescu and Rohrlich [10] we know that all pure entangled states violate some inequality of Bell type. This holds also for the multiparticle states [first discussed in the famous Greenberger-Horne-Zeilinger (GHZ) papers [11]]. However, the recently discovered full set of generalized Bell inequalities for N-particle correlation functions [12–14] is not violated by some entangled states [15]—namely, for the generalized GHZ states

$$\sin \alpha|0,0,\ldots,0\rangle + \cos \alpha|1,1,\ldots,1\rangle, \tag{1}$$

for an odd number of qubits.

Here we show that using a generalization of the approach by Popescu and Rohrlich [10] one can reveal violations of local realism by these states. The Popescu-Rohrlich approach is based on the following idea. If one has an N-particle state and one wants to analyze its relation with local realism, the obvious method is to study N-particle correlations (via suitable Bell inequalities). However, in some cases, when N-particle correlations do not reveal directly violations of local realism, it is still possible to find such violations in $(N-1)$-particle correlations, conditioned on a specific local measurement of one of the observers: e.g., the Nth observer measures a certain observable and obtains a specific result. This (may) cause the $(N-1)$-particle state of the other observers to collapse to such a new state, for which the $(N-1)$-particle correlations (may) violate local realism. Since the result of the Nth observer, in principle, is obtained simultaneously with the measurements of the others, this procedure is not a postselection one, but rather a splitting of the full data in the N-particle experiment into subsets, such that one can pinpoint a subset data which cannot be modeled by local realism.

However, this method does not fully determine the strength of the nonclassicality of the correlations of the (generalized GHZ) states under consideration. To achieve that, one can use the numerical approach first presented in [16]. In this approach, via a linear optimization procedure the computer program seeks to find a local realistic model for the given set of data representing experimental results or quantum predictions. Such an approach, described in more detail later in the text, is equivalent to a test of the full set of Bell inequalities for the problem. Our analysis of the results produced by this procedure leads us to the conclusion that the

Bell inequalities for three- or more-qubit systems that impose the strictest restrictions against local realism are generalizations of the Clauser-Horne (CH) [17] inequality and not of the Clauser-Horne-Shimony-Holt (CHSH) [7] inequality. The distinction that we make here between CH-type and CHSH-type generalizations is the following one. CHSH inequalities involve only two-particle correlation functions. Their generalizations, like the N-particle Mermin-Klyshko inequalities [18] or the ones of Refs. [12–14], later denoted by WWWZB, involve N-particle correlation functions. The CH inequality involves also probabilities or, more precisely, both two-particle correlations and single-particle probabilities. Thus by generalized CH-type inequalities we mean inequalities that involve correlation functions of different order [e.g., N- and $(N-1)$-particle correlation functions]. Our result should motivate further research aimed at finding generalized CH inequalities for multiparticle systems.

This paper is developed as follows: in Sec. II, we briefly reviewed results concerning the violation of Bell inequalities by generalized GHZ states, and in Sec. III, we state and discuss Gisin's theorem. In this paper, we have used powerful numerical methods to maximize or minimize multivariable functions. We discuss our work and the numerical approach in Secs. IV and V. In Sec. VI, we summarize and discuss various implications of our results.

II. VIOLATIONS OF THE FULL SET OF CORRELATION-FUNCTION BELL INEQUALITIES BY GENERALIZED GHZ STATES

In [15,19], it was shown that for any odd number of qubits the generalized GHZ states

$$|\psi_\alpha^{(N)}\rangle = \cos\alpha|0,\ldots,0\rangle + \sin\alpha|1,\ldots,1\rangle \quad (2)$$

possess a surprising property. For $\sin 2\alpha \leqslant 1/\sqrt{2^{N-1}}$ they do not violate any standard correlation-function Bell inequality for $N, N-1,\ldots,2$ qubits. A Bell inequality will be called standard if it involves two settings for each observer. The fact that they do not violate such inequalities for $N-1,\ldots,2$ qubits is a simple consequence of the fact that the reduced density matrices for the subsystems of $N-1$ qubits are separable[1] and thus for all values of α there are no violations of inequalities for $(N-1)$ qubits.

