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ON ZERO-FREE INTERVALS IN \((1, 2)\) OF CHROMATIC POLYNOMIALS OF SOME FAMILIES OF GRAPHS

F. M. DONG\(^\dagger\) AND K. M. KOH\(^\ddagger\)

Abstract. For a family \(S\) of graphs, let \(\omega(S)\) be the supremum of \(t : 1 < t \leq 2\) such that \(P(G, \lambda) \neq 0\) for all \(G \in S\) and all \(\lambda \in (1, t)\). In this paper we show that \(\omega(S) = \omega(S \cap K)\) for any family \(S\) of graphs satisfying certain conditions, where \(K\) is a special family of graphs. This result makes it much easier to determine \(\omega(S)\) for such families \(S\).

Key words. chromatic polynomial, zero-free interval, double subdivision, minor

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1. Introduction. It is well known that \((-\infty, 0)\) and \((0, 1)\) are two zero-free intervals for the chromatic polynomials of all graphs (see [6, 9]). By the results established in Jackson [7] and Thomassen [12], we know that there are no chromatic zeros in \((1, 32/27]\) but chromatic zeros are dense in \([32/27, \infty)\).

Let \(G\) be the family of connected graphs of order at least 2. For any \(G \in G\), we define the following graph-function

\[
Q(G, \lambda) = (-1)^{v(G) + b(G) - 1} P(G, \lambda),
\]

where \(P(G, \lambda)\), \(v(G)\), and \(b(G)\) are the chromatic polynomial, the vertex number (i.e., the order), and the block number of \(G\), respectively. Note that if \(Q(G, \lambda) = 0\) for all \(\lambda \in (1, 2)\), then \(Q(G, \lambda) > 0\) for all \(1 < \lambda < 2\). Jackson [7] proved the following.

Theorem 1.1. For any \(G \in G\), \(Q(G, \lambda) > 0\) for all \(1 < \lambda \leq 32/27\), where the number \(32/27\) cannot be replaced by any larger number.

For any \(S \subseteq G\), if \(S = \emptyset\), let \(\omega(S) = 2\); otherwise, let

\[
\omega(S) = \sup\{1 < t \leq 2 : P(G, \lambda) \neq 0 \text{ for all } \lambda \in (1, t) \text{ and all } G \in S\}.
\]

Since \(Q(G, \lambda)\) is a continuous function for any graph \(G\), by Theorem 1.1,

\[
\omega(S) = \sup\{1 < t \leq 2 : Q(G, \lambda) > 0 \text{ for all } \lambda \in (1, t) \text{ and all } G \in S\}
\]

holds for any family \(S \neq \emptyset\). By Theorem 1.1 and the definition of \(\omega(S)\), the next result follows immediately.

Lemma 1.1. If \(S_1 \subseteq S_2 \subseteq G\), then \(32/27 \leq \omega(S_2) \leq \omega(S_1) \leq 2\).

The following results on \(\omega(S)\) have been obtained, although they are not expressed by \(\omega(S)\):

\(1\) Jackson [7] established that \(\omega(G) = 32/27\).
(ii) Thomassen [11] proved that if $S$ is the family of graphs with a Hamiltonian path, then
$$\omega(S) = \frac{2}{3} + \frac{1}{3} \sqrt[3]{26 + 6\sqrt{33}} + \frac{1}{3} \sqrt[3]{26 - 6\sqrt{33}} = 1.29559 \cdots.$$  

(iii) Dong and Koh [4] proved that $\omega(S) = 32/27$ if $S$ is the family of bipartite planar graphs.

(iv) Dong and Koh [3] showed that if $S$ is the family of graphs with domination number 2, then
$$\omega(S) = 2 - \frac{1}{6} \left( \sqrt[3]{12\sqrt{93} + 108} - \sqrt[3]{12\sqrt{93} - 108} \right) = 1.317672196 \cdots.$$  

Jackson [7] conjectured that $\omega(S) = 2$ for the family $S$ of 3-connected nonbipartite graphs. However, counterexamples to this conjecture were discovered recently by Royle [10].

In [7], Jackson introduced an operation on graphs, i.e., double-subdivision, which was used to construct a special family of graphs. A double-subdivision is an operation on graphs which replaces an edge $uv$ by two $u - v$ paths of length 2. Let $\mathcal{K}$ is the family of graphs defined as follows:

(i) $K_3 \in \mathcal{K}$;

(ii) for any graph $G$ with $v(G) \geq 4$, $G \in \mathcal{K}$ if and only if $G$ can be obtained from a graph $G' \in \mathcal{K}$ by a double-subdivision on some edge in $G'$.

