Title	On zero-free intervals in (1,2) of chromatic polynomials of some families of graphs
Author(s)	F. M. Dong and K. M. Koh
Source	*SIAM Journal on Discrete Mathematics, 24*(2), 370-378. http://dx.doi.org/10.1137/070698294
Published by	Society for Industrial and Applied Mathematics (SIAM).

First Published in *SIAM Journal on Discrete Mathematics* in Volume 24, Issue 2, 2010, published by the Society for Industrial and Applied Mathematics (SIAM).

Citation: Dong, F. M., & Koh, K. M. (2010). On zero-free intervals in (1,2) of chromatic polynomials of some families of graphs. *SIAM Journal on Discrete Mathematics, 24*(2), 370-378. http://dx.doi.org/10.1137/070698294

SIAM J. DISCRETE MATH.
Vol. 24, No. 2, pp. 370–378

ON ZERO-FREE INTERVALS IN (1, 2) OF CHROMATIC POLYNOMIALS OF SOME FAMILIES OF GRAPHS*

F. M. DONG† AND K. M. KOH‡

Abstract. For a family $\mathcal{S}$ of graphs, let $\omega(\mathcal{S})$ be the supremum of $t : 1 < t \leq 2$ such that $P(G, \lambda) \neq 0$ for all $G \in \mathcal{S}$ and all $\lambda \in (1, t)$. In this paper we show that $\omega(\mathcal{S}) = \omega(\mathcal{S} \cap \mathcal{K})$ for any family $\mathcal{S}$ of graphs satisfying certain conditions, where $\mathcal{K}$ is a special family of graphs. This result makes it much easier to determine $\omega(\mathcal{S})$ for such families $\mathcal{S}$.

Key words. chromatic polynomial, zero-free interval, double subdivision, minor

AMS subject classifications. 05C15, 05C31

DOI. 10.1137/070698294

1. Introduction.

It is well known that $(-\infty, 0)$ and $(0, 1)$ are two zero-free intervals for the chromatic polynomials of all graphs (see [6, 9]). By the results established in Jackson [7] and Thomassen [12], we know that there are no chromatic zeros in $(1, 32/27]$ but chromatic zeros are dense in $[32/27, \infty)$.

Let $\mathcal{G}$ be the family of connected graphs of order at least 2. For any $G \in \mathcal{G}$, we define the following graph-function

$$Q(G, \lambda) = (-1)^{v(G)+b(G)-1} P(G, \lambda), \tag{1}$$

where $P(G, \lambda)$, $v(G)$, and $b(G)$ are the chromatic polynomial, the vertex number (i.e., the order), and the block number of G, respectively. Note that $P(G, \lambda) = 0$ if and only if $Q(G, \lambda) = 0$. Also, it can be verified that if $v(G) \leq 4$, then $Q(G, \lambda) > 0$ for all $1 < \lambda < 2$. Jackson [7] proved the following.

THEOREM 1.1. *For any $G \in \mathcal{G}$, $Q(G, \lambda) > 0$ for all $1 < \lambda \leq 32/27$, where the number $32/27$ cannot be replaced by any larger number.*

For any $\mathcal{S} \subseteq \mathcal{G}$, if $\mathcal{S} = \emptyset$, let $\omega(\mathcal{S}) = 2$; otherwise, let

$$\omega(\mathcal{S}) = \sup\{1 < t \leq 2 : P(G, \lambda) \neq 0 \quad \text{for all } \lambda \in (1, t) \text{ and all } G \in \mathcal{S}\}. \tag{2}$$

Since $Q(G, \lambda)$ is a continuous function for any graph G, by Theorem 1.1,

$$\omega(\mathcal{S}) = \sup\{1 < t \leq 2 : Q(G, \lambda) > 0 \quad \text{for all } \lambda \in (1, t) \text{ and all } G \in \mathcal{S}\} \tag{3}$$

holds for any family $\mathcal{S} \neq \emptyset$. By Theorem 1.1 and the definition of $\omega(\mathcal{S})$, the next result follows immediately.

LEMMA 1.1. *If $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{G}$, then $32/27 \leq \omega(\mathcal{S}_2) \leq \omega(\mathcal{S}_1) \leq 2$.*

The following results on $\omega(\mathcal{S})$ have been obtained, although they are not expressed by $\omega(\mathcal{S})$:

(i) Jackson [7] established that $\omega(\mathcal{G}) = 32/27$.

*Received by the editors July 25, 2007; accepted for publication (in revised form) January 14, 2010; published electronically April 14, 2010. This research was partially supported by NIE AcRf funding (RI 5/06 DFM) of Singapore.
http://www.siam.org/journals/sidma/24-2/69829.html
†Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, Singapore 637616 (fengming.dong@nie.edu.sg).
‡Department of Mathematics, National University of Singapore, Singapore 117543 (matkohkm@nus.edu.sg).

