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Scheme for unconventional geometric quantum computation in cavity QED

Xun-Li Feng,1,2 Zisheng Wang,1 Chunfeng Wu,1 L. C. Kwek,1,3 C. H. Lai,1 and C. H. Oh1

1Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117542
2State Key Laboratory of High Field Laser Physics, Shanghai Institute of Optics and Fine Mechanics, Chinese Academy of Sciences, Shanghai 201800, People’s Republic of China
3Nanyang Technological University, National Institute of Education, 1 Nanyang Walk, Singapore 637616
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In this paper, we present a scheme for implementing the unconventional geometric two-qubit phase gate with nonzero dynamical phase based on two-channel Raman interaction of two atoms in a cavity. We show that the dynamical phase and the total phase for a cyclic evolution are proportional to the geometric phase in the same cyclic evolution; hence they possess the same geometric features as does the geometric phase. In our scheme, the atomic excited state is adiabatically eliminated, and the operation of the proposed logic gate involves only the metastable states of the atoms; thus the effect of the atomic spontaneous emission can be neglected. The influence of the cavity decay on our scheme is examined. It is found that the relations regarding the dynamical phase, the total phase, and the geometric phase in the ideal situation are still valid in the case of weak cavity decay. Feasibility and the effect of the phase fluctuations of the driving laser fields are also discussed.

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I. INTRODUCTION

Quantum computation employs the principles of coherent superposition and quantum entanglement to solve certain problems, such as factoring large integers and searching data in an array, much faster than a classical computer [1]. The basic building blocks of a quantum computer are quantum logic gates. It has been shown that any quantum computation can be reduced to a sequence of two classes of quantum gates, namely, universal two-qubit logic gates and one-qubit local operations [2]. The standard paradigm of quantum computation is the dynamical one, where the local interactions between the qubits are controlled in such a way that one can enact a sequence of quantum gates. On the other hand, it has been recognized that the quantum gate operations can also be implemented through the geometric effects on the wave function of the systems; this is the so-called geometric quantum computation [3]. Compared with the dynamical gates, the geometric quantum computation possesses practical advantages. It is well known that geometric phases depend only on some global geometric features, and do not depend on the details of the path, the time spent, the driving Hamiltonian, and the initial and final states of the evolution [4]. Therefore geometric quantum computation is largely insensitive to local inaccuracies and fluctuations, and thus provides us a possible way to achieve fault-tolerant quantum gates.

In the implementation of geometric quantum computation, one practical question we usually meet is how to remove or avoid the dynamical phases, since geometric phases are generally accompanied by dynamical ones that are not robust against local inaccuracies and fluctuations. To this end one simple method is to choose the dark states as the qubit space; thus the dynamical phase is always zero [5]. Another general method is to let the evolution be dragged by the Hamiltonian along several special closed loops; then the dynamical phases accumulated in different loops may be canceled, with the geometric phases being added [6–8]. This is the so-called multiloop scheme.

The geometric quantum computation that is based on the cancellation of dynamical phases is referred to as conventional geometric quantum computation. Correspondingly, several schemes have been presented recently to realize so-called unconventional geometric quantum computation [9–13]. The central idea of unconventional geometric quantum computation is that, for certain quantum evolution of a quantum system of interest, one can implement fault-tolerant quantum computation by using the total phase accumulated in the evolution if it depends only on global geometric features of the evolution. In comparison with conventional geometric gates, unconventional geometric gates do not require additional operations to cancel the dynamical phases and this simplifies the realization operations. Schemes for implementing unconventional geometric gates have been proposed in trapped ion systems [9,11] and in cavity QED systems [12,13]. In the schemes of the cavity QED systems [12,13], the excited states are utilized as ancillary states or as the computational basis during the quantum computation operations; thus spontaneous emission cannot be avoided in such schemes.

In this paper we make use of the two-channel Raman interaction with a small cavity detuning in cavity QED, which is different from the general two-channel Raman resonance [14,15] in that there is a detuning from the Raman resonance in one channel containing the cavity mode, to realize the unconventional geometric gate. In our scheme the atomic excited states are adiabatically eliminated and never excited during the quantum gate operation; therefore atomic spontaneous emission can be avoided. In Sec. II we give the dynamical model of our scheme. Then a two-qubit phase gate is implemented based on the total phase in Sec. III. We show that the dynamical phase acquired in a cyclic evolution is proportional to the geometric phase acquired in the same cyclic evolution. In Sec. IV we examine the influence of the cavity decay on our scheme. It is shown that even in weak cavity decay the dynamical phase and the total phase are also proportional to the geometric phase. In Sec. V we discuss some experimental matters and conclude the paper.
II. THEORETICAL MODEL OF TWO-CHANNEL RAMAN COUPLING IN A CAVITY

