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Entanglement and quantum phase transition of 
spin glass: a renormalization group approach

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Abstract

Using a renormalization group approach, we study the entanglement properties of two spin glass models: the XXZ Heisenberg (with Dzyaloshinskii-Moriya interaction) and Ising transverse field spin glasses. The concurrence for both models are obtained through the Kadanoff’s renormalization group (RG) approach with random $J_i^z$ and $J_i$ respectively. The constant couplings in the RG flow is randomized through the Gaussian distribution. For $\Delta = 0$ corresponding to a non-spin glass material, a first-order transition is expected. By varying $\Delta = 0.05$ to 0.5, the spin glass effect broadens the sharp transition resulting in a second-order-like transition. The fluctuations in the average concurrence for the spin glass case as measured by the standard deviations is also a good indicator of quantum phase transition.
1 Introduction

With just a few percent of a magnetic element randomly distributed in a non-magnetic host such as noble metals, a dilute alloy of a spin glass is formed producing many interesting experimental results. These results have initiated a whole range of new topics specifically in the areas of statistical mechanics and condensed matter physics. Unlike a classical piece of glass, a spin glass consists of magnetic moments or spins which are randomly distributed and quenched. Due to the disorderness, the spins conflict with one another giving rise to frustration effects [1, 2, 3, 4, 5, 6, 7]. These disorders and frustrations produce a complex and rugged free energy landscape. The magnetic element or impurities like manganese (Mn), iron (Fe) or europium (Eu) is introduced into a non-magnetic metal capable of dissolving the impurities. Examples of such diluted alloys are copper and manganese, Cu$_{1-x}$Mn$_x$ [8] or gold and iron, Au$_{1-x}$Fe$_x$ [9]. Alloys with properties of insulation and conduction can also be considered as spin glass and there are europium strontium sulfur Eu$_x$Sr$_{1-x}$S [10] which is a semiconductor and lanthanum gadolinium aluminum La$_{1-x}$Gd$_x$Al$_2$ [11] which is a metal.

Two of the central experimental signatures that are usually used to characterize whether a material is a spin glass are magnetic susceptibility and specific heat capacity. For any typical spin glass, the magnetic susceptibility usually shows a cusp at a certain freezing temperature $T_f$ for low applied magnetic field. By varying the impurity concentration in these alloys, there is a critical temperature which corresponds
to the cusp of the susceptibility. This critical temperature is termed the freezing temperature [12]. For any phase transition to occur, all thermodynamic functions will behave singularly [13, 7]. Hence, the cusp in the magnetic susceptibility suggests that there may be a phase transition at a particular critical temperature. A broader maxima is produced if around 100 G of applied magnetic field is present when the susceptibility is measured [9, 14, 15]. In contrast to the effect of been field dependent, certain spin glasses are also found to be frequency dependent [8, 16]. Even though the magnetic susceptibility of a typical spin glass does exhibit a sharp cusp in low magnetic field, other measurement like the specific heat capacity of Au$_{0.92}$Fe$_{0.08}$ [17] and CuMn [18] were found to have no singularity. This means that only a broad, smooth and rounded maximum is produced instead of a cusp. Moreover, the rounded maximum of the specific heat capacity does not coincide with the transition temperature for the magnetic susceptibility. Beyond the experimental studies, theories like the Edwards-Anderson (EA) model [19] which only allows the spins to interact via nearest-neighbor couplings with no long range order and Sherrington-Kirkpatrick (SK) model [20] for which every spin couples equally with every other spin are formulated in an attempt to explain mainly the cusp in the magnetic susceptibility. The EA model essentially replaces the site disorder and Ruderman-Kittel-Kasuya-Yosida (RKKY) distribution [21, 22, 23] with a random set of bonds. This set of random bonds is usually taken from a distribution like Gaussian. In order to understand the spin glass phase, an order parameter $q$ has been formulated to characterize it. Despite different and new theories have been produced to understand the physics of
spin glass, other problems from the theories have since arise. As an example, original EA equations are not simple to solve and are only soluble in the limits $T \to 0$ and $T \to T_f$. Moreover, the EA equations showed an asymmetric cusp in the magnetic susceptibility and specific heat. In disagreement, the results by Fischer [24] has showed that the theoretical specific heat is different from the experimental result except for the low temperature linear dependence when using spin $S = \frac{1}{2}$. Even though the SK model did exhibit a cusp in the magnetic susceptibility and specific heat, the entropy $S$ becomes negative at $T = 0$ [20]. When $q = 0$, the spin glass state has lower free energy than it has for $q \neq 0$. With such instability in the SK solution, Almeida and Thouless (AT) [25] showed that the stability limits of the SK solution by using the AT line to divide the unstable and stable areas in the spin glass phase diagram. The instability is essentially due to the fact that the SK model treats all the replicas as indistinguishable. Fortunately, Parisi [26, 27, 28, 29, 30] came out with a replica symmetry breaking (RSB) scheme which removes the unphysical negative entropy. However, it was found to be at least marginally stable. Although there are some success in using these models to understand the behaviors of the spin glass, they are unable to account for all the experimental results shown. One possible reason is that these theories are classical in nature and did not consider the quantization of the spins of the impurities [16]. Nevertheless, new insights and mathematical tools developed in this field have been found to be applicable in other areas of condensed matter [6, 5, 31], complex optimization problems [32] and biological problems [33]. Over the recent years, LiHo$_x$Y$_{1-x}$F$_4$ which can be described with a quantum Ising spin glass
model has been experimentally and numerically studied [34, 35, 36, 37, 38, 39]. For an \( x \) concentration of \( \leq 0.25 \), it is believed that a spin glass phase exists. However, it is still an open question of whether a spin glass or an antiglass spin phase exists at lower concentration.