(ii) Thomassen [11] proved that if $\mathcal{S}$ is the family of graphs with a Hamiltonian path, then

$$\omega(\mathcal{S}) = \frac{2}{3} + \frac{1}{3}\sqrt[3]{26+6\sqrt{33}} + \frac{1}{3}\sqrt[3]{26-6\sqrt{33}} = 1.29559\cdots.$$

(iii) Dong and Koh [4] proved that $\omega(\mathcal{S}) = 32/27$ if $\mathcal{S}$ is the family of bipartite planar graphs.

(iv) Dong and Koh [3] showed that if $\mathcal{S}$ is the family of graphs with domination number 2, then

$$\omega(\mathcal{S}) = 2 - \frac{1}{6}\left(\sqrt[3]{12\sqrt{93}+108} - \sqrt[3]{12\sqrt{93}-108}\right) = 1.317672196\cdots.$$

Jackson [7] conjectured that $\omega(\mathcal{S}) = 2$ for the family $\mathcal{S}$ of 3-connected nonbipartite graphs. However, counterexamples to this conjecture were discovered recently by Royle [10].

In [7], Jackson introduced an operation on graphs, i.e., *double-subdivision*, which was used to construct a special family of graphs. A double-subdivision is an operation on graphs which replaces an edge uv by two $u-v$ paths of length 2. Let $\mathcal{K}$ is the family of graphs defined as follows:

(i) $K_3 \in \mathcal{K}$;

(ii) for any graph G with $v(G) \geq 4$, $G \in \mathcal{K}$ if and only if G can be obtained from a graph $G' \in \mathcal{K}$ by a double-subdivision on some edge in G'.

In this paper we find that $\mathcal{K}$ plays an important role in the study of real zeros of chromatic polynomials in the interval $(1, 2)$. In sections 2 and 3 of this paper, we will show that $\omega(\mathcal{S}) = \omega(\mathcal{S} \cap \mathcal{K})$ if $\mathcal{S}$ is a family of connected graphs which satisfy certain conditions (see Theorem 3.2). Clearly this result makes it much easier to determine $\omega(\mathcal{S})$ for such a family $\mathcal{S}$ of graphs. While some families $\mathcal{S}$ of graphs with $\omega(\mathcal{S}) = 2$ are discussed in section 4, we shall apply Theorem 3.2 to determine $\omega(\mathcal{S})$ for some families $\mathcal{S}$ of graphs in section 5.

2. The family $\mathcal{K}$. Let G be any graph. Let $V(G)$ and $E(G)$ be the vertex set and edge set of G, respectively. For any $T \subset V(G)$, let $G - T$ be the graph obtained from G by deleting all vertices in T (of course also all edges incident to vertices in T). For any $e \in E(G)$, let $G - e$ be the graph obtained from G by deleting e and let G/e be the graph obtained from G by contracting e and deleting all multiedges but one. For nonadjacent vertices u and v in G, let $G + uv$ be the graph obtained from G by adding an edge between u and v and let $G \cdot uv$ be the graph obtained from G by identifying u and v and deleting all multiedges but one. Note that if e is an edge between u and v in G, then G/e and $(G - e) \cdot uv$ are the same graph.

For any graph G and $T \subseteq V(G)$, let $G[T]$ (or simply $[T]$) denote the subgraph of G induced by T. If U is the vertex set of a component of $G - T$, then $[T \cup U]$ is called a *T-bridge* of G. Note that if $T \neq V(G)$, then G has exactly $c(G - T)$ T-bridges, where $c(H)$ denotes the number of components of a graph H. If G is connected, then T is a cut-set of G if and only if G has at least two T-bridges. If T is a cut-set of G and $[T]$ is complete, then T is called a *complete cut-set* of G.

By the definition of $\mathcal{K}$ given in section 1, every graph in $\mathcal{K}$ can be obtained from K_3 by a sequence of double-subdivisions. Thus the order of every graph in $\mathcal{K}$ is an odd integer. Some other properties on the graphs in $\mathcal{K}$ are given in Lemma 2.1 below. These properties can be proved by induction (see [7] also). For any $n \geq 1$, let P_n be the path graph of order n.

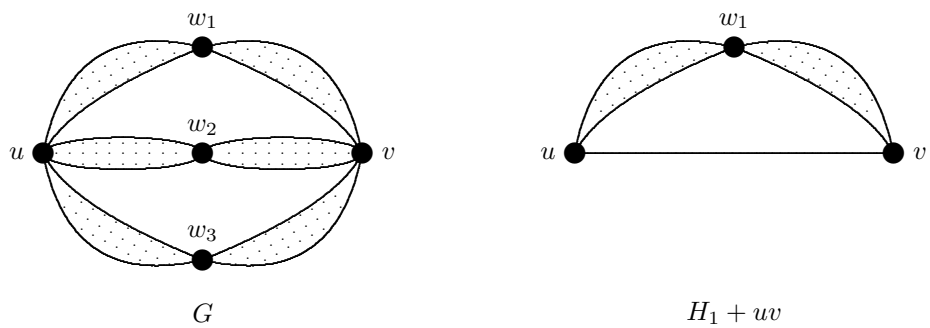

FIG. 1.