We consider two identical three-level atoms in the Λ configuration placed in a high-Q cavity. The level structure of the atoms is shown in Fig. 1, where \( |e_i\rangle \) and \( |g_i\rangle \) \( (i=1,2) \) are metastable states and \( |c_i\rangle \) is an excited state. The transitions \( |c_i\rangle \leftrightarrow |g_i\rangle \) and \( |c_i\rangle \leftrightarrow |e_i\rangle \) are supposed to be dipole allowed. Each atom is off-resonantly excited via two Raman channels by laser fields and the cavity mode. One channel is excited by two classical external fields \( E_p(t) \) and \( E_c(t) \) with the frequencies \( \omega_g \) and \( \omega_e \), respectively. The second channel contains a classical external field \( E_c(t) \) with a frequency \( \omega_e \) and a quantized cavity field \( E_{c}(t) \) of frequency \( \omega_e \). The first channel is assumed to satisfy the usual Raman resonance, that is, \( \omega_1 - \omega_2 = \omega_0 \), while the second has a small detuning \( \delta \) from the Raman resonance, that is, \( \omega_1 - \omega_2 = \omega_0 + \delta \), where \( \omega_0 \) is the energy difference between levels \( |e\rangle \) and \( |g\rangle \). The Hamiltonian of the system is of the form

\[
H = \hbar \omega_g \sum_{j=1}^{2} |g\rangle_{jj} \langle g| + \hbar \omega_e \sum_{j=1}^{2} |e\rangle_{jj} \langle e| + \hbar \omega_a a^\dagger a \\
+ \hbar \left( \Omega e^{i\omega_{eff} t} + \Omega^* e^{-i\omega_{eff} t} \right) \sum_{j=1}^{2} |c\rangle_{jj} \langle g| + \text{H.c.} \\
+ \hbar \left( \eta a + \Omega_2 e^{i\omega_{eff} t} \right) \sum_{j=1}^{2} |c\rangle_{jj} \langle e| + \text{H.c.},
\]

where \( \hbar \omega_i \) \( (i=g,e,c) \) is the energy of the atomic level \( i \) and \( \hbar \omega_a \) has been chosen as zero, \( a \) and \( a^\dagger \) are, respectively, the annihilation and creation operators of the cavity mode, \( \eta \) is the coupling constant of the cavity mode and the atom, and \( \Omega, \Omega_1, \) and \( \Omega_2 \) are the Rabi frequencies of the classical driving fields.

In the case that the detunings \( \delta_1 \) and \( \delta_2 \) (see Fig. 1) are sufficiently large in comparison with \( \eta, \Omega, \Omega_1, \) and \( \Omega_2, \) the atomic excited state \( |c\rangle \) can be adiabatically eliminated. If we further assume \( \delta_1 - \delta_2 \) is large enough and satisfies

\[
\delta_1 - \delta_2 \gg \left\{ \delta_1, \frac{\Omega_1}{\delta_2}, \frac{\Omega_2}{\delta_2}, \frac{\eta \Omega_1}{\delta_2}, \frac{\eta \Omega_2}{\delta_2} \right\} ,
\]

we can take the rotating-wave approximation and obtain an effective Hamiltonian with a small detuning,

\[
H'(t) = \sum_{j=1}^{2} \left( r \sigma^+_j + r^* \sigma^-_j \right) + \sum_{j=1}^{2} (g a \sigma^-_j + g^* a^\dagger \sigma^+_j),
\]

where \( \sigma^+_j = |e\rangle \langle g| \) and \( \sigma^-_j = |g\rangle \langle e| \) are atomic operators, and \( r \) and \( g \) are, respectively, the effective classical and quantum couplings of the forms

\[
r = -\frac{2 \Omega_1 \Omega_2}{\delta_2},
\]

\[
g = -\frac{\eta \Omega}{\delta_1} \left( \frac{1}{\delta_1} + \frac{1}{\delta_1 + \delta} \right) = |g| e^{i(\delta t + \phi_0)},
\]

where \( \phi_0 \) is the initial relative phase between the driving field \( E_g(t) \) and the cavity field \( E_c(t) \).

Under the interacting picture, an effective Hamiltonian can be written as

\[
H_{\text{eff}} = e^{iH_{\text{eff}} t} \sum_{j=1}^{2} \left( g a \sigma^-_j + g^* a^\dagger \sigma^+_j \right) e^{-iH_{\text{eff}} t},
\]

where \( H_0 = \hbar \sum_{j=1}^{2} (r \sigma^+_j + r^* \sigma^-_j) \). For simplicity, we assume that \( r \) is real; then \( H_0 = \hbar r \sum_{j=1}^{2} (\sigma^+_j + \sigma^-_j) \). After a direct calculation we obtain

\[
H_{\text{eff}} = (\hbar/2) \sum_{j=1}^{2} \left( g a + g^* a^\dagger \right) (|+\rangle_j \langle +| - | -\rangle_j \langle -|) \\
+ (\hbar/2) \sum_{j=1}^{2} \left( g a - g^* a^\dagger \right) (e^{-i\omega_r |+\rangle_j \langle +| - e^{i\omega_r | -\rangle_j \langle -|}),
\]

where \( |\pm\rangle_j = (|g\rangle_j \pm |e\rangle_j)/\sqrt{2} \) are eigenstates of \( \sigma^+_j = \sigma^+_j + \sigma^-_j \) with eigenvalues \( \pm 1 \), respectively. In the strong effective classical driving regime \( r \gg |g| \), the terms in Eq. (7) that oscillate with high frequencies can be eliminated in the rotating-wave approximation. Equation (7) can thus be simplified as

\[
H_{\text{eff}} = (\hbar/2) \sum_{j=1}^{2} \left( g a + g^* a^\dagger \right) (|+\rangle_j \langle +| - | -\rangle_j \langle -|) \\
= (\hbar/2) (g a + g^* a^\dagger) (\sigma^+_j + \sigma^-_j).
\]

Similar Hamiltonians have been derived in the strongly driving Jaynes-Cummings model [16] and the two-channel Raman interaction in cavity QED [15]. Note that the theoretical model in this paper is similar to that in Ref. [15]; however, in our model there is a small detuning \( \delta \) from the Raman resonance in the second channel as described above and the effective coupling constant \( g(t) \) is a complex number, while in Ref. [15] the effective coupling constant \( g(t) \) is assumed to be a real function and cannot be adopted in the present paper.
to realize the unconventional geometric quantum computation as shown in the next section.

