An Investigation of Students' Errors in Logarithms

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In this study we set out to investigate the errors made by students in logarithms. A test with 16 items was administered to 89 Secondary three students (Year 9). The errors made by the students were categorized using four categories from a framework by Movshovitz-Hadar, Zaslavsky, and Inbar (1987). It was found that students in the top third were less likely to make 'distorted theorem or definition' type of errors whereas they were more likely to make errors in the other three categories.

Students in Singapore taking Additional Mathematics, which is a more advanced syllabus for O-level mathematics, have to study logarithms in Secondary Three and Four (Years 9 and 10). The basic ideas about logarithms in this syllabus include: the equivalence of $y = a^x$ and $x = \log_a y$, the laws of logarithms and the solution of simple logarithmic equations, as well as some simple uses in the calculus portion of the syllabus when dealing with derivatives and integrals. The topic on logarithms is generally considered as one of the more difficult topics in Additional Mathematics (see Chua & Wood, 2005). In his study involving logarithms at the secondary level in Singapore, Chua (2004) highlighted the difficulty in teaching the topic in schools and claimed that even when students can do the questions that are in the text and on the examinations, "their understanding of the fundamental nature of logarithms remains in doubt" (p. 53). Issues with the teaching and learning of logarithms still persist and seem to be quite widespread as demonstrated by several studies (see Berezvoski, 2004; Tabaghi, 2007). In this study, we set out to investigate some students' errors and their misconceptions in the topic on logarithms. We looked mainly at the following question: what are the types of errors that students make when solving mathematical problems involving logarithms? Due to space limitations, we only broadly focus on why the students make such errors.

Literature Review

Errors and Misconceptions

Olivier (1989) explained the differences between slips, errors and misconceptions. He pointed out that *slips* are the result of students being careless due to their processing. They can be easily detected and corrected. *Errors* on the other hand are the result of planning. They are classified as systematic as they are usually applied again in the same situation. These errors occur due to students' existing conceptual structures. He identified that the underlying beliefs and the principles in these structures as *misconceptions*. Errors are also referred to as "reflections of students' misconceptions and missing conceptions" (Matz, 1982, p. 26).

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Classification of Errors

The study and classification of errors in mathematics is not new. Several studies have looked into this phenomenon. For example, Radatz (1979) used the Information-Processing model to classify errors into five types. Similarly, Matz (1982) in her study in algebra with high school students classified errors into three categories. On the other hand, Movshovitz-Hadar, Zaslavsky and Inbar (1987), in their study on Israeli high school mathematics students developed a model in which the guiding principle was to classify the errors by means of documented performance without appealing to processes in the students' minds that might or might not have yielded the errors committed and without faulting what the students did not do. Accordingly, the authors came up with the following six descriptive categories to classify errors in high school mathematics: (i) misused data, (ii) misinterpreted language, (iii) logically invalid inference, (iv) distorted theorem or definition, (v) unverified solution and, lastly (vi) technical error. The classification of errors by Movshovitz-Hadar et al., that is mentioned above helped us in developing the categories for classifying errors made by the students in this study.

Students' errors in logarithms can be connected to their understanding of the topic which implies knowing about their mathematical thinking and mental constructions. Since we are interested in the errors that students make, it is befitting to look at what it means to understand and, in particular, what it means to understand logarithms and how students acquire this fairly elusive concept. Several authors have looked at the idea of understanding; we will briefly look at the main ideas about understanding from Skemp (1987) and also comment on *Action*, *Process*, *Object* and *Schema* [APOS] theory which has been used in some studies.

Relational and Instrumental Understanding and APOS Theory

Skemp (1987) described that to understand something means to assimilate it into an existing schema. Accordingly, Skemp posited that understanding is subjective and furthermore, it is not an all-or-nothing state. As such, when a student solves a problem using an appropriate rule that he or she has learnt without realising why the rule works, is solving the problem based on his or her *instrumental understanding.* On the other hand, students with *relational understanding* will be able to deduce the appropriate rules and be able to reason why the rules are being used to solve a problem. They will be more adaptable to new situations as they will be able to relate existing concepts to make connections to new concepts.

