Preparation and entanglement purification through two-step measurements

Nan Qiu,[1] Shu-chao Wang,[1,2] Leong Chuan Kwek,[2,3,*] and Xiang-Bin Wang[1,4,†]

[1]*State Key Laboratory of Low Dimensional Quantum Physics, Department of Physics, Tsinghua University, Beijing 100084, People's Republic of China*

[2]*Centre for Quantum Technologies, National University of Singapore, 117543 Singapore*

[3]*National Institute of Education and Institute of Advanced Studies, Nanyang Technological University, 637616 Singapore*

[4]*Jinan Institute of Quantum Technology, Shandong Academy of Information and Communication Technology, Jinan 250101, People's Republic of China*

(Received 22 February 2012; published 13 July 2012)

A method of purification through Zeno-like measurements has been proposed by Nakazato and co-workers [H. Nakazato, T. Takazawa, and K. Yuasa, Phys. Rev. Lett. **90**, 060401 (2003); H. Nakazato, M. Unoki, and K. Yuasa, Phys. Rev. A **70**, 012303 (2004)]. A series of frequent measurements on a quantum system is demonstrated to result in the purification of another quantum system in interaction with the former. Inspired by this idea, we have designed a strategy to obtain entangled states through only two measurements. A deterministic purification method is also provided in this article.

DOI: 10.1103/PhysRevA.86.012313 PACS number(s): 03.67.Bg

I. INTRODUCTION

Entanglement plays a significant and essential role in quantum information processing and quantum commutation [1]. Entangled states possess nonlocal correlations and give rise to a variety of interesting phenomena [2]. But entangled states are often very fragile due to environmental perturbations. It is therefore desirable to prepare entangled states, such as, for instance, the Bell state from some mixed states. There are a lot of methods proposed theoretically and experimentally [3–16]. A very simple but robust method to produce an entangled resource was proposed in 2003 [17]. According to this method, in the case of a discrete spectrum, by doing the same simple repetitive measurement on a system over which we can have control, we can drive another system interacting with the controllable system to a pure state. Moreover, it has been shown that this method is very robust if some certain conditions are fulfilled.

This method has been improved by the same researchers for generating maximally entangled states [18,19]. They have used the scheme to drive entangled states of two distant systems, say **A** and **B**, in interaction with the mediator system **X**, to an entangled state, from an arbitrary initial preparation. However, a large number of measurements are still indispensable in principle. Although they also suggested a method which needs only three measurements [20], there are still limitations which will be pointed out in the following section. Based on the Zeno-like measurement method, several approaches have been proposed in different systems to prepare entanglements [21,22]. In consideration of the decoherence caused by the environment, this method still works [23,24]. In recent years, this method has been extended to the continuous spectrum situation [25,26].

Although this scheme is very robust, it is sometimes not efficient. Despite the large number of measurements it needs, some essential conditions cannot be satisfied all the time. Moreover, successful entanglement preparation is not always guaranteed since the outcomes of measurements may be very different from theoretically expected values. In this article, we proposed an improved method which calls for only two measurements and requires fewer conditions than the original scheme. In addition, a feedback scheme based on this method is given to ensure the generation of required entangled states. We need to stress that an approach of entanglement preparation with a probability 1 using repeated on- and off-resonant scattering of ancilla qubits has also been proposed in a entirely different context [27].

The article is organized as follows. Following the introduction, we briefly provide a framework in Sec II for the Zeno-like measurement method. In Sec. III, we demonstrate our two-step measurement method and show how it improves over the original scheme. In Sec. IV, a deterministic purification method is described based on the feedback of the measurement outcomes and the generation of a maximally entangled state is given in detail as an example. Finally, a short summary is presented in Sec. V.

II. PREPARATION THROUGH ZENO-LIKE MEASUREMENTS

Consider two quantum systems **X** and **A** interacting with each other. A series of frequent measurements on system **X** can result in the system **A**, which is initially in any (mixed) state approaching to a pure state, if certain conditions are satisfied.

The Hamiltonian of the total system is

$$H_{\rm tot} = H_X + H_A + H_{\rm int}, \tag{1}$$

where $H_{X(A)}$ stands for the free Hamiltonian of system **X**(**A**). The system **X** is prepared initially in the state $|\phi\rangle_X\langle\phi|$, while the system **A** is in an arbitrary state ϱ_A. The initial state of the total system is then $\varrho_0 = |\phi\rangle_X\langle\phi| \otimes \varrho_A$. A series of measurements represented by a projection operator $\mathcal{O} = |\phi\rangle_X\langle\phi| \otimes \mathcal{I}_{\mathcal{A}}$ are performed at time interval τ on system **X**. After N such measurements, the probability of finding system

*kwekleongchuan@nus.edu.sg
†xbwang@mail.tsinghua.edu.cn

X is always found in its initial state $|\phi\rangle_X$ is given by

$$\begin{aligned}\mathcal{P}^{(\tau)}(N) &= \mathrm{Tr}[(\mathcal{O}e^{-iH\tau}\mathcal{O})^N\varrho_0(\mathcal{O}e^{iH\tau}\mathcal{O})^N \\ &= \mathrm{Tr}_A[(\mathcal{V}_\phi(\tau))^N\varrho_A(\mathcal{V}_\phi^\dagger(\tau))^N],\end{aligned} \tag{2}$$

where $\mathcal{V}_\phi(\tau) \equiv \langle\phi|e^{-iH_{\mathrm{tot}}\tau}|\phi\rangle_X$ is an operator acting on the Hilbert space of system **A**. The state of the total system is described by the density operator:

$$\begin{aligned}\varrho^{(\tau)}(N) &= (\mathcal{O}e^{-iH\tau}\mathcal{O})^N\varrho_0(\mathcal{O}e^{iH\tau}\mathcal{O})^N/\mathcal{P}^{(\tau)}(N) \\ &= |\phi\rangle_X\langle\phi| \otimes \varrho_A^{(\tau)}(N),\end{aligned} \tag{3}$$

and the reduced density operator of system **A** is

$$\varrho_A^{(\tau)}(N) = [\mathcal{V}_\phi(\tau)]^N\varrho_A(\mathcal{V}_\phi^\dagger(\tau))^N/\mathcal{P}^{(\tau)}(N). \tag{4}$$