III. UNCONVENTIONAL GEOMETRIC TWO-QUBIT PHASE GATE

We choose \( |\pm\rangle = (|g\rangle, \pm |e\rangle)/\sqrt{2} \), the eigenstates of \( \sigma_z^j \) \((j=1,2)\), as the computational basis, so that the Hamiltonian \((8)\) will not give rise to any population changes in such a computational basis when the system is governed by the Hamiltonian \((8)\). In the computational basis \( \{|+\rangle_1|+\rangle_2, |+\rangle_1|-\rangle_2, |\rangle_1|+\rangle_2, |\rangle_1|-\rangle_2\} \), the Hamiltonian \((8)\) is diagonal and takes the form

\[
H_{\text{eff}} = (\hbar/2)(ga + g^*a^\dagger)\text{diag}[\lambda_{++}, \lambda_{+-}, \lambda_{-+}, \lambda_{--}],
\]

(9)

where \( \lambda_{kl} \) \((k,l=+, -)\) are the eigenvalues of \( \sigma_z^1 + \sigma_z^2 \) with \( \lambda_{+-} = -\lambda_{-+} = 0 \). The time evolution matrix \( U(t) \) is thus diagonal,

\[
U(t) = \text{diag}[U_{++}(t), U_{+-}(t), U_{-+}(t), U_{--}(t)],
\]

(10)

where the diagonal matrix elements \( U_{kl}(t) \) \((k,l=+, -)\) can be derived from Eq. (9),

\[
U_{kl}(t) = \hat{T}\exp\left(-i \int_0^t \hat{H}_k(t) \text{d}t \right) = \hat{T}\exp\left(-i \frac{1}{2} \lambda_{kl} \int_0^t [g(\tau)a + g^*(\tau)a^\dagger] \text{d}\tau \right)
\]

\[
= \lim_{N \to \infty} \prod_{\tau = 1}^N \exp\left(-i \frac{1}{2} \lambda_{kl} [g(\tau_n)a + g^*(\tau_n)a^\dagger] \Delta \tau \right)
\]

\[
= \lim_{N \to \infty} \prod_{\tau = 1}^N D[\Delta \alpha_{kl}(\tau_n)],
\]

(11)

where \( \hat{T} \) is the time-ordering operator, and \( \hat{H}_k(t) = \frac{1}{2} \lambda_{kl} [g(t)a + g^*(t)a^\dagger] \) are the matrix elements of the Hamiltonian \( H_{\text{eff}} \) in the computational basis and are still operators of the cavity mode. \( \Delta \tau = t/N \) is the time interval, \( \Delta \alpha_{kl}(\tau_n) = -i \frac{1}{2} \lambda_{kl} g^*(\tau_n) \Delta \tau \), and \( D(\alpha) \) is the displacement operator, which takes the form \( D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \). The displacement operators satisfy the following relation:

\[
D(\alpha) D(\beta) = e^{i \text{Im}(\alpha \beta)} D(\alpha + \beta).
\]

Based on the above formula, Eq. (11) can be further simplified as

\[
U_{kl}(t) = e^{i \gamma_{kl} D} \left( \int_c d\alpha_{kl} \right),
\]

(12)

with \( \gamma_{kl} = \text{Im} \int_c \alpha_{kl}^* d\alpha_{kl} \) and

\[
d\alpha_{kl} = -i \frac{1}{2} \lambda_{kl} g^*(\tau) d\tau.
\]

(13)

From Eqs. (13) and (5) we obtain

\[
\alpha_{kl}(t) = \frac{1}{2\delta} \lambda_{kl} \left| g \right| e^{-i \delta \eta} \left[ (\cos \delta t - 1) - i \sin \delta t \right].
\]

(14)

Suppose the cavity mode is initially in the vacuum state; then after a time interval \( t = 2\pi n / \delta \) \((n \text{ is a positive integer})\), the cavity mode evolves to its initial state and completes a closed path, \( U_{kl}(T) = e^{i \gamma_{kl} D} \). Here \( \gamma_{kl} \) is the total phase acquired by the state \( |k\rangle |l\rangle \) \(( k, l = +, -) \) in the cyclic evolution from \( \tau = 0 \) to \( T \). The total phase \( \gamma_{kl} \) consists of two parts; one part is the geometric phase \( \gamma_{kl}^g \), and the other is the dynamical phase \( \gamma_{kl}^d \) \([4]\). In order to give an explicit form of the geometric phase \( \gamma_{kl}^g \) and the dynamical phase \( \gamma_{kl}^d \), we adopt the the coherent-state path integral method \([11,17,18]\) to derive the formula about \( \gamma_{kl}^g \) and \( \gamma_{kl}^d \). In general, the wave function of the system \( |\Psi(T)\rangle \) acquires a phase in a cyclic evolution process,

\[
|\Psi(T)\rangle = \exp(i\gamma)|\Psi(0)\rangle,
\]

(15)

where the wave function \( |\Psi(t)\rangle \) is governed by the Schrödinger equation

\[
\frac{d|\Psi(t)\rangle}{dt} = -i\hbar \hat{H}_{\text{eff}} |\Psi(t)\rangle.
\]