Some authors have used the ideas of *Action*, *Process*, *Object* and *Schema* or APOS theory (see Dubinsky & McDonald, 2001) to explain the mathematical thinking and mental constructions of students which seem relevant for an analysis of errors. Weber (2002), for instance used the theory to study students' understanding of exponential and logarithm functions. He reported that most of the students involved in the study could only understand exponentiation as an action but could not do so as a process. Tabaghi (2007) also used the APOS theory to analyse students' understanding of logarithms. He reported that the students' understanding of the arithmetic notion of logarithms did not go beyond the *process* level.

Methodology

This qualitative study involved a total of 89 students coming from three different classes in a secondary school in Singapore. All of the students were enrolled for the Additional Mathematics O-level course and had previously studied the topic on logarithms. The students had to sit for a 45-minute written test with 16 items. The students were required to show their detailed work and were not allowed to use a calculator. The instrument was pilot-tested with a group of 20 students and based on the pilot study some items were modified and some sub-parts were dropped so that the test could be completed in 45 minutes. The test was constructed using items in textbooks and past examination papers under the following three groups: (1) logarithms and logarithmic expressions as numbers, (2) operational meaning of logarithms, and (3) logarithms as functions.

A total of 10 students from three groups were interviewed for this study. Three students were selected from the top third (Upper Group-UG), three from the middle third (Middle Group-MG) and four from the bottom third (Lower Group-LG), after the students had been rank-ordered from one to 89 based on the test. Each of the 10 students was assigned an alphabet code from A to J. The main objectives of the interview sessions were to gain more insights into why certain types of errors were committed by the students and to identify their learning gaps should they have any. The interview questions focused around the students' understanding of logarithms and their ability to identify errors in samples provided to them.

Data Analysis

The test was graded out of 45 marks (M = 21.2, SD = 9.04)and the Facility Index (FI) for each of the 16 items in the test was computed by dividing the average marks for the item by the total possible marks for that item and expressing the result as a percentage. In classifying the types of errors made by the different group of students, we used the four categories, namely, *distorted theorem and definition*, *misused data*, *technical errors* and *unverified solution* (see Movshovitz-Hadar et al., 1987) as there were a significant number of errors committed among the three groups in these categories.

Table 1
Percentage Distribution of Students' Errors

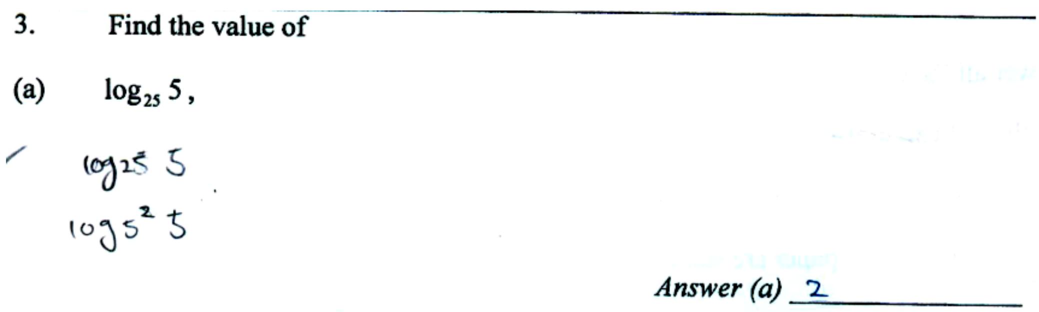
Type of Error	Description of Error	Upper Group M=31.3	Middle Group M=22.3	Lower Group M=11
Distorted Theorem or Definition	Improper or incorrect use of the laws of logarithms	10.1 (9)	25.8 (23)	34.8 (31)
Technical Errors	Manipulation errors due to carelessness	10.1 (9)	6.7 (6)	4.5 (4)
Misused Data	Data used in student written work not the same as data in test item	6.7 (6)	2.2 (2)	4.5 (4)
Unverified Solution	Final solution does not satisfy given equation: Failure to check	26.9 (24)	20.2 (18)	7.9 (7)

(X) – indicates number of students, n = 89, Max. marks for test = 45, M = mean mark for group

In Table 1, all percentages were calculated based on the total number of students who sat for the test which was 89. For each category, we also indicated the total number of students by ability group who made that type of error in brackets.