For the full N-qubit system the whole set of correlation-function Bell inequalities (for Bell experiments allowing two alternative measurement settings for each observer) can be summarized in one generalized inequality [12–14]. For an odd number of qubits, this inequality, as mentioned, can be violated only for $\sin(2\alpha) \geqslant 1/\sqrt{2^{N-1}}$. The classical bound is violated by a factor of $\sin(2\alpha)\sqrt{2^{N-1}}$. The physical interpretation of this "violation factor" can be put as follows. Consider a mixture of the generalized GHZ state $|\psi^N\rangle$ with "white noise" $\rho_{(N)}^{\text{noise}}=(1/2^N)\mathbb{1}$ in the form of

$$\rho_{(N)} = V_{(N)}|\psi_\alpha^{(N)}\rangle\langle\psi_\alpha^{(N)}| + (1-V_{(N)})\rho_{(N)}^{\text{noise}}. \quad (3)$$

It can violate the generalized inequality only for $1 \geqslant V_{(N)} > 1/[\sin(2\alpha)\sqrt{2^{N-1}}]$. The parameter $V_{(N)}$ can be called the "visibility." It quantifies to what extent the properties of the state $|\psi_\alpha^{(N)}\rangle$ are visible operationally, despite the admixed noise. The other way to express the violation threshold is the following one. For a noise admixture $F_{(N)}$ satisfying $0 \leqslant F_{(N)} < 1-1/[\sin(2\alpha)\sqrt{2^{N-1}}] \leqslant 1$, one can always have a violation of the generalized inequality. Note that for $\sin(2\alpha) < 1/\sqrt{2^{N-1}}$ this cannot be the case, because both visibility and noise admixtures are ill defined.

III. GISIN'S THEOREM AND THE POPESCU-ROHRLICH METHOD

The original Gisin's theorem states that any pure nonproduct state of two particles violates the CHSH inequalities. It was later generalized to all pure states. The results summarized in the previous section suggest (superficially) that we have here a counterexample to this theorem. However, the generalized Bell inequality of [12–14] gives a necessary and sufficient condition for the existence of a local realistic model of *only* the N-correlation functions. We shall therefore apply the method of Popescu and Rohrlich [10] (see the Introduction) to show that all multiparticle entangled states violate local realism, provided some additional manipulations are allowed. Here, we briefly sketch their method.

We shall assume that the each qubit is sent to a different observer (i.e., we do not allow observables which apply to pairs of qubits, etc.; we allow only one-qubit measurements). The observers collect their data on a sufficiently big ensemble of N-tuplets of qubits (each being allowed to choose at random between some two local dichotomic observables). After they register the data, they broadcast their settings and results for all emissions. Knowing all the data, they can compute the correlation functions and put them into the generalized inequality. However, for $\sin(2\alpha) \leqslant 1/\sqrt{2^{N-1}}$ this inequality will not be violated.

Nevertheless, they may still perform a more refined analysis. They choose the instances within the full set of data for which one of them (say, observer N) was measuring the observable

$$\hat{O}_N^a = |a\rangle_{NN}\langle a| - |a^\perp\rangle_{NN}\langle a^\perp|, \quad (4)$$

where $\langle a|a^\perp\rangle=0$. After this, they select again, from this subset of their results, those for which the Nth observer obtained the result +1 (i.e., associated with the eigenstate $|a\rangle_N$).