Suppose the operator $\mathcal{V}_\phi(\tau)$ can be decomposed as

$$\mathcal{V}_\phi(\tau) = \sum \lambda_n|\mu_n)(\nu_n|. \tag{5}$$

In general, the operator $\mathcal{V}_\phi(\tau)$ is not Hermitian. $|\mu_n)$ and $(\nu_n|$ are the right and left eigenvectors with λ_n being the corresponding eigenvalue. $|\mu_n)$ and $(\nu_n|$ obey the completeness and orthogonality relations, i.e., $\sum_n |\mu_n)(\nu_n| = \mathcal{I}$ and we have

$$[\mathcal{V}_\phi(\tau)^N] = \sum_n \lambda_n^N|\mu_n)(\nu_n|. \tag{6}$$

It is shown that the eigenvalue λ_n satisfies the condition $0 \leqslant |\lambda_n| \leqslant 1, \forall n$ [18]. In the large N limit, the operator is dominated by a single term provided the largest eigenvalue $\lambda_{\max}$ is unique, discrete, and nondegenerate.

$$[\mathcal{V}_\phi(\tau)]^N \to \lambda_{\max}^N|\mu_{\max})(\nu_{\max}|, \tag{7}$$

where $|\mu_{\max})$ and $(\nu_{\max}|$ are the eigenvectors belonging to the largest eigenvalue $\lambda_{\max}$.

The conclusion is that under the assumption of unique, discrete, and nondegenerate $\lambda_{\max}$, which is the largest eigenvalue of the operator $\mathcal{V}_\phi(\tau)$, it is likely to drive the system **A** to approach a pure state

$$\varrho_A^{(\tau)}(N) \xrightarrow{\text{large } N} |\mu_{\max})(\mu_{\max}|/(\mu_{\max}|\mu_{\max}), \tag{8}$$

with probability

$$\mathcal{P}^{(\tau)}(N) \xrightarrow{\text{large } N} |\lambda_{\max}|^{2N}(\mu_{\max}|\mu_{\max})(\nu_{\max}|\varrho_A|\nu_{\max}), \tag{9}$$

through a series of measurements with the nonvanishing time interval τ, on the system **X** which is constantly interacting with system **A**, for large N.

Furthermore, an efficient purification is obtained if $\lambda_{\max}$ satisfies the condition

$$|\lambda_{\max}| = 1, \tag{10}$$

and the condition

$$|\lambda_n/\lambda_{\max}| \ll 1 \quad (\lambda_n \neq \lambda_{\max}). \tag{11}$$

By adjusting parameters τ, $|\phi\rangle_X$, and those in H_{tot} so that all the required conditions can be satisfied, then the desired pure state of system **A** can be obtained. This method has been used to generate a maximally entangled state $|\psi^-\rangle = \sqrt{\frac{1}{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ [18].

It is noteworthy to emphasize that in this method, a large number of measurements are still needed in principle. Although a scheme which only needs three measurements is proposed [20], there are still limitations. The first one is that the unique, discrete, and nondegenerate largest eigenvalue of operator $\mathcal{V}_\phi(\tau)$ is expected. The other inherent limitation is that the scheme for getting a maximally entangled state is probabilistic.

III. PREPARATION THROUGH TWO-STEP MEASUREMENTS

Based on the previous work by Nakazato and co-workers, which is briefly described in the above section, we proposed an alternative scheme to obtain maximally entangled states $|\psi^-\rangle$ or $|\psi^+\rangle$. As in the previous method, a large number of measurement N is needed; however, in our new proposal, N can be made as small as 2.

As described in Sec. II, the Zeno-like measurement method is achieved by simply repeated projections on the initial state of system **X** which can be easily controlled. Before using our method to generate the entangled states, a simple example to illustrate the framework and the advantages will be given.

We follow the example Ref. [18] uses to show how their mechanism works and apply our proposed technique to show the improvement. Let us consider two two-level systems **X** and **A** interacting with the Hamiltonian:

$$H_{\mathrm{tot}} = \Omega_X\frac{1+\sigma_3^X}{2} + \Omega_A\frac{1+\sigma_3^A}{2} + g(\sigma_+^X\sigma_-^A + \sigma_-^X\sigma_+^A), \tag{12}$$

where σ_i $(i = 1,2,3)$ are the Pauli operators and $\sigma^\pm = \frac{\sigma_1 \pm i\sigma_2}{2}$ are the ladder operators. Ω_X, Ω_A, and the coupling constant g are all real parameters. Since the parameter $g_{\mathbf{XA}}$ and $g_{\mathbf{XB}}$ can be adjusted, we would consider the simple situation in which they are equal to each other. For simplicity, we also assume that g is a positive number.