Distorted Theorem or Definition Error

According to Movshovitz-Hadar et al. (1987), this category includes those errors that deal with a distortion of a specific and identifiable principle, rule, theorem, or definition. In this item (See Figure 1), the student is trying to use the rule that $\log_a a = 1$ to evaluate the logarithm. Using the rule the student changed the base of the logarithm to 5^2 instead of changing 5 to $25^{\frac{1}{2}}$. Although, the student probably knew that there was a need to make both the numbers in the logarithms the same before evaluating, the rule was applied wrongly resulting in an incorrect answer. At least five students made this particular error. Overall, this type of error was more prevalent among the MG students (25.8%) and LG students (34.8) as compared to only 10.1% among the UG students.


3. **Find the value of**

(a) $\log_{25} 5$,

$\log_{25} 5$

$\log_{5^2} 5$

Answer (a) 2

Figure 1. Example of distorted theorem or definition error

The distorted theorem or definition error is illustrated in this interview excerpt of student H.

Question: How would you simplify $\dfrac{\log_y 16}{\log_y 4}$? Explain your answer.

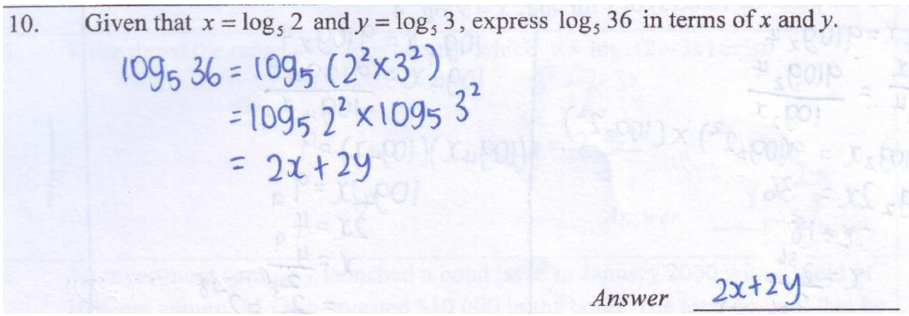
Response: Apply the law $\dfrac{\log_y 16}{\log_y 4} = \log_y(16-4) = \log_y 12$.

Question: Are you very sure?

Response: Yes, because divide means minus.

Technical Error

The technical error category, amongst others, includes computational errors, errors in manipulating algebraic symbols, errors in extracting data from tables, mistakes in executing algorithms and other careless errors (see Movshovitz-Hadar et al., 1987). The technical error is illustrated in Figure 2. This student was careless in using a multiplication sign rather than an addition sign. Although one may argue that this might be due to a misconception, we consider it a technical error based on his answers to other similar problems. It is interesting to note that this type of error was more common among the UG students (10.1%) as compared to 6.7% among the MG students and 4.5% among the LG students.


10. Given that $x = \log_5 2$ and $y = \log_5 3$, express $\log_5 36$ in terms of x and y.

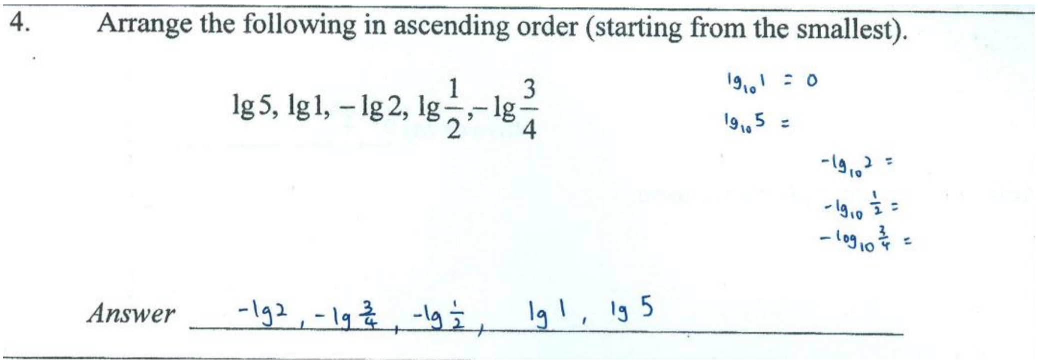
$\log_5 36 = \log_5 (2^2 \times 3^2)$
$= \log_5 2^2 \times \log_5 3^2$
$= 2x + 2y$