Under those circumstances, the results obtained by the other $N-1$ observers are such as if they were observing events produced by the "collapsed" $(N-1)$-qubit state

$$|\psi^{(N-1)}\rangle = \cos\alpha\langle a|0\rangle_N|0,0,\ldots,0\rangle_{(N-1)} + \sin\alpha\langle a|1\rangle_N|1,1,\ldots,1\rangle_{(N-1)}. \quad (5)$$

This can be an entangled state. Moreover, if

[1]For all multiparty cuts.

$$|a\rangle = \sin \alpha|0\rangle + \cos \alpha|1\rangle, \tag{6}$$

the new state is the GHZ state for $N-1$ qubits, in which one observes the strongest violations of the generalized Bell inequalities for correlations function for $(N-1)$-particle systems.

Note that this happens for all values of α, for which $|\psi_\alpha^{(N)}\rangle$ is an entangled state—that is, for $0<\alpha\leq\pi/4$ (values larger than $\pi/4$ are excluded by definition).

Also, if some of the $k-1$ observers choose as one of their observables

$$\hat{O}_N^b = |b\rangle_{NN}\langle b| - |b^\perp\rangle_{NN}\langle b^\perp|, \tag{7}$$

with $\langle b|b^\perp\rangle=0$, and $|b\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, then we can choose a subset of the data from the experiment which contains only those cases in which observer N obtained +1, when measuring $\hat{O}_N^a$, while the observers $N-1, N-2,\ldots,N-k+1$ observing $\hat{O}_i^b$ $(i=N-1,\ldots,N-k)$ received +1. For the selected subset of data of the remaining $N-k$ observers, according to quantum mechanics, must be such as if the remaining qubits were in the $(N-k)$-qubit GHZ state:

$$\frac{1}{\sqrt{2}}|0,0,\ldots,0\rangle_{(N-k)} + \frac{1}{\sqrt{2}}|1,1,\ldots,1\rangle_{(N-k)}. \tag{8}$$

Of course, for suitable local settings such a set of data cannot have a local and realistic model.

To give a quantitative measure of the strength of the violation of local realism for such a data analysis one can again use the method based on testing the noise resistance of the nonclassicality of the correlations. It is straightforward to show that if the Nth observer gets a result +1 when measuring the observable $\hat{O}_N^a$ and $k-1$ observers also get +1 for measuring the observables $\hat{O}_i^b$, the initial noisy state

$$\rho^{(N)} = V^{(N)}|\psi_\alpha^{(N)}\rangle\langle\psi_\alpha^{(N)}| + (1-V^{(N)})\rho^{\text{noise}} \tag{9}$$

collapses into a state which is a product of the pure state of the remaining $N-k$ qubits with[2]

$$\rho^{(N-k)} = V^{(N-k)}|\text{GHZ}\rangle\langle\text{GHZ}| + (1-V^{(N-k)})\rho^{\text{noise}}. \tag{10}$$

The "visibility" $V^{(N-k)}$ is related to the initial one by

$$V^{(N-k)} = \frac{V^{(N)}\sin^2 2\alpha}{(1-V^{(N)}) + V^{(N)}\sin^2 2\alpha}. \tag{11}$$

Since the threshold for a violation of the generalized Bell inequality by a noisy GHZ state is the visibility $V^{(N-k)}=2^{-(N-k-1)/2}$ (see, e.g., [12]), the selected data when the remaining $N-k$ observers violate the inequality is when $V^{(N-k)}$ is above this value. Thus the threshold value of the visibility of the initial N-qubit state, $V_{\text{thr}}^{(N)}$, above which we have this phenomenon reads

$$V_{\text{thr}}^{(N)} = \frac{1}{1+\sin^2 2\alpha(\sqrt{2^{N-1-k}}-1)}. \tag{12}$$

Note (a) that the value of $V_{\text{thr}}^{(N)}$ is smallest for $k=1$ and (b) that $V_{\text{thr}}^{(N)}<1$ for all nonzero values of $\sin(2\alpha)$. This means (a) that the optimal procedure involves analysis of the data of the remaining $N-1$ observers conditioned on the Nth observer finding his qubit in state (6) and (b) that the generalized GHZ state always violates local realism.