The theory of entanglement has been studied and used in both quantum information theory and condensed matter physics. In condensed matter physics, entanglement has been used to study the phase transitions of spin chain at low and finite temperature [40]. For a quantum phase transition, the change occurs at zero temperature where only the quantum fluctuations are involved and not the thermal counterpart. Numerous studies were carried out in investigating the role of entanglement in the proximity of quantum critical point for the different spin chain models [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50]. In quantum information theory, entanglement is viewed as an important resource in applications such as quantum key distribution, quantum teleportation, quantum dense coding, entanglement swapping and others [51, 52, 53, 54, 55, 56, 57]. In particular with the application of entanglement in spin chains, density-matrix renormalization group (DMRG) approach has been utilized to understand the quantum effects for finite spin chain [58, 59, 60, 61, 62, 63]. Even though such approach has been useful and accurate in describing the ground states for finite chain especially for one dimensional case, it is numerical in nature and not many works have studied using DMRG for spin glass. Other works have focused on using the Kadanoff’s block approach in understanding the renormalization of entanglement and phase diagram for the various spin model [64, 65, 66, 67]. Since Kadanoff’s
block allows one to investigate the critical behaviour of the spin chain analytically, one would be curious to know if it could help us in using this approach to better understand the physics of spin glass. With this motivation, we use the Kadanoff’s block approach to obtain the scaled couplings from the effective Hamiltonian of the renormalized $XXZ$ Heisenberg with Dzyaloshinskii-Moriya (DM) interaction and the Ising model with transverse field (ITF). With these new effective Hamiltonians containing the renormalized couplings, we investigate the entanglement of these models to finite chain with increasing renormalized group (RG) iterations. The effective couplings are then used to explore the behaviour of a spin glass for finite chain.

The paper is organized as follows. We begin in Sec. 2 by defining the Hamiltonian for a $XXZ$ Heisenberg model with DM interaction. Using this model, we defined a single RG block and the effective Hamiltonian expressed in terms of the new renormalized coupling constant $J_z^i$ and the DM interaction $D$. In addition, we also look at the Ising model and obtained the new renormalized coupling constant $J$ and the applied magnetic field $B$. For both models, we use the Kadanoff’s block approach to find the projection of each operators in the renormalized space and obtain the projected intra- and inter-block for the new effective Hamiltonian. With the use of a bipartite measure, we use the renormalized expressions to compute the entanglement (concurrence) for the block. With iteration of $n$th steps, we trace the RG flow and reached a steady point for finite size of spin chain. By using the rescaled renormalization equations, we explore how the concurrence changes with each iteration for the case of a spin glass where the couplings is subject to a Gaussian distribution. These results
are presented and discussed in Sec. 3. In Sec. 4, we summarize our results.