LEMMA 2.1. *Let $G \in \mathcal{K}$, e be any edge in G, and let $\{u, v\}$ be any cut-set of G. Then*

(i) *G is 2-connected;*

(ii) *$G-e$ is connected with $b(G-e) = 2$, and $G-e$ either is P_3 or can be obtained from P_3 by a sequence of double-subdivisions; and*

(iii) *$uv \notin E(G)$, $c(G - \{u, v\}) = 3$, and $b(B) = 2$ for each $\{u, v\}$-bridge B of G.*

The operation "double-subdivision" can be generalized to the operation "replacing an edge by a graph" as introduced below. Let G and H be two graphs and let $uv \in E(G)$. Replacing uv in G by H is the combination of the following operations:

(i) Remove uv from G;

(ii) identify u and v, respectively, with two distinct vertices in H.

Thus, a double-subdivision can be regarded as an operation "replacing an edge by a 4-cycle." Notice that a double-subdivision on any edge of a connected graph produces a new connected graph with the same number of blocks. In general, we have the following result.

LEMMA 2.2. *Let G be a connected graph and uv any edge in G. Then the graph obtained from G by replacing uv by a 2-connected graph has the same number of blocks as G.*

Let $\mathcal{W}$ be the family of 2-connected graphs G such that G is not 3-connected, and for each cut-set $\{u, v\}$ of G, we have $uv \notin E(G)$, $c(G - \{u, v\}) = 3$, and $b(B) = 2$ for each $\{u, v\}$-bridge B of G. By Lemma 2.1, $\mathcal{K}$ is a subfamily of $\mathcal{W}$. The next result shows that these two families are actually the same.

THEOREM 2.1. $\mathcal{K} = \mathcal{W}$.

Proof. It suffices to show that $\mathcal{W} \subseteq \mathcal{K}$. We prove this by induction on the order of graphs in $\mathcal{W}$. Let $G \in \mathcal{W}$. By the definition of $\mathcal{W}$, G is 2-connected, and so $v(G) \geq 3$. If $v(G) = 3$, it is clear that $G \cong K_3 \in \mathcal{K}$. Now assume that $v(G) \geq 4$. By the definition of $\mathcal{W}$ again, G has a cut-set $\{u, v\}$, where $uv \notin E(G)$, such that $G - \{u, v\}$ has exactly three $\{u, v\}$-bridges of G, say H_1, H_2, and H_3, and $b(H_i) = 2$ for each $i = 1, 2, 3$. For $i = 1, 2, 3$, let w_i be the only cut-vertex of H_i. It is clear that $w_i \notin \{u, v\}$, as shown in Figure 1.

CLAIM A. *$H_i + uv \in \mathcal{K}$ for $i = 1, 2, 3$.*

We just prove the case that $i = 1$. Note that $v(H_1) \geq 3$ and $H_1 + uv$ is 2-connected. If $v(H_1) = 3$, then $H_1 + uv \cong K_3 \in \mathcal{K}$.

Now assume that $v(H_1) \geq 4$. Then either $\{u, w_1\}$ or $\{v, w_1\}$ is a cut-set of $H_1 + uv$. So $H_1 + uv$ is not 3-connected.

Let $\{x, y\}$ be any cut-set of $H_1 + uv$. As $\{u, v\}$ is not a cut-set of $H_1 + uv$,

$\{x, y\} \neq \{u, v\}$. Note that G can be considered as a graph obtained from $H_1 + uv$ by replacing uv by $G[V(H_2) \cup V(H_3)]$. So $\{x, y\}$ is a cut-set of G. By the definition of $\mathcal{W}$, we have $xy \notin E(G)$, G has exactly three $\{x, y\}$-bridges, and each $\{x, y\}$-bridge of G has exactly two blocks.

Let B_0 be the $\{x, y\}$-bridge of $G_1 + uv$ such that $uv \in E(B_0)$. Note that every $\{x, y\}$-bridge of G either is an $\{x, y\}$-bridge of $H_1 + uv$ or can be obtained from B_0 by replacing uv by the graph $G[V(H_2) \cup V(H_3)]$. On the other hand, for every $\{x, y\}$-bridge B of $H_1 + uv$, if B is not B_0, then B is an $\{x, y\}$-bridge of G. Thus $H_1 + uv$ has exactly three $\{x, y\}$-bridges. By Lemma 2.2, it is clear that each $\{x, y\}$-bridge of $H_1 + uv$ has exactly two blocks. Hence $H_1 + uv \in \mathcal{W}$ by the definition of $\mathcal{W}$. By induction, $H_1 + uv \in \mathcal{K}$, and Claim A holds.

For each $i = 1, 2, 3$, since $H_i + uv \in \mathcal{K}$, by Lemma 2.1(ii), H_i either is P_3 or can be obtained from P_3 by a sequence of double-subdivisions. Thus G either is the complete bipartite graph $K(2, 3)$ or can be obtained from $K(2, 3)$ by a sequence of double-subdivisions. Therefore $G \in \mathcal{K}$. ∎

Note that it can be proved similarly that $\mathcal{K} = \mathcal{W}'$, where $\mathcal{W}'$ is the family of 2-connected graphs G defined as follows:

(i) If $v(G) = 3$, then $G \in \mathcal{W}'$; and
(ii) if $v(G) \geq 4$, then G has a cut-set $\{u, v\}$, where $uv \notin E(G)$ such that $G + uv$ contains exactly three $\{u, v\}$-bridges and each $\{u, v\}$-bridge of $G + uv$ belongs to $\mathcal{W}'$.