The formula (10) implies that a local realistic description of the correlations is now excluded for all values of α—that is, for all generalized GHZ states. However, as we shall show in the next section, formula (10) does not represent the absolute threshold visibility of the nonclassical correlations. The true threshold visibility for $N=3$ and 5 is even lower, and one can conjecture that this must be the case for all (odd) N.

IV. LINEAR OPTIMIZATION METHOD

It has been mentioned above that there is a numerical method for checking if the given correlations can be simulated classically. This method gives necessary and sufficient conditions for the existence of a classical description of the given set of quantum prediction and is more general than the approach using Bell inequalities.

Consider a gedanken experiment involving N observers, each measuring K_i observables $(i=1,2,\ldots,N)$ on some quantum state ρ. The set of observables for the ith observer is denoted by $\hat{A}_{j_i}^i$, where $j_i=1,2,\ldots,K_i$. As we consider qubits, the measurement of each observable has two possible results which we denote by $a_1^i(j_i)$ and $a_2^i(j_i)$ [it is convenient to imagine that each observable has two detectors that are labeled by $a_1^i(j_i)$ and $a_1^i(j_i)$, where $j_i=1,2,\ldots,K_i$; $i=1,2,\ldots,N$]. During the experiment there is a certain probability that N detectors $a_{d_1}^1(j_1)$, $a_{d_2}^2(j_2),\ldots,a_{d_N}^N(j_N)$ will click in coincidence, which is denoted by $P_{\text{QM}}(a_{d_1}^1(j_1),a_{d_2}^2(j_2),\ldots,a_{d_N}^N(j_N))$ $(d_1,d_2,\ldots,d_N=1,2)$. By *correlations* observed in such an experiment we understand the set of $2^N\times K_1\times K_2\times\cdots\times K_N$ such probabilities. They are computed in the standard way—i.e.,

$$P_{\text{QM}}(a_{d_1}^1(j_1),a_{d_2}^2(j_2),\ldots,a_{d_N}^N(j_N)) = \text{Tr}[\rho\hat{P}_{d_1}^1(j_1)\otimes\hat{P}_{d_2}^2(j_2)\otimes\cdots\otimes\hat{P}_{d_N}^N(j_N)], \tag{13}$$

where $\hat{P}_1^i(j_i)$, $\hat{P}_2^i(j_i)$ are mutually orthogonal projectors such that $\hat{A}_{j_i}^i=a_1^i(j_i)\hat{P}_1^i(j_i)+a_2^i(j_i)\hat{P}_2^i(j_i)$ (spectral decomposition).

It has been shown by Fine [20] that correlations such as those in Eq. (13) can be simulated classically if and only if there exists a joint probability distribution $P(a_1^1(j_1),a_2^1(j_1);\ldots;a_1^N(j_N),a_2^N(j_N))$ with marginals:

[2]Indeed, the strength of violation can be defined as the minimal amount $1-V^{(N)}$ $(0\leq V^{(N)}\leq1)$ of pure noise that one adds to the state $|\text{GHZ}\rangle$ so that the resulting state does not admit a local realistic description. This quantity is sometimes refer to as the "visibility."

$$P_{\mathrm{CL}}(a^1_{d_1}(j_1),a^2_{d_2}(j_2),\ldots,a^N_{d_N}(j_N)) = \sum_{a^1_{d_1+1}(j_1)}\sum_{a^2_{d_2+1}(j_2)}\cdots\sum_{a^N_{d_N+1}(j_N)} P(a^1_1(j_1),a^1_2(j_1);a^2_1(j_2),a^2_2(j_2);\ldots;a^N_1(j_N),a^N_2(j_N)), \tag{14}$$

which are equal to these probabilities (the arithmetic on the indices is modulo 2). However, finding this joint probability distribution or proving its existence is a difficult task in general. Usually, one must resort to numerical methods (see the pioneering work in [21]).