In this paper we find that $\mathcal{K}$ plays an important role in the study of real zeros of chromatic polynomials in the interval $(1, 2)$. In sections 2 and 3 of this paper, we will show that $\omega(S) = \omega(S \cap \mathcal{K})$ if $S$ is a family of connected graphs which satisfy certain conditions (see Theorem 3.2). Clearly this result makes it much easier to determine $\omega(S)$ for such a family $S$ of graphs. While some families $S$ of graphs with $\omega(S) = 2$ are discussed in section 4, we shall apply Theorem 3.2 to determine $\omega(S)$ for some families $S$ of graphs in section 5.

2. The family $\mathcal{K}$. Let $G$ be any graph. Let $V(G)$ and $E(G)$ be the vertex set and edge set of $G$, respectively. For any $T \subseteq V(G)$, let $G - T$ be the graph obtained from $G$ by deleting all vertices in $T$ (of course also all edges incident to vertices in $T$). For any $e \in E(G)$, let $G - e$ be the graph obtained from $G$ by deleting $e$ and let $G/e$ be the graph obtained from $G$ by contracting $e$ and deleting all multiedges but one. For nonadjacent vertices $u$ and $v$ in $G$, let $G + uv$ be the graph obtained from $G$ by adding an edge between $u$ and $v$ and let $G \cdot uv$ be the graph obtained from $G$ by identifying $u$ and $v$ and deleting all multiedges but one. Note that if $e$ is an edge between $u$ and $v$ in $G$, then $G/e$ and $(G - e) \cdot uv$ are the same graph.

For any graph $G$ and $T \subseteq V(G)$, let $G[T]$ (or simply $[T]$) denote the subgraph of $G$ induced by $T$. If $U$ is the vertex set of a component of $G - T$, then $[T \cup U]$ is called a $T$-bridge of $G$. Note that if $T \neq V(G)$, then $G$ has exactly $c(G - T)$ $T$-bridges, where $c(H)$ denotes the number of components of a graph $H$. If $G$ is connected, then $T$ is a cut-set of $G$ if and only if $G$ has at least two $T$-bridges. If $T$ is a cut-set of $G$ and $[T]$ is complete, then $T$ is called a complete cut-set of $G$.

By the definition of $\mathcal{K}$ given in section 1, every graph in $\mathcal{K}$ can be obtained from $K_3$ by a sequence of double-subdivisions. Thus the order of every graph in $\mathcal{K}$ is an odd integer. Some other properties on the graphs in $\mathcal{K}$ are given in Lemma 2.1 below. These properties can be proved by induction (see [7] also). For any $n \geq 1$, let $P_n$ be the path graph of order $n$. 
LEMMA 2.1. Let $G \in \mathcal{K}$, $e$ be any edge in $G$, and let $\{u,v\}$ be any cut-set of $G$. Then

(i) $G$ is 2-connected;
(ii) $G - e$ is connected with $b(G - e) = 2$, and $G - e$ either is $P_3$ or can be obtained from $P_3$ by a sequence of double-subdivisions; and
(iii) $uv \notin E(G)$, $c(G - \{u, v\}) = 3$, and $b(B) = 2$ for each $\{u, v\}$-bridge $B$ of $G$.

The operation “double-subdivision” can be generalized to the operation “replacing an edge by a graph” as introduced below. Let $G$ and $H$ be two graphs and let $uv \in E(G)$. Replacing $uv$ in $G$ by $H$ is the combination of the following operations:

(i) Remove $uv$ from $G$;
(ii) identify $u$ and $v$, respectively, with two distinct vertices in $H$.

Thus, a double-subdivision can be regarded as an operation “replacing an edge by a 4-cycle.” Notice that a double-subdivision on any edge of a connected graph produces a new connected graph with the same number of blocks. In general, we have the following result.

LEMMA 2.2. Let $G$ be a connected graph and $uv$ any edge in $G$. Then the graph obtained from $G$ by replacing $uv$ by a 2-connected graph has the same number of blocks as $G$.

Let $\mathcal{W}$ be the family of 2-connected graphs $G$ such that $G$ is not 3-connected, and for each cut-set $\{u, v\}$ of $G$, we have $uv \notin E(G)$, $c(G - \{u, v\}) = 3$, and $b(B) = 2$ for each $\{u, v\}$-bridge $B$ of $G$. By Lemma 2.1, $\mathcal{K}$ is a subfamily of $\mathcal{W}$. The next result shows that these two families are actually the same.