Answer $2x+2y$

Figure 2. Example of a technical error

Misused Data Error

The misused data category of errors is due to a discrepancy between the given data in the item and the way the student treated them (Movshovitz-Hadar et al., 1987). This error involved the students misusing the data given in the problem and was only identified in item 4 of the test instrument. About 15% of the students either ignored the negative sign or added a negative sign to the given data in item 4 that resulted in wrong answers. In this item (see Figure 3), it was expected that the students would be able to identify that $-\lg 2$ and $\lg\frac{1}{2}$ are the same numbers. Almost all students assumed that $-\lg\frac{3}{4}$ is a negative number which may probably be due to the preceding negative sign. This error was made by 6.7% of the UG students as compared to only 2.2% of the MG students and 4.5 % of the LG students.


4. Arrange the following in ascending order (starting from the smallest).

$\lg 5, \lg 1, -\lg 2, \lg\frac{1}{2}, -\lg\frac{3}{4}$

$\lg_{10} 1 = 0$
$\lg_{10} 5 =$
$-\lg_{10} 2 =$
$-\lg_{10} \frac{1}{2} =$
$-\log_{10} \frac{3}{4} =$

Answer $-\lg 2, -\lg\frac{3}{4}, -\lg\frac{1}{2}, \lg 1, \lg 5$

Figure 3. Example of misused data error

Unverified Solution Error

Errors in this category, apply when the steps of the solution are basically correct but the student did not check the answers (Movshovitz-Hadar et al., 1987). In this problem (see Figure 4), the student had to solve for x (see example in Figure 4). While the steps are correct, the student did not check the solutions and accepted both values of x. This type of error was more prevalent among the UG and MG students with 26.9% and 20.2% of students respectively making this type of error. Interestingly, only 7.9% of LG students made this type of error.

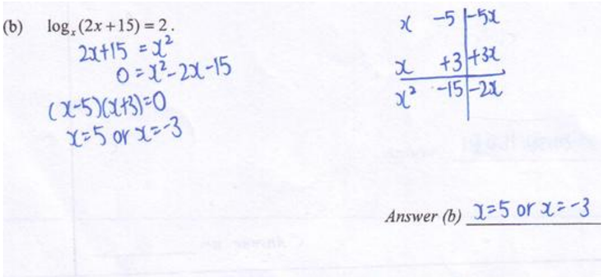


Figure 4. Example of unverified solution error

Discussion

Most of the errors in logarithms made by the students in this study seem to fit nicely into the four categories from Movshovitz et al. (1987) described above. However, finding the reasons behind the errors is not so straightforward. Several ideas about logarithms seem not very clear to students:

1. the definition - a logarithm is defined as an exponent and basically establishes the equivalence of $y = a^x$ and $x = \log_a y$ with the required conditions that $a > 0$, and $a \neq 1$. The restrictions on a were quite often overlooked (see Figure 4 above).
2. the connection to exponents – the presentation of the topic on logarithms in the classroom follows the presentation of the topic on exponents. Such a presentation of the topic on logarithms is at odds with how logarithms developed historically, without any emphasis on exponents (see Tabaghi, 2007). However, others such as Webb, Van der Kooij, and Geist (20011) have proposed that to conceptualise logarithms, students have to understand exponential growth.
3. the connection between additive and multiplicative structures - perhaps the single most important feature of logarithms is the connection that logarithms make between additive and multiplicative structures (see Berezvoski, 2004). A multiplication or division problem can be respectively reduced to an addition or subtraction problem, which is a very significant result in the absence of a calculating device such as a calculator, but not evident to the average student.
4. the symbolism - while logarithms should always be written indicating clearly the base, quite often the base is not included, for example for common logarithms the symbol $\log_{10} x$ is written as $\lg x$ and for natural logarithms the symbol for $\log_e x$ is written as $\ln x$. Students quite often cannot make sense of the symbols used (see Figure 5).