To solve the above problem numerically we apply the following procedure (see, for instance, [16,21,22]). First, we note that the joint probability distribution is a set of $M=2^N \times K_1\times K_2\times\cdots\times K_N$ non-negative numbers (unknowns) that sum up to 1. These numbers must obey the set of $L=4\times K_1\times K_2\times\cdots\times K_N$ *linear* equations given by Eq. (14). Usually, the number of equations is much smaller than the number of unknowns ($L<M$), which means that the solution (if it exists) depends on $M-L$ non-negative parameters. As we are interested in *any* solution to the problem we can apply a very efficient method called linear programming (for details see [23]).

Linear programming is a method for maximizing (minimizing) a multivariable linear function the domain of which is a convex set. Obviously, if the domain is empty, then the maximum (minimum) cannot be found. Let us define a constant function on M variables (in our case these variables are probabilities belonging to a joint probability distribution we are looking for) constrained by the equations in Eq. (14), which define a convex set. We apply a linear programming algorithm to this function; i.e., we look, for instance, for its minimum. If the convex set is nonempty, then the algorithm (which can be implemented on a computer) finds any set of M variables satisfying Eq. (14) (this happens because the function is constant). However, the algorithm cannot find such a solution if the convex set is empty, which means that in this case the joint probability distribution fulfilling Eq. (14) does not exist. This, in turn, means that there is no classical description of quantum correlations.

The above method tells us if a classical description of the quantum correlations exists or not. However, it does not say anything about the strength of the quantum correlations, which we now define in the following way. First, we notice that the pure noise, defined as $\rho_{\mathrm{noise}}=(1/2^N)\mathbb{1}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}$ (in this expression we have the tensor product of N identity matrices), generates correlations that admit a trivial classical description. Having a state ρ that does not admit a classical description, we ask what is the minimal amount of noise $F_{\min}$ ($0\leqslant F_{\min}\leqslant 1$) that must be added to the state ρ so that the resulting state $\rho(F)=(1-F)\rho+F\rho_{\mathrm{noise}}$ can be described classically. The largest $F_{\min}$ is the highest strength of the quantum correlations.

To find $F_{\min}$ for a state ρ and for the given set of observables $\hat{A}^1_1,\hat{A}^1_2;\hat{A}^2_1,\hat{A}^2_2;\ldots;\hat{A}^{N-1}_1,\hat{A}^{N-1}_2;\hat{A}^N_1,\hat{A}^N_2$ we use the linear-programming method. This time the quantum probabilities read

$$\begin{aligned}P^F_{\mathrm{QM}}&(a^1_{d_1}(j_1),a^2_{d_2}(j_2),\ldots,a^N_{d_N}(j_N))\\ &=(1-F)P_{\mathrm{QM}}(a^1_{d_1}(j_1),a^2_{d_2}(j_2),\ldots,a^N_{d_N}(j_N))\\ &\quad+F/2^N.\end{aligned} \tag{15}$$

Here too, one can have a classical hidden-variable description if and only if there exists a joint probability distribution $P(a^1_1(j_1),a^1_2(j_1);\ldots;a^N_1(j_N),a^N_2(j_N))$ [this is not the same probability distribution as in Eq. (14) but we denote it by the same symbol for simplicity] returning the quantum probabilities (15) as marginals—i.e.,

$$\begin{aligned}P^F_{\mathrm{QM}}&(a^1_{d_1}(j_1),a^2_{d_2}(j_2),\ldots,a^N_{d_N}(j_N))\\ &=P_{\mathrm{CL}}(a^1_{d_1}(j_1),a^2_{d_2}(j_2),\ldots,a^N_{d_N}(j_N)),\end{aligned} \tag{16}$$

where $P_{CL}(a^1_{d_1}(j_1),\ldots,a^N_{d_N}(j_N))$ is given by Eq. (14).