(16)

From Eq. (15) one can obtain \( \exp(i\gamma) = \langle \Psi(0)|\Psi(T)\rangle = \langle \Psi(0)|U(T)|\Psi(0)\rangle \). As mentioned above, in the computational basis \( \{|+\rangle_1|+\rangle_2, |+\rangle_1|-\rangle_2, |\rangle_1|+\rangle_2, |\rangle_1|-\rangle_2\} \) the time evolution matrix \( U(t) \) takes the diagonal form (10); thus the total phase \( \gamma_{kl} \) acquired by the basis vector \( |k\rangle |l\rangle \) \(( k, l = +, -) \) is of the form

\[
\exp(i\gamma_{kl}) = \langle \alpha(0)|U_{kl}(T)|\alpha(0)\rangle,
\]

(17)

where \( |\alpha(0)\rangle \) is an initial coherent state of the cavity mode. According to the coherent-state path integral method \([11,17,18]\), we obtain

\[
\gamma_{kl} = \gamma_{kl}^g + \gamma_{kl}^d,
\]

(18)

\[
\gamma_{kl}^g = \int_0^T \langle \alpha_{kl}^* \alpha_{kl} - \alpha_{kl} \alpha_{kl}^* \rangle dt,
\]

(19)

with

\[
H_{kl}(\alpha_{kl}^*, \alpha_{kl}; t) = \langle \alpha_{kl}(t)| H_{kl}(t) |\alpha_{kl}(t)\rangle.
\]

(20)

Substituting Eqs. (9) and (14) into Eq. (20), we get

\[
H_{kl}(\alpha_{kl}^*, \alpha_{kl}; t) = -i \frac{\hbar}{4} \lambda_{kl}^2 \left( g(t) \int_0^t g^*(\tau) d\tau - g^*(t) \int_0^t g(\tau) d\tau \right)
\]

\[
= -i \frac{\hbar}{4} \lambda_{kl}^2 G(t);
\]

(21)

here, for the sake of simplicity, \( G(t) \) is defined as
\[ G(t) = g(t) \int_0^t g^*(\tau)d\tau - g^*(t) \int_0^t g(\tau)d\tau. \] (22)

With Eqs. (14) and (21), the geometric phase \( \gamma_0^t \) and the dynamical phase \( \gamma_0^d \) can be calculated, respectively, according to the formulas (18) and (19),

\[ \gamma_0^t = -\frac{i}{2\lambda^2} \int_0^T G(t)dt, \] (23)

\[ \gamma_0^d = -\frac{i}{4\lambda^2} \int_0^T G(t)dt, \] (24)

and the total phase is given by

\[ \gamma_0^t = \gamma_0^d + \gamma_0^d = \frac{i}{2\lambda^2} \int_0^T G(t)dt. \] (25)

Using Eqs. (23)–(25), one finds

\[ \gamma_0^t = -\frac{1}{2} \gamma_0^d = -\gamma_0^d. \] (26)

It is interesting to note that the relations between the total phase \( \gamma_0^t \), the dynamical phase \( \gamma_0^d \), and the geometric phase \( \gamma_0^d \) indicate that the total phase \( \gamma_0^t \) and the dynamical phase \( \gamma_0^d \) possess global geometric features as does the geometric phase \( \gamma_0^d \). Therefore the cyclic evolution

\[ U(\gamma) = \text{diag}[e^{i\gamma}, e^{-i\gamma}, e^{-i\gamma}, e^{-i\gamma}], \] (27)

with \( \gamma=(i/2)\int_0^T G(t)dt \), is a two-qubit phase gate operation that is largely insensitive to local inaccuracies and fluctuations. This gate is nontrivial if \( \gamma \neq 2n\pi \) (\( n \) is an integer). By substituting Eq. (5) into Eq. (22), we can express \( \gamma \) as

\[ \gamma = \frac{|g|^2}{\delta} \left( \frac{1}{\delta} \sin \delta T - T \right) = -2\pi m \frac{|g|^2}{\delta}. \] (28)

Here the condition of the closed-path evolution for the cavity mode \( T=2m\pi/\delta \) with \( m \) a positive integer has been used. As shown in the preceding section, the effective coupling constant \( g \) and the detuning \( \delta \) can be controlled by adjusting the driving light field, so that a cyclic evolution condition and a certain total phase \( \gamma \) can thus be achieved.

IV. THE INFLUENCE OF THE CAVITY DECAY

In the above discussion, we have neglected the cavity decay and the possibility of spontaneous emission in the atoms, two main sources of decoherence in cavity QED quantum computation. If we consider the case in which the excited states of the atoms are adiabatically eliminated and the quantum phase gate operation involves only atomic metastable states, we may neglect the atomic spontaneous emission. However, we still need to study the effects of the cavity decay. Cavity decay is regarded as the decoherence in the cavity mode due to coupling to a reservoir, so that the system comprised of the atoms and the cavity mode becomes an open system. The system then evolves as a mixed state through interaction with the environment. The mixed-state geometric phase of open systems was extensively studied in recent years [19–21]. In this section, we will resort to the quantum trajectory method [21] to evaluate the geometric phase in the case of weak cavity decay. Although, as was pointed out very recently in Refs. [22,23], this method for the geometric phases of a general situation with stochastic unravelings is no longer applicable because there are infinitely many different ways to unravel the environment with the trajectory method, and each unraveling gives rise to different definitions of the geometric phase, it can still be used to associate a geometric phase to an individual quantum trajectory through measurement of the environment [23]. In particular, the initial state of this trajectory is a pure state. In this section we will focus on a no-jump trajectory corresponding to the situation that no leaky photons from the cavity are detected while monitoring the environment outside the cavity. In practice, in the case of the high-Q cavity we consider here, the no-jump trajectory occurs with the most probability and the jump trajectories occur with a very small probability during the realization of the quantum gate operations. Therefore the situation described here is rather different from the general situation with stochastic unravelings; it is actually an individual trajectory through the measurement of the environment as described in Ref. [23].

