Teaching Heuristics and Metacognition in Mathematical Problem Solving

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Introduction

The recommendation that school mathematics should be primarily concerned with developing children's problem solving skills has been overwhelmingly endorsed both in the United States of America as well as in most other western countries. This great support is evident through the considerable amount of interest shown by educational researchers into the nature of mathematical problem solving in recent years. This surge of activity came about after the National Council of Teachers of Mathematics 1980 An Agenda For *Action* (NCTM Yearbook (1980), followed by the well received Cockcroft Report (Cockcroft (1982), both called for problem solving to be included as an important component in the range of mathematics classroom activities and teaching styles.

Yet to many people, teaching mathematics has always been about teaching students how to solve problems, so what is new about "teaching problem solving" in the classroom? According to Branca (1980) a common interpretation of "problem solving" as a dynamic, ongoing process has implications for classroom practices. What is considered important in this interpretation are the methods, procedures, strategies and heuristics that students use in solving problems. These elements of the problem solving process are seen to be its essence and as such became a focus of the mathematics curriculum advocated for the 1980s. In Singapore, a new mathematics curriculum for the 1990s is being conceptualised based on the theme "mathematical problem solving" with emphasis on two new components: metacognition and heuristic processes, to be incorporated into the primary and secondary school mathematics syllabi (MOE, 1990).

Teaching Heuristics

In the theory of problem solving described by Newell and Simon (1972), heuristics are described as rules for selecting search paths through a problem space by exploiting the information in the task environment. The literature on the learning and teaching of problem solving in the domain of mathematics, concentrates largely on the type of heuristics advocated by Polya (1957). Polya's classic "How To Solve It" provides a list of heuristics for understanding a problem and devising a plan to solve it, including making sure that the conditions and the goal state are understood, reformulating the problem, thinking of a related problem, making the problem more general and breaking the problem into parts. Schoenfeld (1985) defines heuristic strategies as techniques used by problem solvers when they run into difficulty. They are general suggestions that help an individual to understand a problem better or to make progress towards a solution.

Polya examined his own thoughts to find useful patterns of problem solving behaviours. The result was a general prescription of a four-phase model of the problem solving process: understanding the problem, devising a plan, carrying out the plan and looking back. The details of each stage included a range of problem solving heuristics or rules of thumb for making progress on difficult problems. Some of Polya's heuristics are usually set in the form of questions or general hints such as: "What is the unknown?", "What are the data?", "What is the condition?", "Do you know a related problem?", "Draw a figure.", "Introduce suitable notation.", "If you cannot solve the proposed problem try to solve first some related problem." etc.

Mathematicians generally agree that Polya's description of the problem solving strategies are accurate. The bulk of the mathematics education community has adopted Polya's approach as ***the*** approach to problem solving. Problem solving lessons based on Polya's method, designed by school teachers have appeared in the NCTM 1980 and 1983 Yearbooks (Krulik and Reys 1980, Shufelt and Smart 1983). Many other instructional packages prescribed for training students to be good problem solvers have also been produced. Among which are: Rubinstein (1980), Hayes (1981), Mason, Burton and Stacey (1982), Whimbey and Lochhead (1982), Burton (1984), Krulik and Rudnick (1984), Stacey and Grove (1985), Day (1986) and others.

In attempting to teach problem solving skills, most of the instructional programmes use non-routine mathematical process problems for pupils to learn to use some of the following heuristics:

- understand the problem
- try some simple examples
- organise systematically
- make a table
- spot a pattern
- make a guess and check
- make logical deduction
- generalise to a rule
- look back and check

Focus on Metacognitlon

The work of Schoenfeld (1985) has become prominent in mathematical problem solving in recent years because of his belief in the importance of control dr metacognition in the process of solving complex problems. The term "metacognition" was first used by Flavell (1976) to refer to "one's knowledge concerning one's own cognitive processes and products or anything related to them". Schoenfeld recognised the need to do more than equip students with a collection of skills and strategies. Students need to be instructed on how to reflect upon the processes they use to solve problems and control their own resources to make reasonable decisions about which heuristics to try and when to use them. According to Schoenfeld:

> One of the hallmarks of good problem solvers' control behaviour is that, while they are in the midst of working problems,

> such individuals seem to maintain an internal dialogue regarding the way that their solutions evolve.
>
> (Schoenfeld 1985, p. 140)

A Metacognitive-Heuristic Approach in the Classroom

As problem solving is being accepted as an important aspect of the mathematics curriculum in schools, the trend is toward instruction that attempts to create an appropriate atmosphere for effective problem solving, involving changes in the traditional role of the teachers and in classroom techniques. In a traditional lesson when a student sees a teacher explain a problem, he or she sees the results of the teacher's thinking but seldom witnesses the thought process itself. Similarly in most mathematics textbooks, the logic of mathematics is presented by way of elegant well-structured theorems or proofs which seldom reveal much about the often messy ways in which these proofs were originally discovered. Textbooks almost never provide examples of knowledge representation or of heuristics or meta-level processes. Exemplifying these processes is therefore left to the teacher.

The traditional teaching method of exposition-and-imitationis not sufficient given the demands that problem solving makes on the learner, such as independent decision making, exploration, investigation of alternatives, hypothesis testing, discussion and evaluation of processes and results. Teachers may have to move towards a fundamentally different pedagogical style in order to model, as well as to instil in students a repertoire of effective problem solving heuristics and metacognitive strategies that would lead to the solution of any problem. Teachers who encourage their students to solve problems, to inculcate independent thinking and who ask for a wide variety of approaches (rather than merely giving answers) will provide their students with a rich problem solving experience.

Schoenfeld (1985) through his series of studies on a prescriptive approach to the teaching of heuristics at a metacognitive level,. developed four natural "in-class" techniques that are adaptable for use in any classroom for mathematical problem solving instruction.

The first technique was to create awareness of metacognitive issues among his students when he showed them videotapes of others working on problems. He found that it was easier for students to analyse someone else's behaviour and then to see how the analysis applied to themselves. As a result of watching the tapes and discussing them the students became more aware of their own thinking processes.

The second technique used the teacher as a role model for metacognitive behaviour. By working a problem on the board from the beginning, students went along with the teacher through all the stages to the solution: looking at examples, making a few tentative explorations and looking for promising things to do. The teacher would generate a few reasonable approaches, deciding among them and pursue one for a while, occasionally evaluating with "Am I making reasonable progress?" or "Does this seem like the right thing to do?" and so on. Schoenfeld admitted that such a modelling approach could be artificial and recommended that its use should not be extended for too long.

The third technique was class discussion with the teacher serving as a "control". The class worked on the problem as a whole. The teacher did not try to guide the students to the correct solution, but instead helped them make the most of what they themselves generated to reflect on their strategies. The teacher's presence as a moderator forced the class to focus on control decisions made by themselves.

Lastly, students were given plenty of opportunities to be actively engaged in problem solving in small groups of three or four persons. Here the teacher as a facilitator moves from group to group, answering questions, offering advice and asking questions such as "What are you doing?", "Can you describe it precisely?", "How does it fit into the solution?", "What will you do with the outcome?" and so on.

Other small group problem solving techniques such as having students work in pairs have been proposed. Whimbey and Lochhead (1982) advocate working in pairs and "thinking aloud" while trying to solve problems for two reasons: (a) by listening to other people solving problems, one may learn something about techniques that work and those that do not, and (b) exposing one's own thought processes verbally to others and to oneself, makes it possible for one's approach to be analysed and criticised. In a pair-setting it is often that one student acts as the "solver" and the other the "monitor" and then the roles are exchanged. The role of the monitor is to ask questions that clarify the nature of the problem solving ability of the solver. In this way the students would be explicitly encouraged to accept not only their own mistakes but those made by their classmates.

In conclusion, if the aim is to help students become effective problem solvers, then instruction on mathematical problem solving must also address these important meta-cognitive processes. What one needs to become an effective problem solver is a repertoire of heuristics that are likely to be useful in a variety of problem situations, along with meta-knowledge about situations in which specific heuristics are appropriate. Until these processes receive explicit attention in the curriculum students may know fairly well what to do in routine and simple problem situations, but will have little competence in handling unfamiliar or complex problems.

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