In this case, the unknowns are the probabilities $P(a^1_1(j_1),a^1_2(j_1);\ldots;a^N_1(j_N),a^N_2(j_N))$ and the noise fraction F ($0\leqslant F\leqslant 1$). They are constrained by Eq. (15) and form a convex set. Let us define a multivariable linear function, with the domain being this convex set, in such a way that it returns F for any point in its domain. Using the linear-programming method, we find the minimum of this function—i.e., $F_{\min}$—which is the minimal admixture of noise we are looking for. Note that, in contrast to the problem described above, this linear-programming problem always has a solution $F_{\min}\leqslant 1$. This is so, simply because there always exists a joint probability distribution fulfilling Eq. (16) for $F=1$ (noise always has a classical description).

We must stress here that the above method finds $F_{\min}$ for the given set of observables measured by the observers; $F_{\min}$ depends on these observables. Usually, each observable is defined by some set of local parameters that can be changed by the respective observer. Therefore, we can consider $F_{\min}$ as the function of all local parameters associated with the observers' observables and we can find the maximum of $F_{\min}$ over these parameters. In this way we obtain the set of observables that should be measured by the observers on the state ρ so that $F_{\min}$ is as large as possible.

V. RESULTS OF THE NUMERICAL METHOD

Let us consider the experiment described above for three observers each measuring two observables on the quantum state of three qubits $\rho_{|\psi\rangle}=|\psi\rangle\langle\psi|$, with $|\psi\rangle=\cos(\alpha)|000\rangle+\sin(\alpha)|111\rangle$. The measured observables are of the form

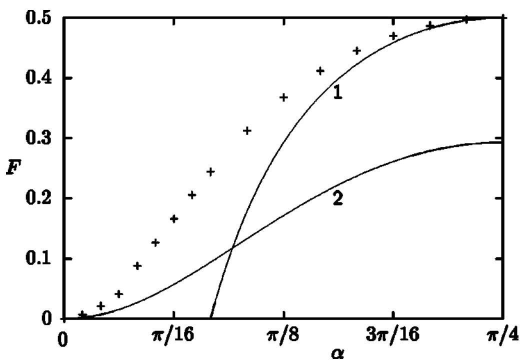

FIG. 1. Strength of nonclassicality of correlation (F) versus the angle (α) for the three-qubit generalized GHZ state. Line 1 shows the noise resistance of violations of local realism revealed by the generalized Bell inequality for correlation functions. Line 2 is the strength of the violation of local realism revealed by selected ensembles of two-particle correlated counts. Plus signs are results from the numerical method.

$$\hat{A}^i_{j_i} = a^i_1(j_i)U^\dagger(\phi^i_{j_i},\theta^i_{j_i})|0\rangle\langle 0|U(\phi^i_{j_i},\theta^i_{j_i}) + a^i_2(j_i)U^\dagger(\phi^i_{j_i},\theta^i_{j_i})|1\rangle\langle 1|U(\phi^i_{j_i},\theta^i_{j_i}), \quad (17)$$

where $a^i_1(j_i)=1$, $a^i_2(j_i)=-1$, and $U(\phi^i_{j_i},\theta^i_{j_i})$ is a SU (2) matrix and the angles $\phi^i_{j_i}$, $\theta^i_{j_i}$ which are the local macroscopic parameters of the ith observer. This means that the observers measure the most general quantum observables in this case [strictly speaking one could measure observables belonging to the U (2) group but it does not change the results presented here].

As shown in [15], none of the Bell inequalities presented in [12,14] are violated for $0<\alpha\leqslant\pi/12$. Let us apply the linear-programming method to this case. More specifically, we consider the state $\rho_{|\psi\rangle}(F)=(1-F)\rho_{|\psi\rangle}+\rho_{\text{noise}}$ for $\alpha\leqslant\pi/12$. We choose eight equally spaced points from the range $(0,\pi/12)$, i.e. $(0,\pi/96,\pi/48,\ldots,\pi/12)$, obtaining eight different asymmetric GHZ states. Each state is fed into the computer program that computes the largest possible $F_{\min}$ (we use HOPDM 2.13 [24] for the linear programming and amoeba [25] for optimization over the observables). The results are presented in Fig. 1. We clearly see that in the whole region $(0,\pi/12)$ (except for $\alpha=0$, in which case we have a separable state) there does not exist a classical description of the quantum correlations. Thus, the result is in concurrence with Gisin's for two particles.