The eigenvalues of the Hamiltonian in Eq. (12) are

$$E^{(0)} = 0, \tag{13a}$$

$$E_\pm^{(1)} = \frac{\Omega_X + \Omega_A}{2} \pm \delta, \tag{13b}$$

$$E^{(2)} = \Omega_X + \Omega_A. \tag{13c}$$

The corresponding eigenstates are

$$|E^{(0)}\rangle_{XA} = |\downarrow\downarrow\rangle, \tag{14a}$$

$$\begin{aligned}|E_\pm^{(1)}\rangle_{XA} = \frac{1}{\sqrt{2}}\Bigg(&\sqrt{1 \pm \frac{\Omega_X - \Omega_A}{2\delta}}|\uparrow\downarrow\rangle_{XA} \\ &\pm\sqrt{1 \mp \frac{\Omega_X - \Omega_A}{2\delta}}|\downarrow\uparrow\rangle_{XA}\Bigg),\end{aligned} \tag{14b}$$

$$|E^{(2)}\rangle_{XA} = |\uparrow\uparrow\rangle_{XA}, \tag{14c}$$

where

$$\delta = \sqrt{(\Omega_X - \Omega_A)^2/4 + g^2}$$

and $|\uparrow(\downarrow)\rangle$ is the eigenstate of σ_3 belonging to the eigenvalue $+1$ (-1).

After a straightforward calculation, we get the matrix of the operator $\mathcal{V}_\phi(\tau)$:

$$\mathcal{V}_\phi(\tau) \equiv \langle\phi|e^{-iH_{\rm tot}\tau}|\phi\rangle_X = \begin{pmatrix} \mathcal{V}_{11} & \mathcal{V}_{12} \\ \mathcal{V}_{21} & \mathcal{V}_{22} \end{pmatrix},$$

$$\mathcal{V}_{11} = \exp[-i(\Omega_X+\Omega_A)\tau]\left\{\cos^2\frac{\theta}{2} + \exp\left[i\frac{(\Omega_X+\Omega_A)\tau}{2}\right] \times \left(\cos\,\delta\tau + i\frac{\Omega_X-\Omega_A}{2\delta}\sin\,\delta\tau\right)\sin^2\frac{\theta}{2}\right\},$$

$$\mathcal{V}_{12} = -i\frac{g}{\delta}\exp\left(\frac{-i(\Omega_X+\Omega_A)\tau}{2}\right)\sin\,\delta\tau\,\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

$$\mathcal{V}_{21} = -i\frac{g}{\delta}\exp\left(\frac{-i(\Omega_X+\Omega_A)\tau}{2}\right)\sin\,\delta\tau\,\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

$$\mathcal{V}_{22} = \sin^2\frac{\theta}{2} + \exp\left[-i\frac{(\Omega_X+\Omega_A)\tau}{2}\right] \times \left(\cos\delta\tau - i\frac{\Omega_X-\Omega_A}{2\delta}\sin\delta\tau\right)\cos^2\frac{\theta}{2}. \quad (15)$$

Here, we assume that the initial state of system **X** can be written as

$$|\phi\rangle_X = \cos\frac{\theta}{2}|\uparrow\rangle_X + \sin\frac{\theta}{2}|\downarrow\rangle_X, \quad (16)$$

since the ability to control and perform operations on the state of system **X** means that it is reasonable to omit the phase information. If $\varrho_A = |\uparrow\rangle_A\langle\uparrow|$ is the state we need, the only thing we should do is adjust the parameters in $\mathcal{V}_\phi(\tau)$ to make it into a projection operator on the state $|\uparrow\rangle_A$. This can be achieved by choosing $\Omega_X = \Omega_A$, $\Omega_X + \Omega_A = 4n\pi$, $\cos\frac{\theta}{2} = \pm 1$, and $\cos\delta\tau = 0$. The operator $\mathcal{V}_\phi(\tau)$ then becomes

$$\mathcal{V}_\phi(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (17)$$

One can check that this operator is a projection operator on the state $|\uparrow\rangle_A$. If we chose $\Omega_X = \Omega_A = 0$, $\theta = 0$, $\tau = \frac{\pi}{2g}$, and prepare the system **X** on the initial state $|\phi\rangle_A = |\uparrow\rangle_A$. At the time $\tau = \frac{\pi}{2g}$, we measure the spin of the qubit **X**, chances are that we get $|\uparrow\rangle_X$, and the system **A** collapses to the state $\varrho_A = |\uparrow\rangle_A\langle\uparrow|$.

In the following, we extend the method to multiparticle cases and provide a two-step method to generate a maximally entangled state. For simplicity, consider a three-qubit system $\mathbf{X}+\mathbf{A}+\mathbf{B}$, with a Hamiltonian

$$H_{\rm tot} = \Omega\frac{1+\sigma_3^X}{2} + \Omega\frac{1+\sigma_3^A}{2} + \Omega\frac{1+\sigma_3^B}{2} + g(\sigma_+^X\sigma_-^A + \sigma_-^X\sigma_+^A) + g(\sigma_+^X\sigma_-^B + \sigma_-^X\sigma_+^B). \quad (18)$$

The eigenvalues are

$$E^{(0)} = 0, \quad (19a)$$

$$E_0^{(1)} = \Omega, \quad E_\pm^{(1)} = \Omega \pm \sqrt{2}g, \quad (19b)$$

$$E_0^{(2)} = 2\Omega, \quad E_\pm^2 = 2\Omega \pm \sqrt{2}g, \quad (19c)$$

$$E^{(3)} = 3\Omega, \quad (19d)$$

and the corresponding eigenstates are

$$|E^{(0)}\rangle_{XAB} = |\downarrow\downarrow\downarrow\rangle_{XAB}, \quad (20a)$$

$$\left|E_0^{(1)}\right\rangle_{XAB} = |\downarrow\psi^-\rangle_{XAB}, \quad (20b)$$

$$|E_\pm^{(1)}\rangle_{XAB} = \frac{1}{\sqrt{2}}[|\downarrow\psi^+\rangle_{XAB} \pm |\uparrow\downarrow\downarrow\rangle_{XAB}], \quad (20c)$$