In the no-jump trajectory the wave function of the system, \( |\Psi^0(t)\rangle \), evolves according to an effective non-Hermitian Hamiltonian due to its coupling to the reservoir:

\[ H_{\text{non}} = H_{\text{eff}} - i\hbar \kappa a^\dagger a, \] (29)

where \( H_{\text{eff}} \) is given by (8) and \( \kappa \) is the cavity decay rate. The detection click of the leaky photon(s) (jump) is accompanied by wave function collapse [24].

In order to obtain the wave function of the system \( |\Psi^0(t)\rangle \) under the condition of no jump (here the superscript 0 represents a no-jump trajectory), let us set \( |\Psi^0(t)\rangle = e^{-\kappa a^\dagger a}|\Phi(t)\rangle \). After substituting into the Schrödinger equation governed by \( H_{\text{non}} \) in Eq. (29), we have

\[ i\hbar \frac{d}{dt}|\Phi(t)\rangle = e^{-\kappa a^\dagger a}H_{\text{eff}}e^{-\kappa a^\dagger a}|\Phi(t)\rangle = \tilde{H}|\Phi(t)\rangle, \] (30)

where

\[ \tilde{H} = \hbar/2 \left[ g(t) e^{-\kappa a^\dagger a} + g^*(t) e^{\kappa a^\dagger a} \right] (\sigma_z^i + \sigma_z^r). \] (31)

Note that \( \tilde{H} \) differs from \( H_{\text{eff}} \) in Eq. (8) only by the following transformations: \( g(t) \rightarrow g(t) e^{-\kappa a^\dagger a} \) and \( g^*(t) \rightarrow g^*(t) e^{\kappa a^\dagger a} \). Similarly, we could derive the effective wave function \( |\Phi(t)\rangle \) as in Sec. III. From (30) we get

\[ |\Phi(t)\rangle = U(t)|\Phi(0)\rangle, \] (32)

where \( U(t) \) is the time evolution operator, which, in general, is not unitary, and \( |\Phi(0)\rangle = |\Phi(0)_c\rangle |\Phi(0)_a\rangle \) is the initial state of the system with the subscripts \( c \) and \( a \) representing the cavity mode and the atoms, respectively. We assume henceforth that initially the cavity mode is in its vacuum state \( |0\rangle_c \) and the atoms are in a state \( |\Phi(0)_a\rangle \). According to Eqs. (30)
and (31), \( \tilde{U}(t) \) takes the following form in the computational basis \( \{|+,\rangle, |+,\rangle_2, |+,\rangle_{1|2}, |-,\rangle_{1|2}, |+,\rangle_{1|2}, |-,\rangle_{1|2} \}:

\[
\tilde{U}(t) = \text{diag} [\tilde{U}_{ll}(t), 1, 1, \tilde{U}_{ll}(t)]
\]

with the diagonal matrix elements \( \tilde{U}_{kl}(t) = (k,l)=\pm,\) given by

\[
\tilde{U}_{kl}(t) = \hat{T} \exp \left( -i \frac{\lambda_{kl}}{2} \int_0^t \left[ g(\tau) e^{-i \tau} a + g^*(\tau) e^{i \tau} a^\dagger \right] d\tau \right)
\]

\[
= \lim_{N \to \infty} \prod_{n=1}^N \exp \left( -i \lambda_{kl} \int_0^t \left[ g(\tau_n) e^{-i \tau_n} a + g^*(\tau_n) e^{i \tau_n} a^\dagger \right] d\tau \right)
\]

\[
= \lim_{N \to \infty} \prod_{n=1}^N \hat{D}(\Delta \beta_{kl}(\tau_n), \Delta \beta_{kl}^*(\tau_n)).
\]

Here,

\[
\hat{D}(\Delta \beta_{kl}(\tau_n), \Delta \beta_{kl}^*(\tau_n)) = \exp[\Delta \beta_{kl}(\tau_n) a + \Delta \beta_{kl}^*(\tau_n) a^\dagger],
\]

(35)

and

\[
\Delta \beta_{kl}(\tau_n) = -i \frac{1}{2} \lambda_{kl} g(\tau_n) e^{-i \tau_n} \Delta \tau,
\]

(36)

\[
\Delta \beta_{kl}^*(\tau_n) = -i \frac{1}{2} \lambda_{kl} g^*(\tau_n) e^{i \tau_n} \Delta \tau.
\]

(37)