We have performed a similar computation for five qubits for $\alpha=\pi/12$ and $\alpha=\pi/24$ with the results $F_{\min}=0.453$ and $F_{\min}=0.0638$, respectively; i.e., these states also violate local realism. As the computations are very time consuming in this case, we choose only two asymmetric GHZ states but there is very little doubt that the other states from the range $(0,\pi/12)$ violate local realism as well.

In both cases $N=3$ and 5, the numerical approach reveals stronger violations of local realism than the method of subensemble analysis.

VI. CONCLUSIONS

We have applied a method that gives necessary and sufficient conditions for the existence of a classical description of quantum correlations. This method, in general, cannot be used without the aid of a computer.

The method has been applied to the correlations obtained in the measurements of two observables by each observer on three and five qubits in states belonging to the family of asymmetric GHZ states that do not violate the generalized Bell inequality [15].

Nevertheless, if one adopts the ideas of Popescu and Rohrlich [10], violations of local realism can be spotted in specific subsets of the data collected by the observers. They are present for all α values, except $\alpha=0$ (which gives the product state). Still, the method based on linear optimization and the general necessary and sufficient condition for the existence of a local realistic model reveals violations of local realism for noise levels for which the subensemble of data analysis method fails.

Since in the case of three qubits the subensemble method is based on the analysis of two-particle correlations conditioned on a specific result obtained for the third one, it is effectively equivalent to the use of the CHSH inequalities for such data. However, it is well known that the full set of CHSH inequalities constitutes the sufficient and necessary condition for a local realistic description of two-particle correlations. Thus, the contribution of two-particle conditional correlations to the violations of local realism is never as strong as the contribution of the three-particle ones (it was shown in [15,26] that in the case of the W state, the situation is completely opposite). This is because there is no more refined analysis of two-particle correlations than the one based on CHSH inequalities.

So we have a seemingly paradoxical situation: the full set of correlation-function inequalities for the three-qubit correlations does not reveal any violation of local realism for $\sin(2\alpha)\leqslant\frac{1}{2}$, but the three-particle correlations are the main reason for the violation of the sufficient and necessary condition for local realism [14]. The solution to this puzzle is as follows. Pitovsky and Svozil [27] have derived, by algorithmic methods, 53 856 Bell inequalities for three-qubit systems. Recently they were reduced to 46 classes of equivalence [28]. The inequalities are a generalization of the Clauser-Horne ones [17], which are for probabilities of certain detection events (and not for N-particle correlation functions and thus in principle can impose a more stringent constraint on local realism than inequalities of the CHSH type). The totality of them forms the sufficient and necessary condition for local realism.

Therefore, we conclude that at least one of the three-particle Pitovsky-Svozil inequalities must be violated by the generalized GHZ state. Thus we have shown that, *in contrast to the case of two qubits, for which, in the case of arbitrary quantum states the Clauser-Horne and CHSH inequalities are fully equivalent, this is not so for three qubits*. The Pitovsky-Svozil CH-type inequalities form a more stringent constraint on local realism than the WWWZB correlation-function ones.

Finally, we stress that our analysis also shows that the violation of local realism for all generalized GHZ states even for the region of α, for which the WWWZB inequalities are not violated, can always be "blamed on" three-particle correlations. Two-particle correlations lead to much weaker violations. Conclusions of a similar kind can be drawn for five-qubit states and one could be tempted to conjecture that they apply to all generalized GHZ states.

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