3. Splitting-closed families. Let $\mathcal{S}$ be a family of connected graphs. Following [3], we say that $\mathcal{S}$ is *splitting-closed* if the following properties hold for each graph $G \in \mathcal{S}$:

(i) If T is a complete cut-set of G with $|T| \leq 2$, then $\mathcal{S}$ includes all T-bridges;
(ii) if G is 2-connected and $\{u, v\}$ is a cut-set of G with $uv \notin E(G)$, then $\mathcal{S}$ includes all $\{u, v\}$-bridges of $G + uv$ and all blocks of $G \cdot uv$.

The above two properties will be referred to as the *splitting-closed conditions* (i) and (ii), respectively.

For example, the family of trees, the family of planar graphs, the family of connected graphs, and the family of connected chordal graphs are all splitting-closed. If $\mathcal{S}$ is any splitting-closed family of graphs, then the family of graphs in $\mathcal{S}$ of order at most k is also splitting-closed for any positive integer k. But the family of connected bipartite graphs is *not* splitting-closed.

The concept of a splitting-closed family was used by Jackson and Thomassen implicitly in tackling their problems in [7] and [11], respectively. Their results implied that the family of connected graphs and the family of graphs having a Hamiltonian path are splitting-closed, respectively.

Let $e(G)$ be the number of edges in a graph G. For any graph $G \in \mathcal{S}$ with $v(G) \geq 2$, define

$$\mathcal{S}^-(G) = \{H \in \mathcal{S} : v(H) \geq 2, e(H) + v(H) < e(G) + v(G)\}. \tag{1}$$

For any $\alpha \in (1, 2)$, define

$$\mathcal{S}(\alpha) = \{G \in \mathcal{S} : Q(G, \alpha) \leq 0 \text{ and } Q(H, \alpha) > 0 \text{ for all } H \in \mathcal{S}^-(G)\}. \tag{2}$$

THEOREM 3.1. *Let $\mathcal{S}$ be any family of connected graphs. Then*

$$\omega(\mathcal{S}) = \omega\left(\bigcup_{\alpha \in (1,2)} \mathcal{S}(\alpha)\right). \tag{3}$$

Proof. Let t be the value of the right-hand side of (3). It suffices to show that $\omega(\mathcal{S}) \geq t$, i.e., $Q(G,\lambda) > 0$ for any $G \in \mathcal{S}$ and any $\lambda \in (1,t)$.

Suppose that there exists $G \in \mathcal{S}$ and $\alpha \in (1,t)$ such that $Q(G,\alpha) \leq 0$. Then there exists $G' \in \mathcal{S}$ such that $G' \in \mathcal{S}(\alpha)$, which implies that

$$\omega\left(\bigcup_{\alpha\in(1,2)} \mathcal{S}(\alpha)\right) \leq \alpha < t,$$

a contradiction. ∎

We need to apply the results in Lemma 3.1 below, which are actually given in Lemmas 2.2, 2.3(iv), 2.4, and 2.5 in [3], respectively.

LEMMA 3.1. *Let $\mathcal{S}$ be a splitting-closed family and let $G \in \mathcal{S}(\alpha)$ for some $\alpha \in (1,2)$. Then*

(i) *G contains no complete cut-set T with $|T| \leq 2$;*
(ii) *for every cut-set $\{u,v\}$ of G, $c(G-\{u,v\})$ is odd;*
(iii) *for any $e \in E(G)$, if both $G-e$ and G/e belong to $\mathcal{S}$, then $b(G-e) = 2$;*
(iv) *for any cut-set $\{u,v\}$ of G and any two components G_1 and G_2 of $G-\{u,v\}$, if both $G-(V(G_1)\cup V(G_2))$ and H belong to $\mathcal{S}$, where H is the graph obtained from $[V(G_1)\cup V(G_2)\cup\{u,v\}]$ by adding a new vertex w and two new edges joining w to u and v, then G has exactly three $\{u,v\}$-bridges and the subgraph $G-(V(G_1)\cup V(G_2))$ is a $\{u,v\}$-bridge of G which is connected with exactly two blocks.*

Let $\mathcal{S}$ be a family of connected graph and let G be 2-connected in $\mathcal{S}$. For a cut-set $\{u,v\}$ of G, we say G satisfies condition W^* at $\{u,v\}$ if the following condition is satisfied:

> For every pair of components G_i and G_j of $G-\{u,v\}$, both $G-(V(G_i)\cup V(G_j))$ and $H_{i,j}$ belong to $\mathcal{S}$, where $H_{i,j}$ is the graph obtained from $[V(G_i)\cup V(G_j)\cup\{u,v\}]$ by adding a new vertex w and two new edges joining w to u and v.