2 Theoretical Formulation for RG Approach

2.1 $XXZ$ Heisenberg Model with DM Interaction

In this subsection, we show how the Kadanoff’s approach is used in obtaining the renormalized couplings by comparing the inter- and intra-block of an $XXZ$ Heisenberg spin chain. The renormalized couplings is obtained by building the projection operators on each block and projecting each block onto the lower energy subspace. The projected inter- and intra-block are mapped to an effective Hamiltonian which can then be compared to the original Hamiltonian. The $XXZ$ Heisenberg model with the DM interaction in the $z$ direction for $N$ sites is in general given as

$$H_{XXZ} = \frac{J}{4} \left[ \sum_{i=1}^{N} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + J_i^z \sigma_i^z \sigma_{i+1}^z \right) + D \left( \sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^x \right) \right]$$ (1)

where $J$ is the constant coupling between the individual sites in the $XXZ$ periodic chain and $\sigma_i^\alpha$ is the Pauli matrices ($\alpha = x, y, z$) for the $i$th spin [40]. The $D$ term stands for the strength of the DM interaction along $z$ axis and the easy-axis anisotropy is represented with $J_i^z$ which is random. The $J_i^z$ are quenched random variables with a probability distribution $P(J_i^z) = \frac{1}{\sqrt{2\pi}\Delta} e^{-\left(\frac{J_i^z-\mu}{\Delta}\right)^2/2\Delta^2}$ where $\Delta$ is the standard deviation for the distribution. In general, the effective Hamiltonian $H'$ is

$$H' = H'_B + H'_{BB}$$ (2)
where $H_B$ represents the Hamiltonian for the intra-block after projection and $H_{BB}$ represents the Hamiltonian for the inter-block after projection. We find the effective Hamiltonian by considering three qubits as a single block $H_B$. The coarse graining of the degrees of freedom - from 3 sites for each single block is converted into a single site which in return form another single block with other 2 sites. This process of coarse graining is shown in Fig. 1(f). In order to take into account the $J_z$ coupling between the third and forth site, we need to consider the effective Hamiltonian for two single blocks. The first single intra-block $I$ which consists of the first three sites is

$$H'_B = \frac{J}{4} \sum_{I} \left\{ (\sigma^x_{1,I} \sigma^y_{2,I} + \sigma^y_{1,I} \sigma^x_{2,I} + \sigma^y_{2,I} \sigma^y_{3,I} + \sigma^x_{2,I} \sigma^x_{3,I}) 
+ J_z I (\sigma^z_{1,I} \sigma^z_{2,I} + \sigma^z_{2,I} \sigma^z_{3,I}) 
+ D (\sigma^x_{1,I} \sigma^y_{2,I} + \sigma^y_{1,I} \sigma^x_{2,I} - \sigma^y_{2,I} \sigma^x_{3,I}) \right\}$$

(3)

and the inter-block Hamiltonian between the two single blocks is

$$H'_{BB} = \frac{J}{4} \sum_{I} \left\{ (\sigma^x_{3,I} \sigma^x_{1,I+1} + \sigma^y_{3,I} \sigma^y_{1,I+1} + J_z I (\sigma^z_{3,I} \sigma^z_{1,I+1}) 
+ D (\sigma^x_{3,I} \sigma^y_{1,I+1} - \sigma^y_{3,I} \sigma^x_{1,I+1}) \right\}$$

(4)

Hence, the effective Hamiltonian for the $XXZ$ Heisenberg model with DM interaction is

$$H'_{XXZ} = H'_B + H'_{BB}$$

(5)

Since the single intra-block will coarse-grain into a single site block, we only need to consider the coupling between the inter-blocks (third and forth sites) which is shown
in the second term of Eq. (5). The Pauli matrices are in the $z$ basis and are as follows

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  \hspace{1cm} (6)$$

The Hamiltonian produces four distinct eigenvalues which are doubly degenerate.