Note that the existence of the factors \( e^{i \chi_n} \) in Eq. (35) means that \( \hat{D}(\Delta \beta_{kl}(\tau_n), \Delta \beta_{kl}^*(\tau_n)) \) is neither a standard displacement operator nor a unitary operator in the case of \( \chi \neq 0 \). By repeatedly using the following relation in Eq. (34):

\[
\hat{D}(\Delta \beta_{kl}(\tau_{n+1}), \Delta \beta_{kl}^*(\tau_{n+1})) \hat{D}(\Delta \beta_{kl}(\tau_n), \Delta \beta_{kl}^*(\tau_n)) = \exp \left[ \sum_{j=n}^{n+1} \Delta \beta_{kl}(\tau_j) a + \sum_{j=n}^{n+1} \Delta \beta_{kl}^*(\tau_j) a^\dagger \right] e^{i \Delta \beta_{kl}(\tau_n) \Delta \beta_{kl}^*(\tau_{n+1}) - \Delta \beta_{kl}(\tau_{n+1}) \Delta \beta_{kl}^*(\tau_n)/2},
\]

we get

\[
\tilde{U}_{kl}(t) = \exp[\beta_{kl}(t) a + \beta_{kl}^*(t) a^\dagger] \exp[\mu_{kl}(t)/2],
\]

(38)

\[
\mu_{kl}(t) = \int_0^t \beta_{kl}(\tau) d\beta_{kl}(\tau) - \int_0^t \beta_{kl}^*(\tau) d\beta_{kl}^*(\tau),
\]

(39)

where \( \beta_{kl}(t) = \int_0^t d\beta_{kl}(\tau) \) with \( d\beta_{kl}(\tau) = -i \frac{1}{2} \lambda_{kl} g(\tau) e^{-i \tau} d\tau \), and \( \beta_{kl}^*(t) = \int_0^t d\beta_{kl}^*(\tau) \) with \( d\beta_{kl}^*(\tau) = -i \frac{1}{2} \lambda_{kl} g^*(\tau) e^{i \tau} d\tau \).

In the case of a no-jump trajectory, the geometric phase in the open system can be written as [21]

\[
\gamma_{kl}^{\mu} = \gamma_{kl}^{\mu} - \gamma_{kl}^{\nu}.
\]

(40)

\[
\gamma_{kl}^{\mu} = -\text{arg}[\langle \psi_{kl}(t) | \psi_{kl}(0) \rangle],
\]

(41)

where \( |\psi_{kl}(t)\rangle \) is a coherent state of the cavity mode with

\[
\gamma_{kl}^{\mu} = -\int_0^T \langle \psi_{kl}(t)| [H_{kl}] |\psi_{kl}(t) \rangle dt,
\]

(42)

where \( |\psi_{kl}(t)\rangle \) is the wave function of the system at time \( t \) evolving from the initial state \( |0\rangle_{l}, |k\rangle_{l} \) with the computational basis \( |k\rangle_{l} \) acquired by \( |k\rangle_{l} \) from \( t = 0 \) to \( T \). \( \gamma_{kl}^{\mu} \) and \( \gamma_{kl}^{\nu} \) correspond, respectively, to the total phase and the dynamical phase. Substituting \( |\psi_{kl}(t)\rangle = e^{-i \mu_{kl}(t)} |\Phi_{kl}(t)\rangle \) with \( |\Phi_{kl}(t)\rangle = \tilde{U}_{kl}(t)|0\rangle_{l}, |k\rangle_{l} \) into Eqs. (41) and (42) we have

\[
\gamma_{kl}^{\mu} = -\text{arg}[\langle \Phi_{kl}(T)|H_{kl}|\Phi_{kl}(t) \rangle],
\]

(43)

\[
\gamma_{kl}^{\mu} = -\frac{1}{2} \lambda_{kl} \int_0^T e^{-i \mu_{kl}(t)} [ \langle g(t) \beta_{kl}^*(t) + |g(t)\rangle \beta_{kl}(t) \rangle ] dt.
\]

(44)

Next, the partial integration is applied to simplify Eq. (39) as

\[
\mu_{kl}(t) = 2 \int_0^t \beta_{kl}(\tau) d\beta_{kl}(\tau) - \beta_{kl}(t) \beta_{kl}^*(t),
\]

where \( \beta_{kl}(0) \beta_{kl}^*(0) = 0 \) is used since the initial value of effective coupling constant \( g(0) \) is zero. Thus Eq. (43) is further simplified as

\[
\gamma_{kl}^{\mu} = -\text{arg} \left[ \exp \left[ \left( \int_0^T \beta_{kl}(t) d\beta_{kl}(t) \right)^* \right] \right]
\]

\[
= -\frac{1}{4} \lambda_{kl} \int_0^T e^{-i \mu_{kl}(t)} [ \langle g(t) \beta_{kl}^*(t) + |g(t)\rangle \beta_{kl}(t) \rangle ] dt.
\]

(45)

According to Eq. (40), we get

\[
\gamma_{kl}^{\mu} = \gamma_{kl}^{\mu} - \gamma_{kl}^{\nu} = -\frac{1}{4} \lambda_{kl} \int_0^T e^{-i \mu_{kl}(t)} [ \langle g(t) \beta_{kl}^*(t) + |g(t)\rangle \beta_{kl}(t) \rangle ] dt.
\]

(46)

It is not difficult to check that, when \( \chi = 0 \), the above results for \( \gamma_{kl}^{\mu}, \gamma_{kl}^{\mu} \) and \( \gamma_{kl}^{\mu} \) revert to the ones for the ideal (no cavity decay) case. From Eqs. (45), (44), and (46), one can find that in the case of weak cavity decay the dynamical phase and the total phase are both proportional to the geometric phase. In other words, even in the situation of weak cavity decay, the dynamical phase and the total phase still possess a geometric feature, which is the same as that in the ideal case studied in Sec. III.