Now, we have the following.

THEOREM 3.2. *Let $\mathcal{S}$ be a splitting-closed family. Assume that every graph $G \in \mathcal{S}$ satisfies at least one of the following conditions:*

(i) *$v(G) \leq 4$;*
(ii) *G has a complete cut-set T with $|T| \leq 2$;*
(iii) *G has a cut-set $\{u,v\}$ such that $c(G-\{u,v\})$ is even;*
(iv) *there exists $e \in E(G)$ such that $b(G-e) \neq 2$ and both $G-e$ and G/e belong to $\mathcal{S}$;*
(v) *G is not 3-connected and satisfies condition W^* at every cut-set $\{u,v\}$ of G.*

Then $\mathcal{S}(\alpha) \subseteq \mathcal{K}$ for every $\alpha \in (1,2)$. Therefore $\omega(\mathcal{S}) = \omega(\mathcal{S}\cap\mathcal{K})$.

Proof. Let $\alpha \in (1,2)$ and $G \in \mathcal{S}(\alpha)$. If $v(G) \leq 4$, it is easy to verify that $P(G,\lambda)$ has no real roots in $(1,2)$, implying that $G \notin \mathcal{S}(\alpha)$, a contradiction. By Lemma 3.1(i)–(iii), G is 2-connected and does not satisfy conditions (ii)–(iv) in this theorem. Thus G satisfies conditions (v). Now we need to show the following claim, which implies that $G \in \mathcal{W}$.

CLAIM B. *For any cut-set $\{u,v\}$, we have $uv \notin E(G)$, $c(G-\{u,v\}) = 3$, and $b(B) = 2$ for every $\{u,v\}$-bridge B of G.*

Let $\{u,v\}$ be any cut-set of G. As G does not satisfy condition (ii), we have $uv \notin E(G)$. Assume that $c(G-\{u,v\}) = s$. It is clear that $s \geq 2$. So G has exactly s $\{u,v\}$-bridges. Then $s \geq 3$, as condition (iii) is not satisfied. Let G_i and G_j be

any two components of $G - \{u, v\}$. Since G satisfies condition W^*at $\{u, v\}$, both $G - (V(G_i) \cup V(G_j))$ and $H_{i,j}$ always belong to $\mathcal{S}$, where $H_{i,j}$ is the graph obtained from $[V(G_i) \cup V(G_j) \cup \{u, v\}]$ by adding a new vertex w and two new edges joining w to u and v. By Lemma 3.1(iv), $s = 3$ and $G - (V(G_i) \cup V(G_j))$ is a $\{u, v\}$-bridge of G which has exactly two blocks. As this result is true for any two components G_i and G_j of $G - \{u, v\}$, Claim B holds.

Thus $G \in \mathcal{W}$ by Claim B and the definition of $\mathcal{W}$. Hence $\mathcal{S}(\alpha) \subseteq \mathcal{W}$ for each $\alpha \in (1, 2)$. By Theorem 2.1, $\mathcal{S}(\alpha) \subseteq \mathcal{K}$ for each $\alpha \in (1, 2)$, implying that

$$\bigcup_{\alpha \in (1,2)} \mathcal{S}(\alpha) \subseteq \mathcal{K} \cap \mathcal{S}, \tag{4}$$

and so

$$\omega\left(\bigcup_{\alpha \in (1,2)} \mathcal{S}(\alpha)\right) \geq \omega(\mathcal{K} \cap \mathcal{S}). \tag{5}$$

By Theorem 3.1,

$$\omega(\mathcal{S}) = \omega\left(\bigcup_{\alpha \in (1,2)} \mathcal{S}(\alpha)\right) \geq \omega(\mathcal{K} \cap \mathcal{S}). \tag{6}$$

Therefore $\omega(\mathcal{S}) = \omega(\mathcal{K} \cap \mathcal{S})$. ∎

COROLLARY 3.1. $\omega(\mathcal{K}) = 32/27$.

Proof. This result is actually included in [7]. It also follows directly from Theorems 3.2 and 1.1. Recall that $\mathcal{G}$ is the family of connected graphs. It is obvious that $\mathcal{G}$ is splitting-closed and that every graph in $\mathcal{G}$ satisfies one of the conditions in Theorem 3.2. Thus, by Theorem 3.2,

$$\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K}) = \omega(\mathcal{K}).$$

By Theorem 1.1, $\omega(\mathcal{G}) = 32/27$. ∎

Remarks. (i) Indeed, Jackson [7] and Thomassen [11] have, respectively, shown that the following two families of graphs are splitting-closed and satisfy conditions in Theorem 3.2:

(a) The family of connected graphs;

(b) the family of graphs with a Hamiltonian path.

Thus, by Theorem 3.2, $\omega(\mathcal{S}) = \omega(\mathcal{S} \cap \mathcal{K})$ for each of these two families $\mathcal{S}$. However, $\mathcal{S} \cap \mathcal{K}$ is infinite for these two families, and both Jackson and Thomassen determined $\omega(\mathcal{S} \cap \mathcal{K})$ by considering the limits of chromatic zeros in $(1, 2)$ of graphs in $\mathcal{S} \cap \mathcal{K}$.