The normalized doubly degenerate ground states are

$$|\psi_0\rangle = \frac{1}{\sqrt{2q(q + J_z^i)(1 + D^2)}} \left\{ 2 (D^2 + 1) |001\rangle - (1 - iD) (J_z^i + q) |010\rangle 
- 2 \left[ 2iD + (D^2 - 1) \right] |100\rangle \right\}$$

$$|\psi'_0\rangle = \frac{1}{\sqrt{2q(q + J_z^i)(1 + D^2)}} \left\{ 2 (D^2 + 1) |011\rangle - (1 - iD) (J_z^i + q) |101\rangle 
- 2 \left[ 2iD + (D^2 - 1) \right] |110\rangle \right\}$$  \hspace{1cm} (7)$$

with eigenvalue $E_0 = -\frac{J}{4} (J_z + q)$. Note: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The projectors are

$$P = |\psi_0\rangle\langle 0| + |\psi'_0\rangle\langle 1|$$

$$P^\dagger = |0\rangle\langle \psi_0| + |1\rangle\langle \psi'_0|$$  \hspace{1cm} (8)$$

and the effective Hamiltonians for $H_B$ and $H_{BB}$ in terms of the projectors are

$$H'_B = PH_B P^\dagger, \quad H'_{BB} = PH_{BB} P^\dagger$$  \hspace{1cm} (9)$$

Taking $q = \sqrt{(J_z^i)^2 + 8(1 + D^2)}$, the projection of each operators in the renormalized space are

$$P \sigma^x_{i,j} P^\dagger = \frac{-2}{q} (\sigma^x_i + D \sigma^y_i)$$  \hspace{1cm} (10)$$
\begin{align*}
P\sigma^z_{2,1}P^\dagger &= \frac{4}{q} \left( D^2 + 1 \right) \sigma^z_i \\
P\sigma^z_{3,1}P^\dagger &= -\frac{2q}{q} \left( \sigma^z_i - D\sigma^y_i \right) \\
P\sigma^y_{1,1}P^\dagger &= -\frac{2q}{q} \left( D\sigma^y_i - \sigma^y_i \right) \\
P\sigma^y_{2,1}P^\dagger &= -\frac{4}{q} \left( D^2 + 1 \right) \sigma^y_i \\
P\sigma^y_{3,1}P^\dagger &= \frac{2}{q} \left( D\sigma^y_i + \sigma^y_i \right) \\
P\sigma^z_{1,1}P^\dagger &= P\sigma^z_{3,1}P^\dagger \\
&= -\frac{q + J_i^z}{2q} \sigma^z_i \\
P\sigma^z_{2,1}P^\dagger &= \frac{J_i^z}{q} \sigma^z_i
\end{align*}

By replacing all the projected terms into Eq. (4), the inter-block Hamiltonian is given as

\begin{align*}
PH'_{BB}P^\dagger &= \frac{J_i}{4} \left\{ \frac{4}{q^2} \left( \sigma^x_{5,1} - D\sigma^y_{5,1} \right) \left( \sigma^x_{1,1} + D\sigma^y_{1,1} \right) \\
&\quad - \frac{4}{q^2} \left( D\sigma^x_{3,1} + \sigma^y_{3,1} \right) \left( D\sigma^x_{1,1} - \sigma^y_{1,1} \right) \\
&\quad + J_i^z \left( -\frac{J_i^z + q}{2q} \right) \sigma^z_{3,1} \sigma^z_{1,1} + 1 \\
&\quad + 4 \left[ \frac{4}{q^2} \left( \sigma^x_{5,1} - D\sigma^y_{5,1} \right) \left( D\sigma^x_{1,1} - \sigma^y_{1,1} \right) \\
&\quad + \frac{4}{q^2} \left( D\sigma^x_{3,1} + \sigma^y_{3,1} \right) \left( \sigma^x_{1,1} + D\sigma^y_{1,1} \right) \right] \right\}
\end{align*}
By expanding Eq. (18) and rearranging all the similar terms together, the projected inter-block Hamiltonian is

\[
P H'_{BB} P^t = J \left[ \frac{4}{q^2} \left( \sigma_{3,i}^z \sigma_{1,i+1}^z + \sigma_{3,i}^y \sigma_{1,i+1}^y \right) + \frac{J_i^z (J_i^z + q)^2}{4q^2} \sigma_{3,i}^z \sigma_{1,i+1}^z \right]
\]

\[
+ \frac{4D}{q^2} \left( 1 + D^2 \right) \left( \sigma_{3,i}^x \sigma_{1,i+1}^y - \sigma_{3,i}^y \sigma_{1,i+1}^x \right) \right] \quad (19)
\]