THEOREM 2.1. $\mathcal{K} = \mathcal{W}$.

Proof. It suffices to show that $\mathcal{W} \subseteq \mathcal{K}$. We prove this by induction on the order of graphs in $\mathcal{W}$. Let $G \in \mathcal{W}$. By the definition of $\mathcal{W}$, $G$ is 2-connected, and so $v(G) \geq 3$. If $v(G) = 3$, it is clear that $G \cong K_3 \in \mathcal{K}$. Now assume that $v(G) \geq 4$.

By the definition of $\mathcal{W}$ again, $G$ has a cut-set $\{u, v\}$, where $uv \notin E(G)$, such that $G - \{u, v\}$ has exactly three $\{u, v\}$-bridges of $G$, say $H_1, H_2$, and $H_3$, and $b(H_i) = 2$ for each $i = 1, 2, 3$. For $i = 1, 2, 3$, let $w_i$ be the only cut-vertex of $H_i$. It is clear that $w_i \notin \{u, v\}$, as shown in Figure 1.

CLAIM A. $H_i + uv \in \mathcal{K} \text{ for } i = 1, 2, 3$.

We just prove the case that $i = 1$. Note that $v(H_1) \geq 3$ and $H_1 + uv$ is 2-connected. If $v(H_1) = 3$, then $H_1 + uv \cong K_3 \in \mathcal{K}$.

Now assume that $v(H_1) \geq 4$. Then either $\{u, w_1\}$ or $\{v, w_1\}$ is a cut-set of $H_1 + uv$. So $H_1 + uv$ is not 3-connected.

Let $\{x, y\}$ be any cut-set of $H_1 + uv$. As $\{u, v\}$ is not a cut-set of $H_1 + uv$,
\( \{x, y\} \neq \{u, v\} \). Note that \( G \) can be considered as a graph obtained from \( H_1 + uv \) by replacing \( uv \) by \( G[V(H_2) \cup V(H_3)] \). So \( \{x, y\} \) is a cut-set of \( G \). By the definition of \( W \), we have \( xy \notin E(G) \), \( G \) has exactly three \( \{x, y\} \)-bridges, and each \( \{x, y\} \)-bridge of \( G \) has exactly two blocks.

Let \( B_0 \) be the \( \{x, y\} \)-bridge of \( G_1 + uv \) such that \( uv \in E(B_0) \). Note that every \( \{x, y\} \)-bridge of \( G \) either is an \( \{x, y\} \)-bridge of \( H_1 + uv \) or can be obtained from \( B_0 \) by replacing \( uv \) by the graph \( G[V(H_2) \cup V(H_3)] \). On the other hand, for every \( \{x, y\} \)-bridge \( B \) of \( H_1 + uv \), if \( B \) is not \( B_0 \), then \( B \) is an \( \{x, y\} \)-bridge of \( G \). Thus \( H_1 + uv \) has exactly three \( \{x, y\} \)-bridges. By Lemma 2.2, it is clear that each \( \{x, y\} \)-bridge of \( H_1 + uv \) has exactly two blocks. Hence \( H_1 + uv \in W \) by the definition of \( W \). By induction, \( H_1 + uv \in K \), and Claim A holds.

For each \( i = 1, 2, 3 \), since \( H_i + uv \in K \), by Lemma 2.1(ii), \( H_i \) either is \( P_3 \) or can be obtained from \( P_3 \) by a sequence of double-subdivisions. Thus \( G \) either is the complete bipartite graph \( K(2, 3) \) or can be obtained from \( K(2, 3) \) by a sequence of double-subdivisions. Therefore \( G \in K \). \( \square \)

Note that it can be proved similarly that \( K = W' \), where \( W' \) is the family of 2-connected graphs \( G \) defined as follows:

(i) If \( v(G) = 3 \), then \( G \in W' \); and

(ii) if \( v(G) \geq 4 \), then \( G \) has a cut-set \( \{u, v\} \), where \( uv \notin E(G) \) such that \( G + uv \) contains exactly three \( \{u, v\} \)-bridges and each \( \{u, v\} \)-bridge of \( G + uv \) belongs to \( W' \).

3. Splitting-closed families. Let \( S \) be a family of connected graphs. Following [3], we say that \( S \) is splitting-closed if the following properties hold for each graph \( G \in S \):

(i) If \( T \) is a complete cut-set of \( G \) with \( |T| \leq 2 \), then \( S \) includes all \( T \)-bridges;

(ii) if \( G \) is 2-connected and \( \{u, v\} \) is a cut-set of \( G \) with \( uv \notin E(G) \), then \( S \) includes all \( \{u, v\} \)-bridges of \( G + uv \) and all blocks of \( G \cdot uv \).