Now let us study the influences of cavity decay. In the case of a no-jump trajectory,

\[
|\psi_{kl}(t)\rangle = e^{-i \mu_{kl}(t)} |\tilde{U}_{kl}(t)|0\rangle_{l}, |k\rangle_{l} \rangle
\]

\[
= \exp \left[ \left( \int_0^t \beta_{kl}(\tau) d\beta_{kl}(\tau) + \frac{1}{2} \left| f_{kl}(t) \right|^2 \right) |\tilde{U}_{kl}(t)|0\rangle_{l}, |k\rangle_{l} \rangle.
\]

(47)
\[ f_{kl}(t) = \frac{\lambda_{kl}|g|(|\delta - i\kappa|e^{-i\phi_k} - i\sin \delta t). \]

(48)

Obviously, when \( \kappa \neq 0 \) and \( \lambda_{kl} \neq 0 \), we have \( f_{kl}(t) \neq 0 \) at any time \( t \). Hence the cavity mode cannot return to its original state in the case of cavity decay. When \( \kappa \) is very small in comparison with \( \delta \) and \( |g| \), we can choose \( t = T = 2m \pi / \delta \) (\( m \) being a positive integer) as the time interval for an approximate closed path with a small positive coherent amplitude \( f_{kl}(T) = (1 - e^{-\kappa T}) \); obviously \( f_{kl}(T) \) approaches zero when \( \kappa T \) approaches zero. In any case, nonclosure of the path of the cavity mode will reduce the fidelity of the quantum gate operation because there is an entanglement between the atoms and the cavity mode at the end of the gate operation, so we should choose \( m \) as small as possible (e.g., \( m = 1 \), and \( T_{\min} = 2\pi / \delta \)), such that the coherent amplitude \( f_{kl}(T_{\min}) \) is the smallest.

On the other hand, the cavity decay will decrease the amplitude of the computational basis state. The damping factor of amplitude of the computational basis state, denoted by \( R_{kl} \) for the basis state \( |k\rangle |l\rangle \) in the following, is characterized by the real part of the exponential in Eq. (47). Substituting Eq. (5) and \( T = 2m \pi / \delta \) into Eq. (47), we get

\[ R_{kl} = \frac{\lambda_{kl}^2|g|^2}{8(\delta + \delta^2)^2}[\delta^2(1 + 2\kappa T - e^{2\kappa T})]. \]

(49)

Under the first-order approximation when the cavity decay rate \( \kappa \) is very small in comparison with \( \delta \) and \( |g| \), \( R_{kl} \) is simplified as

\[ R_{kl} = -\frac{\lambda_{kl}^2|g|^2 \delta^2 x T}{2(\delta^2 + \delta^2)^2}. \]

(50)

Apparently a small value for \( \kappa T \) is required to reduce the damping of the amplitude of the computational basis.

Now let us study the two-qubit phase gate operation at the time \( t = T = 2m \pi / \delta \). Substituting (5) and \( T = 2m \pi / \delta \) into (45), the total phase is obtained as

\[ \gamma_{kl}^0 = \frac{\lambda_{kl}^2|g|^2 \delta}{4(\delta^2 + \delta^2)^2}[2\kappa(1 - e^{-x T}) - (\delta^2 + \delta^2)T]. \]

(51)

Working to first-order approximation, we have

\[ \gamma_{kl}^0 = \frac{\lambda_{kl}^2|g|^2(\delta^2 - \delta^2) \delta T}{4(\delta^2 + \delta^2)^2}. \]

(52)

We suppose that, as an example, the resultant total phase for the state \( |+\rangle |+\rangle \) \((|--\rangle |--\rangle \rangle \) is \(-\pi\). Then the two-qubit phase gate operation (33) is approximately \( U(T) = \text{diag}[-\exp(R_{++}), 1, 1, \exp(R_{--})] \) with the relation \( R_{+-} = R_{-+} \). In order to achieve a \(-\pi\) total phase, the relation for \( |g| \) and \( \delta \) can be determined according to Eq. (52): \( 2m|g|^2(\delta^2 - \delta^2) = (\delta^2 + \delta^2)^2 \).

Finally, we examine the fidelity \( F_{kl} \) of the computational basis via the gate operation during a time interval \([0, T]\). \( F_{kl} \) is defined as

\[ F_{kl} = |\langle \Psi_{kl}^{(0)}(T) | \Psi_{kl}^{(0)}(T) \rangle|^2, \]

(53)

where \( |\Psi_{kl}^{(0)}(T)\rangle = |\Psi_{kl}^{(0)}(T)\rangle_{x = 0} \) corresponds to the basis states of the system at time \( T \) in the ideal case. Obviously, \( F_{+-} = F_{-+} = 1 \) and \( F_{++} = F_{--} \) are satisfied, so it is sufficient to calculate \( F_{++} = F \). In Fig. 2 we plot the fidelity \( F \) as a function of \( \kappa / |g| \) when \( m = 1, 2, 3, 4 \). We find that the fidelity \( F \) decreases as \( \kappa / |g| \) increases. The result shows that \( \kappa / |g| \) should be sufficiently small to keep a reasonable fidelity.