$$\left|E_0^{(2)}\right\rangle_{XAB} = |\uparrow\psi^-\rangle_{XAB}, \quad (20d)$$

$$|E_\pm^{(2)}\rangle_{XAB} = \frac{1}{\sqrt{2}}[|\uparrow\psi^+\rangle_{XAB} \pm |\downarrow\uparrow\uparrow\rangle_{XAB}], \quad (20e)$$

$$|E^{(3)}\rangle_{XAB} = |\uparrow\uparrow\uparrow\rangle_{XAB}. \quad (20f)$$

The evolution operator $\mathcal{U} = \exp(-iH_{\rm tot}\tau)$ can be expanded as a matrix in the Hilbert space of **X**,

$$\mathcal{U} = \begin{pmatrix} \mathcal{U}_{11} & \mathcal{U}_{12} \\ \mathcal{U}_{21} & \mathcal{U}_{22} \end{pmatrix}, \quad (21)$$

where $\mathcal{U}_{ij}$ $(i,j = 1,2)$ are operators acting on the Hilbert space of $\mathbf{A}\otimes\mathbf{B}$. They are given by

$$\mathcal{U}_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\sqrt{2}g\tau & 0 \\ 0 & 0 & 0 & \cos\sqrt{2}g\tau \end{pmatrix}, \quad (22a)$$

$$\mathcal{U}_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -i\,\sin\sqrt{2}g\tau & 0 & 0 \\ 0 & 0 & -i\,\sin\sqrt{2}g\tau & 0 \end{pmatrix}, \quad (22b)$$

$$\mathcal{U}_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i\,\sin\sqrt{2}g\tau & 0 \\ 0 & 0 & 0 & -i\,\sin\sqrt{2}g\tau \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (22c)$$

$$\mathcal{U}_{22} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\sqrt{2}g\tau & 0 & 0 \\ 0 & 0 & \cos\sqrt{2}g\tau & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (22d)$$

Here, from left to right, we choose $|\psi^-\rangle$, $|\uparrow\uparrow\rangle$, $|\psi^+\rangle$, and $|\downarrow\downarrow\rangle$ as the bases of the Hilbert space $\mathbf{A}\otimes\mathbf{B}$.

We assume that the qubit **X** is initially prepared in the state

$$|\phi_1\rangle_X = \cos\frac{\theta_1}{2}|\uparrow\rangle_X + \sin\frac{\theta_1}{2}|\downarrow\rangle_X. \quad (23)$$

The first step of our two-step preparation method is the projection measurement of the system **X** on the initial state $|\phi_1\rangle_X$. The corresponding $\mathcal{V}$ operator on the Hilbert space $\mathbf{A}\otimes\mathbf{B}$ reads

$$\mathcal{V}_{\phi_1}(\tau_1) = {}_X\langle\phi_1|\mathcal{U}(\tau_1)|\phi_1\rangle_X = {}_X\langle\uparrow|\,M(\tau_1,\theta_1)|\uparrow\rangle_X = {}_X\langle\uparrow|\begin{pmatrix} M_{11}(\tau_1,\theta_1) & M_{12}(\tau_1,\theta_1) \\ M_{21}(\tau_1,\theta_1) & M_{22}(\tau_1,\theta_1) \end{pmatrix}|\uparrow\rangle_X, \quad (24)$$

where

$$M(\tau_1,\theta_1)=\begin{pmatrix}\cos\frac{\theta_1}{2} & \sin\frac{\theta_1}{2}\\ \sin\frac{\theta_1}{2} & -\cos\frac{\theta_1}{2}\end{pmatrix}\begin{pmatrix}\mathcal{U}_{11}(\tau_1) & \mathcal{U}_{12}(\tau_1)\\ \mathcal{U}_{21}(\tau_1) & \mathcal{U}_{22}(\tau_1)\end{pmatrix}\begin{pmatrix}\cos\frac{\theta_1}{2} & \sin\frac{\theta_1}{2}\\ \sin\frac{\theta_1}{2} & -\cos\frac{\theta_1}{2}\end{pmatrix}. \tag{25}$$

One can see that (for the calculation of the M matrix, see the Appendix)

$$\mathcal{V}_{\phi_1}(\tau_1)=M_{11}(\tau_1,\theta_1)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos^2\frac{\theta_1}{2}+\sin^2\frac{\theta_1}{2}\cos\sqrt{2}g\tau_1 & -i\frac{\sin\theta_1}{2}\sin\sqrt{2}g\tau_1 & 0\\ 0 & -i\frac{\sin\theta_1}{2}\sin\sqrt{2}g\tau_1 & \cos\sqrt{2}g\tau_1 & -i\frac{\sin\theta}{2}\sin\sqrt{2}g\tau_1\\ 0 & 0 & -i\frac{\sin\theta_1}{2}\sin\sqrt{2}g\tau_1 & \sin^2\frac{\theta_1}{2}+\cos^2\frac{\theta_1}{2}\cos\sqrt{2}g\tau_1\end{pmatrix}. \tag{26}$$

By adjusting the parameters, for example, if we choose $\theta_1=\frac{\pi}{2},\sqrt{2}g\tau_1=\pi$, i.e.,

$$|\phi_1\rangle_X=\frac{1}{\sqrt{2}}|\uparrow\rangle_X+\frac{1}{\sqrt{2}}|\downarrow\rangle_X, \tag{27}$$

then the matrix $M_{11}(\tau_1,\theta_1)$ reads

$$M_{11}(\tau_1,\theta_1)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{28}$$

It is a projection operator in the subspace spanned by $|\psi^-\rangle$ and $|\psi^+\rangle$. The probability that we get $|\phi_1\rangle_X$ after the measurement is

$$\begin{aligned}p(|\phi_1\rangle_X\to|\phi_1\rangle_X)&=\mathrm{Tr}[M_{11}(\tau_1,\theta_1)\rho M_{11}^{\dagger}(\tau_1,\theta_1)]\\&=\rho_{11}+\rho_{33}.\end{aligned} \tag{29}$$

Here, we assume that the initial density matrix made up of **A** and **B**, $\varrho_{AB}(0)\equiv\rho$, and ρ_{ij} represents its elements.