The above two properties will be referred to as the splitting-closed conditions (i) and (ii), respectively.

For example, the family of trees, the family of planar graphs, the family of connected graphs, and the family of connected chordal graphs are all splitting-closed. If \( S \) is any splitting-closed family of graphs, then the family of graphs in \( S \) of order at most \( k \) is also splitting-closed for any positive integer \( k \). But the family of connected bipartite graphs is not splitting-closed.

The concept of a splitting-closed family was used by Jackson and Thomassen implicitly in tackling their problems in [7] and [11], respectively. Their results implied that the family of connected graphs and the family of graphs having a Hamiltonian path are splitting-closed, respectively.

Let \( e(G) \) be the number of edges in a graph \( G \). For any graph \( G \in S \) with \( v(G) \geq 2 \), define

\[
S^-(G) = \{ H \in S : v(H) \geq 2, e(H) + v(H) < e(G) + v(G) \}.
\]

For any \( \alpha \in (1, 2) \), define

\[
S(\alpha) = \{ G \in S : Q(G, \alpha) \leq 0 \text{ and } Q(H, \alpha) > 0 \text{ for all } H \in S^-(G) \}.
\]

Theorem 3.1. Let \( S \) be any family of connected graphs. Then

\[
\omega(S) = \omega \left( \bigcup_{\alpha \in (1, 2)} S(\alpha) \right).
\]
Proof. Let \( t \) be the value of the right-hand side of (3). It suffices to show that 
\[ \omega(S) \geq t, \]
i.e., \( Q(G, \lambda) > 0 \) for any \( G \in S \) and any \( \lambda \in (1, t) \).

Suppose that there exists \( G \in S \) and \( \alpha \in (1, t) \) such that \( Q(G, \alpha) \leq 0 \). Then there exists \( G' \in S \) such that \( G' \in S(\alpha) \), which implies that 
\[ \omega \left( \bigcup_{\alpha \in (1,2)} S(\alpha) \right) \leq \alpha < t, \]
a contradiction. \( \square \)

We need to apply the results in Lemma 3.1 below, which are actually given in 
Lemmas 2.2, 2.3(iv), 2.4, and 2.5 in [3], respectively.

**Lemma 3.1.** Let \( S \) be a splitting-closed family and let \( G \in S(\alpha) \) for some \( \alpha \in (1,2) \). Then

(i) \( G \) contains no complete cut-set \( T \) with \(|T| \leq 2\);
(ii) for every cut-set \( \{u, v\} \) of \( G \), \( c(G - \{u, v\}) \) is odd;
(iii) for any \( e \in E(G) \), if both \( G - e \) and \( G/e \) belong to \( S \), then \( b(G - e) = 2 \);
(iv) for any cut-set \( \{u, v\} \) of \( G \) and any two components \( G_1 \) and \( G_2 \) of \( G - \{u, v\} \), if both \( G - (V(G_1) \cup V(G_2)) \) and \( H \) belong to \( S \), where \( H \) is the graph obtained from \( [V(G_1) \cup V(G_2) \cup \{u, v\}] \) by adding a new vertex \( w \) and two new edges joining \( w \) to \( u \) and \( v \), then \( G \) has exactly three \( \{u, v\} \)-bridges and the subgraph \( G - (V(G_1) \cup V(G_2)) \) is a \( \{u, v\} \)-bridge of \( G \) which is connected with exactly two blocks.

Let \( S \) be a family of connected graph and let \( G \) be 2-connected in \( S \). For a cut-set \( \{u, v\} \) of \( G \), we say \( G \) satisfies condition \( W^* \) at \( \{u, v\} \) if the following condition is satisfied:

For every pair of components \( G_i \) and \( G_j \) of \( G - \{u, v\} \), both \( G - (V(G_i) \cup V(G_j)) \) and \( H_{i,j} \) belong to \( S \), where \( H_{i,j} \) is the graph obtained from \( [V(G_i) \cup V(G_j) \cup \{u, v\}] \) by adding a new vertex \( w \) and two new edges joining \( w \) to \( u \) and \( v \).

Now, we have the following.

**Theorem 3.2.** Let \( S \) be a splitting-closed family. Assume that every graph \( G \in S \) satisfies at least one of the following conditions:

(i) \( v(G) \leq 4 \);
(ii) \( G \) has a complete cut-set \( T \) with \(|T| \leq 2\);
(iii) \( G \) has a cut-set \( \{u, v\} \) such that \( c(G - \{u, v\}) \) is even;
(iv) there exists \( e \in E(G) \) such that \( b(G - e) \neq 2 \) and both \( G - e \) and \( G/e \) belong to \( S \);
(v) \( G \) is not 3-connected and satisfies condition \( W^* \) at every cut-set \( \{u, v\} \) of \( G \).