V. DISCUSSION AND CONCLUSION

Let us first briefly assess the feasibility of our scheme under the current experimental technique. It is important to note that the detunings \( \delta_1 \) and \( \delta_2 \) (see Fig. 1) should be sufficiently large compared with \( \eta \), \( \Omega_1 \), and \( \Omega_2 \) in order that the atomic excited state can be adiabatically eliminated. In this paper, we take \( \delta_1 = 100 |\eta| \) and \( \delta_2 = 50 |\eta| \). Since the value of \( \eta \) is half the value of the single photon Rabi frequency, we can further assume \( |\Omega_1| = 10 |\eta| \), \( |\Omega_2| = 20 |\eta| \) \((i = 1, 2)\), where \( \Omega_1 \), \( \Omega_2 \), and \( \Omega_2 \) are the Rabi frequencies of the classical driving fields. With these choices, it is easy to check that the conditions in Eq. (2) and \( |r| > |g| \) are satisfied, and that \( |g| = |\eta| / 5 \). As an example, suppose we wish to achieve a total phase \(-\pi\) with \( m = 1 \). In the ideal case, according to Eq. (28) we can set the relation between \( \delta \) and \( |g| \): \( \delta = \sqrt{2} |g| \). Furthermore, the gate time is approximately \( T = 2 \pi / \delta = 2 \pi / |g| \). In the microwave cavity QED experiments of Haroche and co-workers \( [25] \), the coupling constant of the cavity mode and the atom is \( |\eta| = 2 \pi \times 49 \text{ kHz} \) and the photon lifetime is \( T = 1 \text{ ms} \). Thus the gate time \( T_{\min} \) is approximately \( 0.072 \text{ ms} \), which is generally much shorter than the photon lifetime \( T_e \). In fact, in another microwave cavity QED experiment of Walther and co-workers \( [26] \), the photon lifetime reached 0.3 s. Therefore our scheme can be realized under the current experimental techniques.

In addition to cavity decay, the phase fluctuations of the driving laser fields could also have a decoherence effects. In fact, if there are phase fluctuations in the driving laser fields, the effective classical coupling \( r \) in Eq. (4) is generally a complex number \( r = |r| \exp[i(\theta_0 + \theta)] \), where \( \theta_0 \) is a controllable phase of the driving laser fields and \( \theta \) is a random
phase factor standing for the phase fluctuations. The mean of
the phase fluctuations is zero, \( \langle \theta \rangle = 0 \). For simplicity we as-
sume \( \theta_0 = 0 \). After substituting into Eq. (6) and then taking the
rotating-wave approximation as we have done in Sec. II, we
get the following effective Hamiltonian:

\[
\hat{H}_{\text{eff}} = \frac{\hbar}{2} \left( \vec{g} + \vec{a} \right) \sum_{j=1}^{2} \left[ \cos \langle \theta \rangle \left| j \right| \left\{ \left| j \right\rangle + \left| - j \right\rangle \right\} \\
+ i \sin \langle \theta \rangle \left| j \right\rangle \left\{ \left| - j \right\rangle - \left| j \right\rangle \right\} \right],
\]

(54)

where \( \vec{g} = g e^{i \theta} \). In comparison with Eq. (8), Eq. (54) is no
longer the same form (diagonalized) due to the presence of
the last term. However, in the above Hamiltonian, the cavity
mode operators and the atomic operators still remain separ-
able, and all the conclusions about the dynamical phase, the
total phase, and the geometric phase derived in Sec. III are
still valid. From the expression for \( G \) in Eq. (22) and the
formulas about the various phases thereafter, one sees that the
factor \( \theta \) that appears in \( g \) has no effect on these phases.
However, the eigenstates of the atomic operator in Eq. (54)
for the \( j \)th atom are replaced by \( \cos \theta \left| 2 \right\rangle + i \sin \theta \left| - 2 \right\rangle \) \( j \)
and \( -i \sin \theta \left| 2 \right\rangle + \cos \theta \left| - 2 \right\rangle \); moreover, they are no
longer the eigenstates of atomic operator \( \sigma^j_x = \sigma^x + i \sigma^y \).
In other words, the presence of phase fluctuations in the driving
laser fields will give rise to unwanted atomic transitions un-
less the phase fluctuations are kept small, \( \langle \theta \rangle \ll 1 \). Therefore,
our scheme requires a high-quality laser fields with little
phase fluctuation acting as the driving field in order to avoid
the undesirable atomic transitions. In fact, this condition is
also necessary for most of quantum-information processing
which involves the coupling between the atoms and the laser
light.

In summary, we present a scheme for implementing the
unconventional geometric two-qubit phase gate with nonzero
dynamical phase based on two-channel Raman interaction of
two atoms in a cavity. We show that the dynamical phase and
the total phase for a cyclic evolution are proportional to the
geometric phase in the same cyclic evolution; hence they
possess the same geometric features as does the geometric
phase. For a noisy system, we argued that the atomic excited
state can be adiabatically eliminated and that the operation of
the proposed logic gate involves only the metastable states of
the atom. Thus the effect of the atomic spontaneous emission
can be neglected. The influence of the cavity decay on our
scheme can therefore also be examined using the quantum
trajectory method. It is found that the relations regarding the
dynamical phase, the total phase, and the geometric phase in
the ideal situation are still valid in the case of weak cavity
decay. The presence of cavity decay will reduce the ampli-
ditude of the quantum computational basis and give rise to a
nonclosure of the cavity mode evolution, and thereby reduce
the fidelity of the gate operation. The feasibility and the ef-
flect of the phase fluctuations of the driving laser fields are
also discussed.

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