In the next step, a different base state, say $|\psi_2\rangle_X=\cos\frac{\theta_2}{2}|\uparrow\rangle_X+\sin\frac{\theta_2}{2}|\downarrow\rangle_X$ will be used. After the first measurement, a rotation from $|\phi_1\rangle_X$ to $|\phi_2\rangle_X$ is done on the system **X**. After a time interval τ_2, we perform a second measurement on system **X** using $|\psi_2\rangle_X$. With some probability, we may find the system **X** in the state $|\psi_2\rangle_X$ so that the system $\mathbf{A}+\mathbf{B}$ is driven to a pure state $|\psi^-\rangle$. Due to the very short time that the rotation operation acts on the systems, we could ignore its influence on the evolution of the system $\mathbf{A}+\mathbf{B}$. If we adjust the parameter so that $\cos\sqrt{2}g\tau_2=0$, $\sin\frac{\theta_2}{2}=0$, i.e., we can choose

$$|\phi_2\rangle_X=|\uparrow\rangle_X, \tag{30}$$

then the corresponding M matrix reads

$$M_{11}(\tau_2,\theta_2)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{31}$$

Thus, after two measurements plus a rotation on system **X**, and assuming that both the anticipated outcomes are obtained, we achieve the operation $\mathcal{P}$ on the system $\mathbf{A}+\mathbf{B}$ where

$$\mathcal{P}=M_{11}(\tau_2,\theta_2)M_{11}(\tau_1,\theta_1)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{32}$$

It is exactly a projection operator on the state $|\psi^-\rangle\langle\psi^-|$. Thus, all the components of the original density matrix $\varrho_{AB}(0)$ except for the $|\psi^-\rangle\langle\psi^-|$ part, are effectively eliminated. The state of $\mathbf{A}+\mathbf{B}$ is purified to the maximally entangled state and the probability we can obtain this is

$$p=\mathrm{Tr}(\mathcal{P}\rho\mathcal{P}^{\dagger})=\rho_{11}=\langle\psi^-|\varrho_{AB}(0)|\psi^-\rangle. \tag{33}$$

This is exactly the weight of $|\psi^-\rangle$ in the initial state of $\mathbf{A}+\mathbf{B}$.

This technique has at least two appreciable advantages: (i) Compared with the large measurements number N needed in the Zeno-like measurement method, only two measurements and a rotation are sufficient in this scheme. (ii) In the previous Zeno-like measurement method, the existence of the unique, discrete, and nondegenerate largest eigenvalue of operator $\mathcal{V}_{\phi}(\tau)$ is indispensable. However, this requirement is not crucial in the improved strategy. The simple thing that should be done is to design two (as the system becomes larger, more may be needed) states of system **X** as base states to be projected on in order to make the operator $\mathcal{P}=\mathcal{V}_{\phi_2}(\tau_2)\mathcal{V}_{\phi_1}(\tau_1)$ exactly a projection operator on the desired maximally entangled state (for example, $|\psi^-\rangle$).

Lastly, we give a brief summary of our method. We first prepare the system **X**, which we can fully control and on which we can operate, in an initial state $|\phi_1\rangle_X$. At an appropriate time τ_1, we perform a measurement on the state $|\phi_1\rangle_X$. If we get the anticipated state $|\phi_1\rangle_X$, then a rotation from $|\phi_1\rangle_X$ to $|\phi_2\rangle_X$ is performed on **X** immediately. After a time interval τ_2, another measurement on state $|\phi_2\rangle_X$ is performed on **X**. If the same state $|\phi_2\rangle_X$ is obtained, then the system $\mathbf{A}+\mathbf{B}$ has been purified to the desired state; for example, the maximally entangled states. In our example, i.e., in a three-qubit system with the Hamiltonian given by Eq. (12), if we choose $|\phi_1\rangle_X=\frac{1}{\sqrt{2}}|\uparrow\rangle_X+\frac{1}{\sqrt{2}}|\downarrow\rangle_X$, $\tau_1=\frac{\pi}{\sqrt{2}g}$, $|\phi_2\rangle_X=|\uparrow\rangle_X$, and $\tau_2=\frac{\sqrt{2}\pi}{g}$, we get the pure entangled state $|\psi^-\rangle=\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle)$ with a probability $\langle\psi^-|\varrho_{AB}(0)|\psi^-\rangle$, just as in the Zeno-like measurement method.

IV. A DETERMINATE METHOD: A FEEDBACK IMPROVEMENT

Although the method described in Sec. III has some advantages, it is still not a deterministic process because there are finite probabilities that the states $|\phi_1^\perp\rangle_X$ and $|\phi_2^\perp\rangle_X$ are obtained instead of the required states: $|\phi_1\rangle_X$ and $|\phi_2\rangle_X$. If **X** is found in the state $|\phi_1^\perp\rangle_X$ or $|\phi_2^\perp\rangle_X$, the previous scheme would not be suitable for the situation mentioned above. In this section, we give a deterministic method which further improves on the previous scheme. To demonstrate the scheme, we continue to use the example using the Hamiltonian given by Eq. (18) above.