Then \( S(\alpha) \subseteq K \) for every \( \alpha \in (1, 2) \). Therefore \( \omega(S) = \omega(S \cap K) \).

**Proof.** Let \( \alpha \in (1, 2) \) and \( G \in S(\alpha) \). If \( v(G) \leq 4 \), it is easy to verify that \( P(G, \lambda) \) has no real roots in \((1, 2)\), implying that \( G \notin S(\alpha) \), a contradiction. By Lemma 3.1(i)–(iii), \( G \) is 2-connected and does not satisfy conditions (ii)–(iv) in this theorem. Thus \( G \) satisfies conditions (v). Now we need to show the following claim, which implies that \( G \in W \).

**Claim B.** For any cut-set \( \{u, v\} \), we have \( uv \notin E(G) \), \( c(G - \{u, v\}) = 3 \), and \( b(B) = 2 \) for every \( \{u, v\} \)-bridge \( B \) of \( G \).

Let \( \{u, v\} \) be any cut-set of \( G \). As \( G \) does not satisfy condition (ii), we have \( uv \notin E(G) \). Assume that \( c(G - \{u, v\}) = s \). It is clear that \( s \geq 2 \). So \( G \) has exactly \( s \) \( \{u, v\} \)-bridges. Then \( s \geq 3 \), as condition (iii) is not satisfied. Let \( G_i \) and \( G_j \) be
any two components of $G - \{u, v\}$. Since $G$ satisfies condition $W^*$ at $\{u, v\}$, both $G - (V(G_i) \cup V(G_j))$ and $H_{i,j}$ always belong to $\mathcal{S}$, where $H_{i,j}$ is the graph obtained from $[V(G_i) \cup V(G_j) \cup \{u, v\}]$ by adding a new vertex $w$ and two new edges joining $w$ to $u$ and $v$. By Lemma 3.1(iv), $s = 3$ and $G - (V(G_i) \cup V(G_j))$ is a $\{u, v\}$-bridge of $G$ which has exactly two blocks. As this result is true for any two components $G_i$ and $G_j$ of $G - \{u, v\}$, Claim B holds.

Thus $G \in \mathcal{W}$ by Claim B and the definition of $\mathcal{W}$. Hence $\mathcal{S}(\alpha) \subseteq \mathcal{W}$ for each $\alpha \in (1, 2)$. By Theorem 2.1, $\mathcal{S}(\alpha) \subseteq \mathcal{K}$ for each $\alpha \in (1, 2)$, implying that

$$
\bigcup_{\alpha \in (1, 2)} \mathcal{S}(\alpha) \subseteq \mathcal{K} \cap \mathcal{S},
$$

and so

$$
\omega \left( \bigcup_{\alpha \in (1, 2)} \mathcal{S}(\alpha) \right) \geq \omega(\mathcal{K} \cap \mathcal{S}).
$$

By Theorem 3.1,

$$
\omega(\mathcal{S}) = \omega \left( \bigcup_{\alpha \in (1, 2)} \mathcal{S}(\alpha) \right) \geq \omega(\mathcal{K} \cap \mathcal{S}).
$$

Therefore $\omega(\mathcal{S}) = \omega(\mathcal{K} \cap \mathcal{S})$.

**Corollary 3.1.** $\omega(\mathcal{K}) = 32/27$.

**Proof.** This result is actually included in [7]. It also follows directly from Theorems 3.2 and 1.1. Recall that $\mathcal{G}$ is the family of connected graphs. It is obvious that $\mathcal{G}$ is splitting-closed and that every graph in $\mathcal{G}$ satisfies one of the conditions in Theorem 3.2. Thus, by Theorem 3.2,

$$
\omega(\mathcal{G}) = \omega(\mathcal{G} \cap \mathcal{K}) = \omega(\mathcal{K}).
$$

By Theorem 1.1, $\omega(\mathcal{G}) = 32/27$.

**Remarks.** (i) Indeed, Jackson [7] and Thomassen [11] have, respectively, shown that the following two families of graphs are splitting-closed and satisfy conditions in Theorem 3.2:

(a) The family of connected graphs;

(b) the family of graphs with a Hamiltonian path.