By comparing Eq. (19) with the original Hamiltonian in Eq. (1), the renormalized coupling \(J', (J_i^z)'\) and \(D'\) are given as

\[
J' = J \left( \frac{2}{q} \right)^2 \left( 1 + D^2 \right) \quad (20)
\]

\[
(J_i^z)' = \frac{J_i^z}{1 + D^2} \left( \frac{J_i^z + q}{4} \right)^2 \quad (21)
\]

\[
D' = D \quad (22)
\]

We adopt a bipartite measure for a two-level system called concurrence [68, 69] to compute the entanglement of the RG block under finite iterations. A concurrence with a value of zero means that the state is unentangled and for a maximally entangled state, the concurrence is unity (= 1). The concurrence for a bipartite two-level system is defined as

\[
C(\rho_s) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \quad (23)
\]

where \(\lambda_n (n = 1, 2, 3, 4)\) are the square roots of the eigenvalues arranged in decreasing order. The eigenvalues are obtained from the matrix \(\rho_s \tilde{\rho}_s\). The eigenvalues \(\lambda_n\) are real and non-negative. \(\rho_s\) is the density matrix of the 2-qubit single block and \(\tilde{\rho}_s\) is
defined as the “spin-flipped” state, expressed as

\[ \tilde{\rho}_s = (\sigma_y \otimes \sigma_y) \rho_s^*(\sigma_y \otimes \sigma_y) \]  

(24)

where \( \sigma_y \) is a Pauli matrix expressed as

\[ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]  

(25)

Hence,

\[ \rho_s \tilde{\rho}_s = \rho_s(\sigma_y \otimes \sigma_y \rho_s^* \sigma_y \otimes \sigma_y) \]  

(26)

where \( \rho_s^* \) is the complex conjugate of the \( \rho_s \). The density matrix of the single block is

\[ \rho_{XXZ} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & a & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \]  

(27)

and

\[
\begin{align*}
a &= \frac{1}{q'} \\
b &= \frac{-i + D}{q'(-i + D)} \\
c &= \frac{i + D}{q'(-i + D)} \\
d &= \frac{(J_i^z + \sqrt{q})^2}{4q'(-i + D)(i + D)}
\end{align*}
\]

(28)

where \( q' = 2 + \frac{(J_i^z + \sqrt{q})^2}{4(-i + D)(i + D)} \).
2.2 Ising Model with Transverse Magnetic Field

In this subsection, we show how the Kadanoff’s approach is used to obtain the renormalized couplings by comparing the inter- and intra-block of the Ising model with applied transverse magnetic field (ITF). The renormalized couplings is obtained by building the projection operators on each block and projecting each block onto the lower energy subspace. The coarse graining process of the ITF is simply illustrated in Fig. 2(f) where the 2-site block is reduced to a single site. This single site is later combined with another site to form a single block and it continues. The projected inter- and intra-block are mapped to an effective Hamiltonian which can then be compared to the original Hamiltonian. The ITF Hamiltonian is given as

\[ H_{ITF} = \sum_{i=1}^{N} -J_i (\sigma_i^z \sigma_{i+1}^z + g\sigma_i^x) \] \hspace{1cm} (29)

where \( J_i \) are the random variables and \( \sigma_i^\alpha \) denotes the Pauli matrices (\( \alpha = x, z \)) of the \( i \)th spin [40] which is subject to the open boundary condition \( \sigma_i^\alpha \neq \sigma_1^\alpha \). In this case, the \( i \)th spin represents the individual site number from the spin glass chain. The exchange energies \( J_i \) are quenched random variables with a probability distribution

\[ P(J_i) = \frac{1}{\sqrt{2\pi\Delta}} e^{-(J_i-\mu)^2/2\Delta^2} \] where \( \Delta \) is the standard deviation for the distribution.