(ii) For any integer $k \geq 2$, let $\mathcal{D}_k$ be the family of graphs with domination number at most k. It was shown in [3] that $\mathcal{D}_k$ is splitting-closed. It is not difficult to prove that every graph in $\mathcal{D}_k$ satisfies one of the conditions in Theorem 3.2. By Theorem 3.2, $\omega(\mathcal{D}_k) = \omega(\mathcal{D}_k \cap \mathcal{K})$. It can be shown that $\mathcal{D}_k \cap \mathcal{K}$ is a finite set, and so it is not difficult to determine $\omega(\mathcal{D}_k \cap \mathcal{K})$.

4. Graphs with no chromatic zeros in $(1, 2)$. In this section, we study certain families $\mathcal{S}$ of graphs with $\omega(\mathcal{S}) = 2$. We first apply Lemma 3.1 to prove the following result.

THEOREM 4.1. *Let $\mathcal{S}$ be a splitting-closed family of connected graphs. Assume that every $G \in \mathcal{S}$ satisfies one of the following conditions:*

(i) $v(G) \leq 4$;

(ii) *G contains a complete cut-set T with $|T| \leq 2$;*

(iii) *$c(G - \{x, y\})$ is even for some cut-set $\{x, y\}$ of G with $xy \notin E(G)$;*

(iv) *for some $e \in E(G)$, $c(G - e) \neq 2$ and both $G - e$ and G/e belong to $\mathcal{S}$.*

Then $\omega(\mathcal{S}) = 2$.

Proof. Let G be any graph in $\mathcal{S}$. Since G satisfies one of the conditions in Theorem 4.1, it is obvious that G also satisfies one of the conditions in Theorem 3.2. Thus $\omega(\mathcal{S}) = \omega(\mathcal{S} \cap \mathcal{K})$ by Theorem 3.2. By Lemma 2.1, every graph in $\mathcal{K}$ does not satisfy condition (ii)–(iv) in this theorem, and only K_3 satisfies condition (i) in this theorem. Thus $\mathcal{S} \cap \mathcal{K} \subseteq \{K_3\}$. Therefore $\omega(\mathcal{S}) = 2$. $\square$

Remark. For a graph G of order $n \geq 2$, an ordering $(x_1, x_2, \cdots, x_n)$ of the vertices in G is called a *double-link* ordering if $x_1x_2 \in E(G)$ and $|N(x_i) \cap V_i| \geq 2$ for all $i = 3, 4, \cdots, n$, where $V_i = \{x_1, x_2, \cdots, x_{i-1}\}$ and $N(x_i)$ is the set of neighbors of x_i in G. For a double-link ordering $(x_1, x_2, \cdots, x_n)$ of G, if there exist $u_i, v_i \in N(x_i) \cap V_{i-1}$ with $u_i \neq v_i$ for all $i = 3, 4, \cdots, n$ such that the inequality

$$|I| > |\{i : \{u_i, v_i\} \subseteq I, 3 \leq i \leq n\}| \tag{1}$$

holds for every nonempty independent set I of G, then $(x_1, x_2, \cdots, x_n)$ is called a *γ-ordering*. Let Γ be the family of graphs which admits a γ-ordering. It was proved in [5] that every graph in Γ satisfies one of the conditions in Theorem 4.1, and hence $\omega(\Gamma) = 2$.

The family Γ includes all graphs with a 2-tree as a spanning subgraph, all graphs G with a double-link ordering $(x_1, x_2 \cdots x_n)$ such that $x_1x_2 \cdots x_n$ is a Hamiltonian path in G, and all complete t-partite graphs with $t \geq 3$.

A graph G is said to be *α-tough* if $c(G - S) \leq |S|$ for every nonempty independent set S of G. In [5], Dong and Koh proved that every graph in Γ is α-tough, and they conjectured the following.

CONJECTURE 4.1. *Every α-tough graph contains no chromatic zeros in* $(1, 2)$.

Thomassen [11] conjectured that every Hamiltonian graph contains no chromatic zeros in $(1, 2)$. As every Hamiltonian graph is α-tough, Thomassen's conjecture is implied by Conjecture 4.1.

Now we apply Theorem 4.1 to show that Conjecture 4.1 actually follows directly from the next conjecture.

CONJECTURE 4.2. *Every 3-connected α-tough graph G contains an edge e such that both $G - e$ and G/e are α-tough.*

Let $\mathcal{T}$ be the family of α-tough graphs.

LEMMA 4.1. *$\mathcal{T}$ is splitting-closed.*

Proof. Let $G \in \mathcal{T}$. So G is connected and has no cut-vertex. Assume that $\{u, v\}$ is a cut-set of G.

Assume that $uv \in E(G)$. If some $\{u, v\}$-bridge G' of G has an independent set S such that $c(G' - S) > |S|$, then S is also an independent set of G and $c(G - S) = c(G' - S) > |S|$, a contradiction. So splitting-closed condition (i) holds for $\mathcal{T}$.