Thus, by Theorem 3.2, $\omega(\mathcal{S}) = \omega(\mathcal{S} \cap \mathcal{K})$ for each of these two families $\mathcal{S}$. However, $\mathcal{S} \cap \mathcal{K}$ is infinite for these two families, and both Jackson and Thomassen determined $\omega(\mathcal{S} \cap \mathcal{K})$ by considering the limits of chromatic zeros in $(1, 2)$ of graphs in $\mathcal{S} \cap \mathcal{K}$.

(ii) For any integer $k \geq 2$, let $\mathcal{D}_k$ be the family of graphs with domination number at most $k$. It was shown in [3] that $\mathcal{D}_k$ is splitting-closed. It is not difficult to prove that every graph in $\mathcal{D}_k$ satisfies one of the conditions in Theorem 3.2. By Theorem 3.2, $\omega(\mathcal{D}_k) = \omega(\mathcal{D}_k \cap \mathcal{K})$. It can be shown that $\mathcal{D}_k \cap \mathcal{K}$ is a finite set, and so it is not difficult to determine $\omega(\mathcal{D}_k \cap \mathcal{K})$.

4. Graphs with no chromatic zeros in $(1, 2)$. In this section, we study certain families $\mathcal{S}$ of graphs with $\omega(\mathcal{S}) = 2$. We first apply Lemma 3.1 to prove the following result.
THEOREM 4.1. Let \( S \) be a splitting-closed family of connected graphs. Assume that every \( G \in S \) satisfies one of the following conditions:

(i) \( v(G) \leq 4 \);
(ii) \( G \) contains a complete cut-set \( T \) with \( |T| \leq 2 \);
(iii) \( c(G - \{x, y\}) \) is even for some cut-set \( \{x, y\} \) of \( G \) with \( xy \notin E(G) \);
(iv) for some \( e \in E(G) \), \( c(G - e) \neq 2 \) and both \( G - e \) and \( G/e \) belong to \( S \).

Then \( \omega(S) = 2 \).

Proof. Let \( G \) be any graph in \( S \). Since \( G \) satisfies one of the conditions in Theorem 4.1, it is obvious that \( G \) also satisfies one of the conditions in Theorem 3.2. Thus \( \omega(S) = \omega(S \cap K) \) by Theorem 3.2. By Lemma 2.1, every graph in \( K \) does not satisfy condition (ii)--(iv) in this theorem, and only \( K_3 \) satisfies condition (i) in this theorem. Thus \( S \cap K \subseteq \{K_3\} \). Therefore \( \omega(S) = 2 \).

Remark. For a graph \( G \) of order \( n \geq 2 \), an ordering \( (x_1, x_2, \ldots, x_n) \) of the vertices in \( G \) is called a double-link ordering if \( x_1x_2 \in E(G) \) and \( |N(x_i) \cap V_i| \geq 2 \) for all \( i = 3, 4, \ldots, n \), where \( V_i = \{x_1, x_2, \ldots, x_{i-1}\} \) and \( N(x_i) \) is the set of neighbors of \( x_i \) in \( G \). For a double-link ordering \( (x_1, x_2, \ldots, x_n) \) of \( G \), if there exist \( u_i, v_i \in N(x_i) \cap V_{i-1} \) with \( u_i \neq v_i \) for all \( i = 3, 4, \ldots, n \) such that the inequality

\[
|I| > \left| \left\{ i : \{u_i, v_i\} \subseteq I, 3 \leq i \leq n \right\} \right|
\]

holds for every nonempty independent set \( I \) of \( G \), then \( (x_1, x_2, \ldots, x_n) \) is called a \( \gamma \)-ordering. Let \( \Gamma \) be the family of graphs which admits a \( \gamma \)-ordering. It was proved in [5] that every graph in \( \Gamma \) satisfies one of the conditions in Theorem 4.1, and hence \( \omega(\Gamma) = 2 \).

The family \( \Gamma \) includes all graphs with a 2-tree as a spanning subgraph, all graphs \( G \) with a double-link ordering \( (x_1, x_2 \ldots x_n) \) such that \( x_1x_2 \ldots x_n \) is a Hamiltonian path in \( G \), and all complete \( t \)-partite graphs with \( t \geq 3 \).

A graph \( G \) is said to be \( \alpha \)-tough if \( c(G - S) \leq |S| \) for every nonempty independent set \( S \) of \( G \). In [5], Dong and Koh proved that every graph in \( \Gamma \) is \( \alpha \)-tough, and they conjectured the following.

CONJECTURE 4.1. Every \( \alpha \)-tough graph contains no chromatic zeros in (1, 2).