The external magnetic field \( B \) is applied transversely across the individual spin site \( \sigma_i^z \) and \( B = gJ_i \). The first single intra-block \( I \) consists of the first two sites is

\[ H'_B = \sum_{I=1}^{N/2} -J_i (\sigma_{1,I}^z \sigma_{2,I}^z + g\sigma_{1,I}^x) \] \hspace{1cm} (30)
and the inter-block Hamiltonian between the two single blocks is

\[ H'_{BB} = \frac{N}{2} \sum_{I=1}^{N/2} -J_i \left( \sigma^z_{2,I} + \sigma^z_{1,I+1} + g\sigma^x_{2,I} \right) \]  

(31)

Hence, the effective Hamiltonian for the ITF model is

\[ H'_{ITF} = H'_B + H'_{BB} \]  

(32)

With the same procedure of obtaining the eigenvalues and eigenvectors for the XXZ model, we take the two lowest ground state energies to obtain the projectors using Eq. (8) and find that the renormalized coupling constants [67] are

\[ J'_i = J_i \frac{2 \left( \sqrt{g^2 + 1} + g \right)}{1 + \left( \sqrt{g^2 + 1} + g \right)^2} \]  

(33)

\[ g' = g^2 \]  

(34)

Using the same measurement of entanglement (concurrence) in Eq. (23) to compute the concurrence of the spin glass chain for a renormalized two sites block, the density matrix of the block is

\[ \rho_{ITF} = \begin{pmatrix} a & b & b & a \\ b & c & c & b \\ b & c & c & b \\ a & b & b & a \end{pmatrix} \]  

(35)
and

\[ a = \frac{1}{4} (1 + \alpha) \]
\[ b = \frac{1}{2} \beta \]
\[ c = \frac{1}{4} (1 - \alpha) \] (36)

where \( \alpha = \frac{J_i}{\sqrt{4B^2 + J_i^2}} \), \( \beta = \frac{B}{\sqrt{4B^2 + J_i^2}} \) and \( B = gJ \). With the density matrix, we can compute the concurrence by substituting the renormalized coupling constants for each iteration and trace the RG flow to a fixed point. In order to understand how the renormalized coupling constants change under the RG flow for the spin glass case, we use the renormalized couplings to define a new distribution for the coupling \( J_i^z \) and \( J_i \). The mean concurrence of each of this new \( J_i^z \) and \( J_i \) coupling distributions are obtained with an iteration of 1000 times. The mean renormalized concurrence with the corresponding values of \( D \) and \( \frac{B}{J_i} \) are plotted with a variation in \( \Delta \). The results are presented in Sec. 3.

3 Results and Discussion

The renormalization group equations for both the XXZ Heisenberg and Ising with transverse field models are first obtained. We then present and discuss the numerical results for the XXZ with DM case followed by the ITF case. With the couplings \( J_i^z \) set to a constant mean of \( \mu = \sqrt{2} \) and \( \Delta = 0 \) (i.e. no spin glass), we see that the average concurrence exhibit a phase transition for \( D = 1 \) after 15 iterations. This
Figure 1: With $\mu = \sqrt{2}$ and varying standard deviation $\Delta$ for the $J_i^z$ coupling, the four plots (a, b, c, d) show how the average concurrence evolves over the DM interaction ($D$). (e) shows how the standard deviation of the concurrence varies with the DM interaction and displaying a phase transition at $D = 1$. (f) shows a simplified diagram of how the RG condition $J_{i'} \sim N(\mu, \sigma^2)$ is implemented.
Figure 2: With mean of $J_i = 1$ and varying standard deviation, the four plots (a, b, c, d) show how the average concurrence evolves over $\frac{B}{J_i}$. (e) shows how the standard deviation of the concurrence varies with the $\frac{B}{J_i}$ and displaying a phase transition at $\frac{B}{J_i} = 1$. (f) shows a simplified diagram of how the RG approach is used in renormalizing the
plot of the concurrence with a first-order phase transition is shown in Fig. 1(a). With a small fluctuation introduced into the XXZ model ($\Delta = 0.05$), we observe a large fluctuation in the average concurrence at around $D = 1$. This is presented in Fig. 1(b). With an increase in the randomness by varying the $\Delta$ from 0.1 to 0.5, the sharp first-order transition at $D = 1$ in Fig. 1(a) disappears and gives way to a second-order type transition. As illustrated in Figs. 1(c) and (d), increasing the standard deviation $\Delta$ from 0.1 to 0.5 results in greater fluctuation in the average concurrence away from the critical $D$ value. From the graphs, it appears that there is a smearing of the sharp first-order phase transition of a XXZ with DM interaction model at $D = 1$ due to the introduction of disorder and frustration in a spin glass.