Now assume that $uv \notin E(G)$. Since $G \in \mathcal{T}$, we have $c(G - \{u, v\}) = 2$. It is obvious that $G + uv \in \mathcal{T}$. So every $\{u, v\}$-bridge of $G + uv$ belongs to $\mathcal{T}$. Suppose that $G \cdot uv$ has a block B such that $B \notin \mathcal{T}$. Then B has an independent set S such that $c(B - S) > |S|$. Let x be the new vertex in $G \cdot uv$ after identifying u and v. If $x \notin S$, then S is also an independent set of G and $c(G - S) = c(B - S) > |S|$, a

contradiction. Thus $x \in S$. But then $S' = (S \setminus \{x\}) \cup \{u, v\}$ is an independent set of G and $c(G - S') = c(B - S) + 1 > |S'|$, a contradiction too. Thus splitting-closed condition (ii) also holds for $\mathcal{T}$.

Hence $\mathcal{T}$ is splitting-closed. ∎

THEOREM 4.2. *Conjecture* 4.2 *implies Conjecture* 4.1.

Proof. Assume that Conjecture 4.2 is true. By Lemma 4.1, $\mathcal{T}$ is splitting-closed. Let G be any graph in $\mathcal{T}$. If G is not 3-connected, then G satisfies condition (i), (ii), or (iii) in Theorem 4.1. As Conjecture 4.2 is assumed to be true, condition (iv) in Theorem 4.1 is satisfied if G is 3-connected. By Theorem 4.1, $\omega(\mathcal{T}) = 2$. Hence Conjecture 4.1 is true. ∎

Remark. Let $\mathcal{H}$ be the family of Hamiltonian graphs. Notice that $\mathcal{H}$ is splitting-closed and that for any $G \in \mathcal{H}$, if G is not 3-connected, then it satisfies one of the conditions in Theorem 4.1. If every 3-connected graph $G \in \mathcal{H}$ satisfies condition (iv) in Theorem 4.1, i.e., G contains an edge e such that both $G - e$ and G/e belong to $\mathcal{H}$, then $\omega(\mathcal{H}) = 2$ by Theorem 4.1.

5. Determine $\omega(\mathcal{S})$ for minor-closed families $\mathcal{S}$. For any graphs G and H, H is called a *minor* of G, denoted by $H \preceq G$, if H can be obtained from G by a sequence of operations: (a) Removing an edge, (b) contracting an edge, and (c) removing a vertex. Let $H \not\preceq G$ denote that H is not a minor of G. A family $\mathcal{S}$ of graphs is said to be *minor-closed* if all minors of G also belong to $\mathcal{S}$ for every $G \in \mathcal{S}$.

THEOREM 5.1. *Let $\mathcal{S}$ be a family of connected graphs. If $\mathcal{S}$ is minor-closed, then $\mathcal{S}$ is splitting-closed, and therefore $\omega(\mathcal{S}) = \omega(\mathcal{S} \cap \mathcal{K})$.*

Proof. By the definition of $\mathcal{S}$, $\mathcal{S}$ is splitting-closed. Let G be any graph in $\mathcal{S}$. Assume that G does not satisfy conditions (i)–(iii) in Theorem 3.2. If G is 3-connected, then both $G - e$ and G/e are 2-connected and belong to $\mathcal{S}$, and thus G satisfies condition (iv) in Theorem 3.2. Now assume that G is not 3-connected and $\{u, v\}$ is any cut-set of G. Since G does not satisfy condition (iii) in Theorem 3.2, we have $c(G - \{u, v\}) \geq 3$. Let G_i and G_j be any two components of $G - \{u, v\}$. As $\mathcal{S}$ is minor-closed, we have $G - (V(G_i) \cup V(G_j)) \in \mathcal{S}$ and $H_i \in \mathcal{S}$, where H_i is the graph stated in the definition of condition W^*. Thus G satisfies condition W^* at $\{u, v\}$, implying that G satisfies condition (iv) in Theorem 3.2.

Thus the result follows from Theorem 3.2. ∎

For any positive integers m and n, let $K(m, n)$ denote the complete bipartite graph with bipartition A and B such that $|A| = m$ and $|B| = n$.

COROLLARY 5.1. *Let $\mathcal{S}$ be a minor-closed family of connected graphs. Then*

$$\omega(\mathcal{S}) \begin{cases} \leq \beta & \text{if } K(2,3) \in \mathcal{S}, \\ = 2 & \text{otherwise}, \end{cases} \tag{1}$$

where $\beta = 1.430159709\cdots$ is the real zero of $P(K(2,3), \lambda)$ in $(1, 2)$ (i.e., the real zero of the equation $x^3 - 5x^2 + 10x - 7 = 0$).