(1) We perform the measurement on **X** to confirm it is in the state $|\phi_1\rangle_X = \frac{1}{\sqrt{2}}|\uparrow\rangle_X + \frac{1}{\sqrt{2}}|\downarrow\rangle_X$ at the time interval $\tau_1 = \frac{\pi}{\sqrt{2}g}$. But it may be in the state $|\phi_1^\perp\rangle_X = \frac{1}{\sqrt{2}}|\uparrow\rangle_X - \frac{1}{\sqrt{2}}|\downarrow\rangle_X$ with the probability $\rho_{22} + \rho_{44}$. In that case, it is equivalent to an operation $M_{21}(\tau_1)$ on the system $\mathbf{A} + \mathbf{B}$, where

$$M_{21}(\tau_1) = {}_X\langle\phi_1^\perp|\mathcal{U}(\tau_1)|\phi_1\rangle_X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (34)$$

Thereafter, the components $|\uparrow\uparrow\rangle\langle\uparrow\uparrow|$ and $|\downarrow\downarrow\rangle\langle\downarrow\downarrow|$ are projected out and the other components are all eliminated. The existence of nondiagonal elements $|\psi^+\rangle\langle\uparrow\uparrow|$ and $|\psi^+\rangle\langle\downarrow\downarrow|$ in the operator $\mathcal{V}_\phi(\tau)$ provides the recovery of $|\psi^+\rangle$ which is also one of the maximally entangled states, in the case of the evolution with total Hamiltonian and measurement on **X** in Eq. (26). So, by adjusting the parameters in the operator $\mathcal{V}_\phi(\tau)$, the state $|\psi^+\rangle$ can be generated from the residual $|\uparrow\uparrow\rangle\langle\uparrow\uparrow|$ and $|\downarrow\downarrow\rangle\langle\downarrow\downarrow|$ components.

After the state $|\phi_1^\perp\rangle_X$ is obtained, we still choose $|\phi_1\rangle_X = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ as the base state. But the parameter τ_1' should be adjusted to satisfy $\sin\sqrt{2}g\tau_1' = 1$. In this case, (i) the state $|\phi_1\rangle_X$ is obtained with the probability $\mathrm{Tr}[M_{12}(\tau_1')\rho M_{12}^\dagger(\tau_1')] = \frac{1}{2}[\rho_{22} + \rho_{44} + \mathrm{Re}(\rho_{24})]$, where $M_{12}(\tau_1')$ is the corresponding operator on system $\mathbf{A} + \mathbf{B}$,

$$M_{12}(\tau_1') = {}_X\langle\phi_1|\mathcal{U}(\tau_1')|\phi_1^\perp\rangle_X = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & i & 0 & -i \\ 0 & 0 & i & -1 \end{pmatrix}. \quad (35a)$$

(ii) The state $|\phi_1^\perp\rangle_X$ may be obtained with the probability $\mathrm{Tr}[M_{22}(\tau_1')\rho M_{22}^\dagger(\tau_1')] = \frac{1}{2}(\rho_{22} + \rho_{44} - Re(\rho_{24}))$, where $M_{22}(\tau_1')$ is the corresponding operator on system $\mathbf{A} + \mathbf{B}$,

$$M_{22}(\tau_1') = {}_X\langle\phi_1^\perp|U(\tau_1')|\phi_1^\perp\rangle_X = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & 0 & i \\ 0 & 0 & i & 1 \end{pmatrix}. \quad (35b)$$

From Eq. (35), one can notice that both the operators $M_{12}(\tau_1')$ and $M_{22}(\tau_1')$ can generate $|\psi^+\rangle$ from the residual $|\uparrow\uparrow\rangle$ or $|\downarrow\downarrow\rangle$ components (notice that $|\psi^-\rangle$ cannot revive). Therefore, the crucial step is to measure the system **X** to find out whether it is in state $|\phi_1\rangle_X$ with the time intervals τ_1 and τ_1' in turn. We will give a flow chart to explain the process in Fig. 1.