Besides observing the evolution of a phase transition for a spin glass XXZ model through the measurement of the average concurrence, the critical point of the phase transition could also be determined through the fluctuations in the average concurrence as measured by its standard deviation. At $D = 1$, the standard deviation of the average concurrence exhibits a sharp cusp at low $\Delta = (0.05, 0.1)$. For $\Delta = 0.05$ and 0.1, it is represented with green diamond and blue inverted triangle in Fig. 1(e). With $\Delta = 0.5$, the cusp is broadened as shown in Fig. 1(e). This is analogous to the broadening of the magnetic susceptibility of a spin glass with increasing magnetic field. In this analogy, the standard deviation $\Delta$ in the random variate for the couplings plays the same role as the applied magnetic field for the measurement of magnetic susceptibility.

Iterating the RG equations ten times, the Ising model with a transverse field
$(\Delta = 0)$ has a critical phase transition point at $\frac{B}{J_i} = 1$ as shown in Fig. 2(a). In this plot, the renormalized concurrence is plotted with varying $\frac{B}{J_i}$ where $J_i$ is equal to a fixed constant of 1. With a mean $J_i = 1$ and $\Delta = 0.05$ in Fig. 2(b), we observed that the average concurrence again fluctuates around the critical point as compared to Fig. 2(a). By increasing the $\Delta$ from 0.1 to 0.5 in Figs. 2(c) and (d), the average concurrence for the ITF case starts to fluctuate even more. The fluctuations in the average concurrence gets bigger, spreading out from the critical point of $\frac{B}{J_i} = 1$.

By comparing the four plots [(a), (b), (c), (d)] in Fig. 2 with those in Fig. 1, the analysis of the average concurrence seems to behave in a qualitatively similar manner with the respective order parameter except that for the $XXZ$ model, the average concurrences are larger for larger $D$ whereas for the ITF case, the average concurrences are smaller for larger order parameter (i.e. $\frac{B}{J_i}$). Just as the case of the $XXZ$ Heisenberg magnet, Fig. 2(e) illustrates how standard deviation in the average concurrence is another criteria for probing the critical phase transition in the ITF model.

4 Conclusion

The interesting experimental results like the magnetic susceptibility and specific heat capacity of a typical spin glass have introduced many interesting and open questions in the statistical and condensed matter fields. These questions have motivated many researchers to invent various new and novel methods to reconcile the results between
the experiments and theories of a spin glass. In this study, we have introduced a new way of exploring and understanding the possible phase transition of a spin glass in $XXZ$ and ITF models by studying concurrence between adjacent blocks using the RG equations. The renormalization group approach has provided a novel way to overcome the exponential increase in the computational resources in the dimension of the Hilbert space by coarse graining the number of degrees of freedom in the chain. By considering random couplings in the RG equations, we have introduced the idea of frustration and disorder in the quantum systems resulting in a spin glass-like effect. With the aid of the RG approach and the theory of quantum entanglement, we are able to study the phase transition of a spin glass in both the $XXZ$ and ITF to $N$ qubits in one-dimension systems. In general, an increase of disorderness by increasing the standard deviation $\Delta$ causes the first-order transition at the critical order parameter to change into a second-order-like transition. In short, the introduction of spin glass effects into a system results in a second-order phase transition in the order parameter. This is probably due to reduced symmetry in the system. Further fluctuation due to the increase in $\Delta$ for the quantum system results in the disappearance of the phase transition as observed through the measurement of entanglement (concurrence). In addition to using the concurrence to compute and probe the order of transition for such models, the understanding of the phase transition may also be carried out by plotting the standard deviation of the average concurrence with $D$ or $\frac{B}{J_i}$. Thus it would be interesting to device direct measurements of these fluctuations in the average concurrences.
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References