Proof. By definition, if $K(2,3) \in \mathcal{S}$, then $\omega(\mathcal{S}) \leq \beta$. Now assume that $K(2,3) \notin \mathcal{S}$. By Theorem 5.1, $\omega(\mathcal{S}) = \omega(\mathcal{S} \cap \mathcal{K})$. Note that every graph in $\mathcal{K}$, except K_3, contains $K(2,3)$ as a minor. Since $\mathcal{S}$ is minor-closed, we have $\mathcal{S} \cap \mathcal{K} \subseteq \{K_3\}$. Hence $\omega(\mathcal{S}) = 2$. ∎

For any graph H, let

$$\mathit{Forb}_{\preceq}(H) = \{G : G \text{ is connected and } H \not\preceq G\}.$$

Note that $\mathit{Forb}_{\preceq}(H)$ is minor-closed, and thus it is splitting-closed by Theorem 5.1. By Theorem 5.1 and Corollary 5.1, we have the following.

COROLLARY 5.2. *For any graph H,*

$$\omega(Forb_{\preceq}(H)) = \omega(Forb_{\preceq}(H) \cap \mathcal{K}) \begin{cases} \leq \beta & \text{if } H \not\preceq K(2,3), \\ = 2 & \text{if } H \preceq K(2,3). \end{cases} \tag{2}$$

By Corollary 5.2, we have $\omega(Forb_{\preceq}(K_4 - e)) = 2$, where $K_4 - e$ is the graph obtained from G by removing an edge. In the following we show that $\omega(Forb_{\preceq}(K_4)) = 32/27$.

LEMMA 5.1. *For graphs H and G, if $H \preceq G$, then $\omega(Forb_{\preceq}(G)) \leq \omega(Forb_{\preceq}(H))$.*

Proof. If $H \preceq G$, then $Forb_{\preceq}(H) \subseteq Forb_{\preceq}(G)$. Thus the result follows from Lemma 1.1. ∎

THEOREM 5.2. *Let H be any graph.*

(i) *If $K_4 \preceq H$, then $\omega(Forb_{\preceq}(H)) = \frac{32}{27}$.*

(ii) *$\omega(Forb_{\preceq}(K(2,4))) = \beta$.*

Proof. (i) Since all graphs in $\mathcal{K}$ are series/parallel graphs, we have $\mathcal{K} \subseteq Forb_{\preceq}(K_4)$. If $K_4 \preceq H$, then by Lemma 5.1,

$$\frac{32}{27} \leq \omega(Forb_{\preceq}(H)) \leq \omega(Forb_{\preceq}(K_4)) \leq \omega(\mathcal{K}) = \frac{32}{27},$$

implying that $\omega(Forb_{\preceq}(H)) = \frac{32}{27}$.

(ii) Notice that $K(2,4) \preceq G$ for every $G \in \mathcal{K}$ with $v(G) \geq 7$. So $Forb_{\preceq}(K(2,4)) \cap \mathcal{K} = \{K_3, K(2,3)\}$. As β is the only zero of $P(K(2,3), \lambda)$ in $(1,2)$, the result follows from Theorem 5.1. ∎

Remark. By a similar method, it is not difficult to determine $\omega(Forb_{\preceq}(K(2,k)))$ for any given $k \geq 5$.

Acknowledgments. The authors wish to thank the referees for their very helpful suggestions and constructive comments.

REFERENCES

[1] R. DIESTEL, *Graph Theory*, Springer-Verlag, Berlin, 2005.

[2] F. M. DONG AND K. M. KOH, *On planar and nonplanar graphs having no chromatic zeros in the interval* $(1,2)$, Discrete Math., 308 (2008), pp. 3897–3905.

[3] F. M. DONG AND K. M. KOH, *Domination numbers and zeros of chromatic polynomials*, Discrete Math., 308 (2008), pp. 1930–1940.

[4] F. M. DONG AND K. M. KOH, *A maximal zero-free interval for chromatic polynomials of bipartite planar graphs*, Discrete Math., 308 (2008), pp. 2285–2287.

[5] F. M. DONG AND K. M. KOH, *On graphs having no chromatic zeros in* (1, 2), SIAM J. Discrete Math., 20 (2006), pp. 799–810.

[6] F. M. DONG, K. M. KOH, AND K. L. TEO, *Chromatic Polynomials and Chromaticity of Graphs*, World Scientific, Singapore, 2005.

[7] B. JACKSON, *A zero-free interval for chromatic polynomials of graphs*, Combin. Probab. Comput., 2 (1993), pp. 325–336.

[8] R. C. READ, *An introduction to chromatic polynomials*, J. Combin. Theory, 4 (1968), pp. 52–71.

[9] R. C. READ AND W. T. TUTTE, *Chromatic polynomials*, in Selected Topics in Graph Theory III, L. W. Beineke and R. J. Wilson, eds., Academic Press, New York, 1988, pp. 15–42.

[10] G. ROYLE, *Graphs with chromatic roots in the interval* (1, 2), Electron. J. Combin., 14 (2007), Note 18 (electronic).

[11] C. THOMASSEN, *Chromatic roots and Hamiltonian paths*, J. Combin. Theory Ser. B, 80 (2000), pp. 218–224.

[12] C. THOMASSEN, *The zero-free intervals for chromatic polynomials of graphs*, Combin. Probab. Comput., 6 (1997), pp. 497–506.