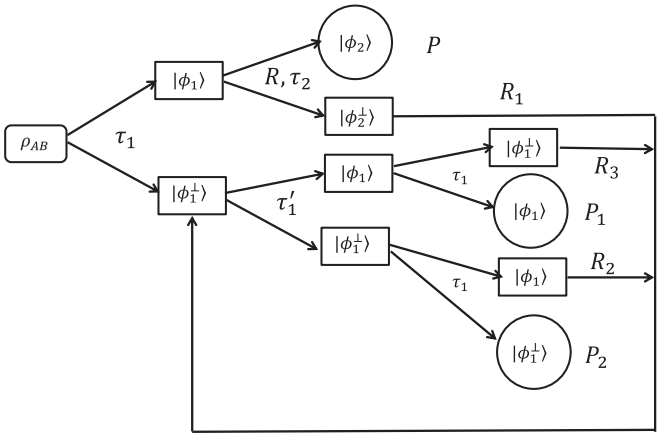

FIG. 1. This is a flow chart to illustrate the process to obtain a maximally entangled state. In the first step, we measure **X** to find out whether it is in the state $|\phi_1\rangle_X$ or in the state $|\phi_1{}^\perp\rangle_X$ with time interval τ_1. In the case that $|\phi_1\rangle_X$ is detected, a rotation (R) is exerted on **X** to rotate it to the state $|\phi_2\rangle_X$. Then, a measurement on $|\phi_2\rangle_X$ is performed after a time interval τ_2. If the outcome is $|\phi_2\rangle_X$, the maximally entangled state $|\psi^-\rangle$ is obtained whose probability is $P = \rho_{11} = \langle\psi^-|\varrho_{AB}(0)|\psi^-\rangle$. If the result comes out to be $|\phi_2{}^\perp\rangle_X$, then a rotation (R_1) is made, as the figure shows. When **X** is found in the state in which we do not confirm it, **X** can be driven to the state $|\phi_1{}^\perp\rangle_X$ (or the density matrix of **AB** has an equivalent structure) by measurement or by rotation. In this case, as the flow chart shows, we have a probability $P_1 + P_2 = \frac{1}{2}(P_1 = \frac{\rho_{22}+\rho_{44}+2\,\mathrm{Re}(\rho_{24})}{4(\rho_{22}+\rho_{44})}$, $P_2 = \frac{\rho_{22}+\rho_{44}-2\,\mathrm{Re}(\rho_{24})}{4(\rho_{22}+\rho_{44})})$ to get the maximally entangled state $|\psi^+\rangle$ in the first circle. That is to say, even in the situation where we fail to obtain the state $|\psi^-\rangle$, as the process goes on, a maximally entangled state $|\psi^+\rangle$ can be obtained in several steps. So, this method is a deterministic one.

(2) We confirm **X** in the state $|\phi_1\rangle_X$ with a probability $\rho_{11} + \rho_{33}$. Then, after a rotation from $|\phi_1\rangle_X$ to $|\phi_2\rangle_X$, we perform the measurement using $|\phi_2\rangle_X$ as the base state. In our case, $|\phi_1\rangle_X = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$, $\cos\sqrt{2}g\tau_1 = -1$, $|\phi_2\rangle_X = |\uparrow\rangle$, and $\cos\sqrt{2}g\tau_2 = 0$.

(i) The system **X** is detected on the state $|\phi_2\rangle_X$ with the probability $\frac{\rho_{11}}{\rho_{11}+\rho_{33}}$. Then, the purification process is completed and we get the maximally entangled state $|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle_{AB} - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle_{AB}$. This is exactly the case described in Sec. III.

(ii) The system **X** can be found to be on the state $|\phi_2^\perp\rangle_X = |\downarrow\rangle$ with the probability $\frac{\rho_{33}}{\rho_{11}+\rho_{33}}$. Then, we can rotate the state of **X** from $|\phi_2^\perp\rangle_X$ to $|\phi_1^\perp\rangle_X$ and continue as (1) describes.

V. CONCLUSION

In this article, a two-step measurement method to generate maximally entangled states is demonstrated using a concrete example. The scheme can be extended in principle to other systems. The basic idea is to design several (in the example here, the number is two) different states of the systems (in this article, **X**) that can be easily controlled. The states are chosen

to make the effective operator $[\mathcal{V}_\phi(\tau)]$ acting on the systems $(\mathbf{A}+\mathbf{B})$ which are interacting with the system $(\mathbf{X})$, into a projection operator on the desired entangled state. Unlike the Zeno-like measurement method which only projected on the same state (the initial state of $\mathbf{X}$), this method needs only two measurements instead of a large N number of measurements.

Another point that needs to be emphasized here is formulation of a feedback strategy that ensures a determinate entanglement purification. This is a powerful tool that can guarantee an entangled state whenever a mixed state is available. However, its extension to other systems for this feedback strategy, though possible, is not explicitly demonstrated here.

Since the Hamiltonian given by Eqs. (12) and (18) [18] is very typical where rotating-wave approximation can be used, it can be facilitated in systems such as the XY coupling in the spin network [28,29] or flux qubits [30]. In the spin network, if spin i and spin j are connected, then the interaction between them can be described by the XY model:

$$H_{\rm int} = \sigma_X^i \sigma_X^j + \sigma_Y^i \sigma_Y^j, \tag{36}$$

which is exactly equivalent to $\sigma_+^i \sigma_-^j + \sigma_-^i \sigma_+^j$ that Eqs. (12) and (18) use. In this case, by performing measurements on a spin, we can drive spins connected to it into entangled states. The XY model is also an experimentally realizable method in superconducting systems [30]. And our scheme can also be applied by performing measurement [31] on a flux qubit. In addition, in the ion trap [32–34] and cavity QED system [35–37], similar Hamiltonians exist, where the σ_+ and σ_- of the electronic field been replaced by the creation and annihilation operators of photons $a^\dagger$ and a. The field can be used as the medium to entangle atoms in distance. However, the difficulty of using electronic fields may lie in the measurement and exact operation of it since it is a continuous-variable system.

ACKNOWLEDGMENTS

This work was supported in part by the 10000-Plan of Shandong province, the National High-Tech Program of China Grants No. 2011AA010800 and No. 2011AA010803 and NSFC Grants No. 11174177 and No. 60725416. L.C.K. and S.-c.W. acknowledge support from National Research Foundation and the Ministry of Education, Singapore.

APPENDIX: THE CALCULATION OF THE M MATRIX

Let us give a detailed calculation of the M matrix in Eq. (24) in this section. Under the assumptions spelled out in Eq. (23), the initial state of system $\mathbf{X}$ can be described by $|\phi_1\rangle_X = \cos\frac{\theta_1}{2}|\uparrow\rangle_X + \sin\frac{\theta_1}{2}|\downarrow\rangle_X$, the M matrix whose blocks represent different effective operators (acting on the Hilbert space of $\mathbf{A}\otimes\mathbf{B}$) corresponding to the different outcomes of the measurements can be calculated as follows:

$$\begin{aligned}
&\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \\
&= \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \mathcal{U}_{11} & \mathcal{U}_{12} \\ \mathcal{U}_{21} & \mathcal{U}_{22} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\frac{\theta}{2}\mathcal{U}_{11} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}\mathcal{U}_{21} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathcal{U}_{12} + \sin^2\frac{\theta}{2}\mathcal{U}_{22} & \sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathcal{U}_{11} + \sin^2\frac{\theta}{2}\mathcal{U}_{21} - \cos^2\frac{\theta}{2}\mathcal{U}_{12} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}\mathcal{U}_{22} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}U_{11} - \cos^2\frac{\theta}{2}\mathcal{U}_{21} + \sin^2\frac{\theta}{2}\mathcal{U}_{12} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}\mathcal{U}_{22} & \sin^2\frac{\theta}{2}\mathcal{U}_{11} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}\mathcal{U}_{21} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathcal{U}_{12} + \cos^2\frac{\theta}{2}\mathcal{U}_{22} \end{pmatrix}.
\end{aligned} \tag{A1}$$

By using the $\mathcal{U}$ matrix given by Eq. (22), it is easy to compute the elements of the M matrix.