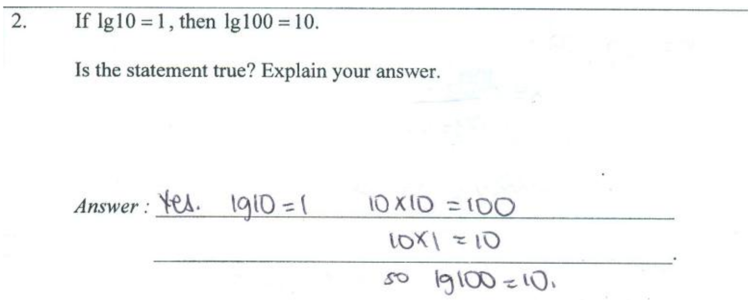


Figure 5. Error when using the symbol lg

There may be several reasons why students make particular types of errors, as highlighted by Radatz (1979):

> In classifying errors according to pupils' individual difficulties, one should, of course, acknowledge that errors are also a function of other variables in the educational process (the teacher, the curriculum, the environment, and possible interactions among these variables). Errors in the learning of mathematics are the result of very complex processes. A sharp separation of the possible causes of a given error is often quite difficult because there is such a close interaction among causes. (p. 164)

As such it is difficult to attribute errors to very specific causes. An overview of Table 1 shows that UG students were least likely of the three groups of students to make errors classified as *Distorted Theorem or Definition* but on the other hand were more likely to make errors in the other three categories. This is probably due to the fact that UG students rely less on instrumental understanding (see Skemp, 1987) compared to students from the other groups. On the other hand, a possible cause for the errors classified under distorted theorem or definition by MG and LG students may be due to students' lack of relational understanding of the laws of logarithms. Although instrumental understanding is necessary for students to apply algorithms and rules correctly, relational understanding will help them better in problem solving and longer retention of what is taught. This explains why LG students commit more of the errors in application questions as they might have only attained instrumental understanding of the laws (see interview of student J below). The higher prevalence among the UG students of *Technical Errors*, *Misused Data Errors* and *Unverified Solution Errors* may be attributed to some kind of overconfidence which leads to some carelessness and oversight.

The MG and UG revealed lower mastering of concepts and procedures. These students are probably not beyond the process stage based on APOS theory. Perhaps, students found the relationship between the multiplicative and additive structures within logarithms confusing. This confusion caused them to make errors such as $\lg a + \lg b = \lg a \times \lg b$ and $\lg a - \lg b = \dfrac{\lg a}{\lg b}$, which are essentially *Distorted Theorem or Definition* type of errors. The cause for this type of error may also be due to the ways in which students memorise these laws and rules. In this case, the students may just remember that "subtract means divide" and "add means multiply" and when the situation arises they apply the rules wrongly. This is highlighted in the response of student J, in an interview when asked to evaluate $\log_5 15 - \log_5 3$.

> Answer: Minus in logarithm is equivalent to division as in exponent; you subtract the power when you divide
>
> Question: Ok, so what is the answer?
>
> Answer: 1
>
> Question: Can you write down the steps for me?
>
> Answer: $\log_5 15 - \log_5 3 = \frac{\log_5 15}{\log_5 3} = \log_5 5 = 1$.

Student J gave the correct answer in the test but did not show any working and in the interview again gave the correct answer but gave an incorrect explanation. During the interview we found that the student vaguely remembered some sort of relationship between division and subtraction in logarithms. So the moment there was a need to apply the law, the student looked for the subtraction and addition operations and replaced them with division and multiplication operations. The interview also demonstrates that a correct response is no guarantee of a lack of misconception.

Conclusion

Errors can be used to create new learning opportunities for students if used appropriately in mathematics instruction (Borasi, 1987). Teachers can use the errors that have been identified in this study as a platform to change their instructions and adopt new teaching strategies so that students can avoid these errors. The errors will also allow teachers to understand how their students learn logarithms. Students must have a good grasp of exponents and the laws of indices. Introducing logarithms in two different representations namely symbolic and graphical will also allow students to have a better understanding of logarithms. It was not possible to describe all of the errors made by the students in this short paper or to comment on individual misconceptions in detail, but an attempt was made to document and classify the errors in logarithms which is generally considered as a hard topic of study by students.

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