Thomassen [11] conjectured that every Hamiltonian graph contains no chromatic zeros in (1, 2). As every Hamiltonian graph is \( \alpha \)-tough, Thomassen’s conjecture is implied by Conjecture 4.1.

Now we apply Theorem 4.1 to show that Conjecture 4.1 actually follows directly from the next conjecture.

CONJECTURE 4.2. Every \( 3 \)-connected \( \alpha \)-tough graph \( G \) contains an edge \( e \) such that both \( G - e \) and \( G/e \) are \( \alpha \)-tough.

Let \( T \) be the family of \( \alpha \)-tough graphs.

LEMMA 4.1. \( T \) is splitting-closed.

Proof. Let \( G \in T \). So \( G \) is connected and has no cut-vertex. Assume that \( \{u, v\} \) is a cut-set of \( G \).

Assume that \( uv \in E(G) \). If some \( \{u, v\} \)-bridge \( G' \) of \( G \) has an independent set \( S \) such that \( c(G' - S) > |S| \), then \( S \) is also an independent set of \( G \) and \( c(G - S) = c(G' - S) > |S| \), a contradiction. So splitting-closed condition (i) holds for \( T \).

Now assume that \( uv \notin E(G) \). Since \( G \in T \), we have \( c(G - \{u, v\}) = 2 \). It is obvious that \( G + uv \in T \). So every \( \{u, v\} \)-bridge of \( G + uv \) belongs to \( T \). Suppose that \( G \cdot uv \) has a block \( B \) such that \( B \notin T \). Then \( B \) has an independent set \( S \) such that \( c(B - S) > |S| \). Let \( x \) be the new vertex in \( G \cdot uv \) after identifying \( u \) and \( v \). If \( x \notin S \), then \( S \) is also an independent set of \( G \) and \( c(G - S) = c(B - S) > |S| \), a
contradiction. Thus \( x \in S \). But then \( S' = (S \setminus \{x\}) \cup \{u, v\} \) is an independent set of \( G \) and \( c(G - S') = c(B - S) + 1 > |S'| \), a contradiction too. Thus splitting-closed condition (ii) also holds for \( T \).

Hence \( T \) is splitting-closed. \( \square \)

**Theorem 4.2.** *Conjecture 4.2 implies Conjecture 4.1.*

**Proof.** Assume that Conjecture 4.2 is true. By Lemma 4.1, \( T \) is splitting-closed. Let \( G \) be any graph in \( T \). If \( G \) is not 3-connected, then \( G \) satisfies condition (i), (ii), or (iii) in Theorem 4.1. As Conjecture 4.2 is assumed to be true, condition (iv) in Theorem 4.1 is satisfied if \( G \) is 3-connected. By Theorem 4.1, \( \omega(T) = 2 \). Hence Conjecture 4.1 is true. \( \square \)

**Remark.** Let \( H \) be the family of Hamiltonian graphs. Notice that \( H \) is splitting-closed and that for any \( G \in H \), if \( G \) is not 3-connected, then it satisfies one of the conditions in Theorem 4.1. If every 3-connected graph \( G \in H \) satisfies condition (iv) in Theorem 4.1, i.e., \( G \) contains an edge \( e \) such that both \( G - e \) and \( G/e \) belong to \( H \), then \( \omega(H) = 2 \) by Theorem 4.1.

5. **Determine \( \omega(S) \) for minor-closed families \( S \).** For any graphs \( G \) and \( H \), \( H \) is called a minor of \( G \), denoted by \( H \preceq G \), if \( H \) can be obtained from \( G \) by a sequence of operations: (a) Removing an edge, (b) contracting an edge, and (c) removing a vertex. Let \( H \not\preceq G \) denote that \( H \) is not a minor of \( G \). A family \( S \) of graphs is said to be minor-closed if all minors of \( G \) also belong to \( S \) for every \( G \in S \).

**Theorem 5.1.** Let \( S \) be a family of connected graphs. If \( S \) is minor-closed, then \( S \) is splitting-closed, and therefore \( \omega(S) = \omega(S \cap \mathcal{K}) \).

**Proof.** By the definition of \( S \), \( S \) is splitting-closed. Let \( G \) be any graph in \( S \). Assume that \( G \) does not satisfy conditions (i)–(iii) in Theorem 3.2. If \( G \) is 3-connected, then both \( G - e \) and \( G/e \) are 2-connected and belong to \( S \), and thus \( G \) satisfies condition (iv) in Theorem 3.2. Now assume that \( G \) is not 3-connected and \( \{u, v\} \) is any cut-set of \( G \). Since \( G \) does not satisfy condition (iii) in Theorem 3.2, we have \( c(G - \{u, v\}) \geq 3 \). Let \( G_i \) and \( G_j \) be any two components of \( G - \{u, v\} \).