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[2] Ryszard Horodecki, Pawel Horodecki, Michal Horodecki, and Karol Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[3] H. Krauter, C. A. Muschik, K. Jensen, W. Wasilewski, J. M. Petersen, J. I. Cirac, and E. S. Polzik, Phys. Rev. Lett. **107**, 080503 (2011).
[4] M. S. Rudner, L. M. K. Vandersypen, V. Vuletic, and L. S. Levitov, Phys. Rev. Lett. **107**, 206806 (2011).
[5] Amir Leshem and Omri Gat, Phys. Rev. A **84**, 052303 (2011).
[6] Chun-Hua Yuan, L. Q. Chen, Z. Y. Ou, and Weiping Zhang, Phys. Rev. A **83**, 054302 (2011).
[7] M. S. Rudner, L. M. K. Vandersypen, V. Vuletic, and L. S. Levitov, Phys. Rev. Lett. **107**, 206806 (2011).
[8] Motoaki Bamba and Hajime Ishihara, Phys. Rev. Lett. **105**, 123906 (2010).
[9] Thomas Zell, Friedemann Queisser, and Rochus Klesse, Phys. Rev. Lett. **102**, 160501 (2009).
[10] Davide Rossini and Giuliano Benenti, Phys. Rev. Lett. **100**, 060501 (2008).
[11] Y. S. Weinstein and C. S. Hellberg, Phys. Rev. Lett. **95**, 030501 (2005).
[12] D. G. Cory and T. F. Havel, Science **4**, 1456 (2004).
[13] Patrick Hayden, Nature (London) **474**, 41 (2011).
[14] Adetunmise C. Dada, Jonathan Leach, Gerald S. Buller, Miles J. Padgett, and Erika Andersson, Nat. Phys. **7**, 677 (2011).
[15] Jonathan Home, Nat. Phys. **6**, 938 (2010).
[16] Elias Amselem and Mohamed Bourennane, Nat. Phys. **6**, 827 (2010).

[17] H. Nakazato, T. Takazawa, and K. Yuasa, Phys. Rev. Lett. **90**, 060401 (2003).
[18] H. Nakazato, M. Unoki, and K. Yuasa, Phys. Rev. A **70**, 012303 (2004).
[19] G. Compagno, A. Messina, Hiromichi Nakazato, A. Napoli, Makoto Unoki, and Kazuya Yuasa, Phys. Rev. A **70**, 052316 (2004).
[20] Kazuya Yuasa and Hiromichi Nakazato, Prog. Theor. Phys. **114**, 523 (2005).
[21] L.-A. Wu, D. A. Lidar, and S. Schneider, Phys. Rev. A **70**, 032322 (2004).
[22] Kanji Yoh, Kazuya Yuasa, and Hiromichi Nakazato, Physica E **29**, 674 (2005).
[23] B. Militello, K. Yuasa, H. Nakazato, and A. Messina, Phys. Rev. A **76**, 042110 (2007).
[24] H. Nakazato, K. Yuasa, B. Militello, and A. Messina, Phys. Rev. A **77**, 042114 (2008).
[25] X. Q. Shao, L. Chen, S. Zhang, Y. F. Zhao, and K. H. Yeon, J. Phys. B , **43**, 135502 (2010).
[26] B. Bruno, C. Giuseppe, N. Hiromichi, and K. Yuasa, Phys. Rev. A **82**, 060101 (2010).
[27] K. Yuasa, D. Burgarth, G. Vittorio, and H. Nakazato, New J. Phys. **11**, 123027 (2009).
[28] Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl, Phys. Rev. Lett. **92**, 187902 (2004).
[29] Tetsufumi Tanamoto, Yu-xi Liu, Xuedong Hu, and Franco Nori, Phys. Rev. Lett. **102**, 100501 (2009).
[30] Yu-xi Liu, L. F. Wei, J. S. Tsai, and Franco Nori, Phys. Rev. Lett. **96**, 067003 (2006).
[31] A. Izmalkov, M. Grajcar, E. Ilichev, T. Wagner, H. G. Meyer, A. Y. Smirnov, M. H. S. Amin, A. Maassen van den Brink, and A. M. Zagoskin, Phys. Rev. Lett. **93**, 037003 (2004).
[32] J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
[33] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, Rev. Mod. Phys. **75**, 281 (2003).
[34] Rainer Blatt and David Wineland, Nature (London) **453**, 1008 (2008).
[35] A. Imamoğlu, D. D. Awschalom, G. Burkard, D. P. DiVincenzo, D. Loss, M. Sherwin, and A. Small, Phys. Rev. Lett. **83**, 4204 (1999).
[36] J. M. Fink, M. Goppl, M. Baur, R. Bianchetti, P. J. Leek, A. Blais, and A. Wallraff, Nature (London) **454**, 315 (2008).
[37] E. Verhagen, S. Delglise, S. Weis, A. Schliesser, and T. J. Kippenberg, Nature (London) **482**, 63 (2012).