As \( S \) is minor-closed, we have \( G - (V(G_i) \cup V(G_j)) \in S \) and \( H_i \in S \), where \( H_i \) is the graph stated in the definition of condition \( W^* \). Thus \( G \) satisfies condition \( W^* \) at \( \{u, v\} \), implying that \( G \) satisfies condition (iv) in Theorem 3.2.

Thus the result follows from Theorem 3.2. \( \square \)

For any positive integers \( m \) and \( n \), let \( K(m, n) \) denote the complete bipartite graph with bipartition \( A \) and \( B \) such that \( |A| = m \) and \( |B| = n \).

**Corollary 5.1.** Let \( S \) be a minor-closed family of connected graphs. Then

\[
\omega(S) \begin{cases} 
\leq \beta & \text{if } K(2, 3) \in S, \\
2 & \text{otherwise},
\end{cases}
\]

where \( \beta = 1.430159709 \cdots \) is the real zero of \( P(K(2, 3), \lambda) \) in (1, 2) (i.e., the real zero of the equation \( x^3 - 5x^2 + 10x - 7 = 0 \)).

**Proof.** By definition, if \( K(2, 3) \in S \), then \( \omega(S) \leq \beta \). Now assume that \( K(2, 3) \notin S \). By Theorem 5.1, \( \omega(S) = \omega(S \cap \mathcal{K}) \). Note that every graph in \( \mathcal{K} \), except \( K_3 \), contains \( K(2, 3) \) as a minor. Since \( S \) is minor-closed, we have \( S \cap \mathcal{K} \subseteq \{K_3\} \). Hence \( \omega(S) = 2 \).

For any graph \( H \), let

\[
\text{Forb}_{\preceq}(H) = \{G : G \text{ is connected and } H \not\preceq G\}.
\]

Note that \( \text{Forb}_{\preceq}(H) \) is minor-closed, and thus it is splitting-closed by Theorem 5.1.

By Theorem 5.1 and Corollary 5.1, we have the following.
Corollary 5.2. For any graph \( H \),

\[
\omega(\text{Forb}_\omega(H)) = \omega(\text{Forb}_\omega(H) \cap K) \begin{cases} 
\leq \beta & \text{if } H \not\preceq K(2, 3), \\
= 2 & \text{if } H \preceq K(2, 3).
\end{cases}
\]

By Corollary 5.2, we have \( \omega(\text{Forb}_\omega(K_4 - e)) = 2 \), where \( K_4 - e \) is the graph obtained from \( G \) by removing an edge. In the following we show that \( \omega(\text{Forb}_\omega(K_4)) = 32/27 \).

Lemma 5.1. For graphs \( H \) and \( G \), if \( H \preceq G \), then \( \omega(\text{Forb}_\omega(G)) \leq \omega(\text{Forb}_\omega(H)) \).

Proof. If \( H \preceq G \), then \( \text{Forb}_\omega(H) \subseteq \text{Forb}_\omega(G) \). Thus the result follows from Lemma 1.1.

Theorem 5.2. Let \( H \) be any graph.

(i) If \( K_4 \preceq H \), then \( \omega(\text{Forb}_\omega(H)) = \frac{32}{27} \).

(ii) \( \omega(\text{Forb}_\omega(K(2, 4))) = \beta \).

Proof. (i) Since all graphs in \( K \) are series/parallel graphs, we have \( K \subseteq \text{Forb}_\omega(K_4) \).

If \( K_4 \preceq H \), then by Lemma 5.1,

\[
\frac{32}{27} \leq \omega(\text{Forb}_\omega(H)) \leq \omega(\text{Forb}_\omega(K_4)) \leq \omega(K) = \frac{32}{27},
\]

implying that \( \omega(\text{Forb}_\omega(H)) = \frac{32}{27} \).

(ii) Notice that \( K(2, 4) \preceq G \) for every \( G \in K \) with \( v(G) \geq 7 \). So \( \text{Forb}_\omega(K(2, 4)) \cap K = \{K_3, K(2, 3)\} \). As \( \beta \) is the only zero of \( P(K(2, 3), \lambda) \) in \( (1, 2) \), the result follows from Theorem 5.1.

Remark. By a similar method, it is not difficult to determine \( \omega(\text{Forb}_\omega(K(2, k))) \) for any given \( k \geq 5 \).

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REFERENCES
