Characterising the Cognitive Processes in Mathematical Investigation

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Abstract

Many educators believe that mathematical investigation is open and it involves both problem posing and problem solving, but some teachers have taught their students to investigate during problem solving. The confusion about the relationship between investigation and problem solving may affect how teachers teach their students and how researchers conduct their research. Moreover, there is a research gap in studying the thinking processes in mathematical investigation, partly because it is not easy to define these processes. Therefore, this article seeks to address these issues by first distinguishing between investigation as a task, a process and an activity; and then providing an alternative characterisation of the process of investigation in terms of its core cognitive processes: specialising, conjecturing, justifying and generalising. These will help to clarify the relationship between investigation and problem solving: an open investigative activity involves both problem posing and problem solving; but the problem-solving process entails solving by the process of investigation and/or by using 'other heuristics'. In other words, mathematical investigation does not have to be open. The article concludes with some implications of this alternative view of mathematical investigation on teaching and research.

1. INTRODUCTION

Many educators are of the opinion that mathematical investigation must be open. For example, Orton and Frobisher (1996) described investigation as an open problem which should not specify any goal in its task statement while Delaney (1996) believed in the more open spirit of the process-dominated investigation. Bailey (2007) defined investigation as "an open-ended problem or statement that lends itself to the possibility of multiple mathematical pathways being explored, leading to a variety of mathematical ideas and/or solutions" (p. 103). In other words, these authors believe that mathematical investigation is open with respect to its goal, processes and answer.

The term 'investigation' is also used to mean different things by different educators. For example, Orton and Frobisher (1996) used the term 'investigation' to refer to the task when they compared investigations with problems while Evans (1987) used the term 'investigation' to mean the process when he contrasted investigation with problem solving. Ernest (1991) observed that there had been a fairly widespread adoption of the term 'investigation' as the task itself when investigation is actually a process. This is what Jakobsen (1956, cited in Ernest, 1991) called a metonymic shift in meaning which replaces the whole activity by one of its components. Thus many educators seem not to distinguish between the investigative task and the process of investigation.

Although many educators agree that there are overlaps between investigation and problem solving, they usually end up separating them into two distinct activities: investigation must involve 'open investigative tasks' with an open goal (Orton & Frobisher, 1996) and an open answer (Pirie, 1987) and thus is divergent; while problem solving is restricted to 'closed problem-solving tasks' with a closed goal and a closed answer and thus is convergent (Evans, 1987; HMI, 1985). Others believe that investigation involves both problem posing and problem solving (Cai & Cifarelli, 2005), i.e., problem solving is a subset of investigation. However, Yeo and Yeap (2009) have observed some teachers telling their students to investigate when solving a 'closed problem-solving task'. If teachers are confused between the similarities and differences between investigation and problem solving, then they may not be able to teach their students effectively (Frobisher, 1994).

Similarly, this may also affect how researchers conduct their research if they are not able to define clearly what an investigation is. Many studies on mathematical investigation only reported its general benefits, such as the students becoming more interested (Davies, 1980) or more open to working mathematically (Tanner, 1989). Boaler (1998) went one step further to study the effectiveness of process-based teaching using open-ended investigative activities by looking at how the students fared in a new form of GCSE examination that rewarded problem solving (this examination was discontinued in 1994). But there are very few studies that examine the thinking processes when students engage in investigation, partly because it is hard to identify the processes that constitute mathematical investigation.

Therefore, it is the purpose of this article to define the cognitive processes of mathematical investigation so as to inform both teaching and research (it is beyond the scope of this article to examine the metacognitive processes). We will begin by separating investigation into task, process and activity; and then we will characterise the process of mathematical investigation in terms of its cognitive processes. A mathematical investigation model will be used to demonstrate the interaction among these cognitive processes during an open investigative activity. The relationship between investigation and problem solving will also be clarified. We will conclude with some implications of this alternative characterisation of mathematical investigation on teaching and research.

2. MATHEMATICAL INVESTIGATION: TASK, PROCESS AND ACTIVITY

We will begin by distinguishing between a task and an activity although these two terms are often treated as synonyms (Mason & Johnston-Wilder, 2006). A task refers to what the teacher sets while the activity refers to what the student does in response to the task (Christiansen & Walther, 1986). For example, Task 1 below is an *open* investigative task because the goal is open: students have the freedom to choose any goal to pursue (Orton & Frobisher, 1996). As there are many correct answers, the task is also said to have an open answer (Becker & Shimada, 1997).

Task 1: Powers of 3 (Open Investigative Task)
Powers of 3 are 3^1, 3^2, 3^3, 3^4, 3^5, … Investigate.

When students attempt an open investigative task, they are engaged in a mathematical activity which we will call an open investigative activity. The first thing students should do is to understand the task, which is similar to the first phase of Pólya's (1957) problem-solving model for closed problem-solving tasks: understand the problem. But unlike Pólya's model, students need to pose their own problems for this type of open investigative tasks. There are generally two approaches. Students may set a specific goal by posing a *specific* problem to solve (Cai & Cifarelli, 2005) but they may not have any idea what problems to pose. So they may just set a general goal by searching for any pattern (Height, 1989). The latter can be called the posing of the *general* problem "Is there any pattern?" Both approaches can be collectively called problem posing. Therefore, an open investigative activity involves both problem posing and problem solving (Cai & Cifarelli, 2005).

We would like to make a further distinction between investigation as an activity and investigation as a process. An analogy may be helpful. In Pólya's (1957) problem-solving model for closed problem-solving tasks, the first phase of understanding the problem is what a person should do before problem solving, while the *actual* problem-solving process begins in the second phase of devising a plan and continues into the third phase of carrying out the plan. The fourth phase of looking back is what the person should do after problem solving. But *all* the four phases are considered part of Pólya's problem-solving model. So there is a need to differentiate between the actual *process* of problem solving and the entire mathematical *activity* of problem solving.

Similarly, an open investigative activity includes what a person should do before investigation, the actual process of investigation, and what the person should do after investigation. It is clear that the first phase of understanding the task is what a person should do before investigation. Since the purpose of the second phase of problem posing is to pose problems to investigate, it seems evident that this is what the person should do *before* investigation. Then the third phase is the actual process of investigation. Therefore, problem posing is not part of the *process* of investigation although problem posing is an integral part of an open investigative activity. The decoupling of problem posing from the process of investigation is important since we would like to argue later that the process of investigation can occur during problem solving, because if investigation involves both problem posing and problem solving (i.e., problem solving is a subset of investigation), then it is not possible for investigation to occur during problem solving (i.e., investigation will now become a subset of problem solving). This is necessary for a proper understanding of the relationship between investigation and problem solving which we will discuss later in more details. From this point onwards, the term 'investigation' will be used to refer to the process unless otherwise stated.

3. MATHEMATICAL INVESTIGATION: COGNITIVE PROCESSES

There is a need to clarify the use of the word 'process'. From one angle, an investigation is one whole process, but from another perspective, there are many processes in an investigation (Frobisher, 1994). Shufelt (1983) has earlier observed the same thing about problem solving: although it is one whole process, it contains many processes. In this article, we will use both perspectives of the term 'process'. So, what types of processes does the process of mathematical investigation involve?

In Task 1, students may start by evaluating some powers of 3 and then comparing them to find out if there is any pattern. This involves examining specific examples, or sometimes special cases, in order to generalise. Mason, Burton and Stacey (1985) called these processes 'specialising' and 'generalising'. If a pattern is found, this is only a conjecture to be proven or refuted. If it is proven, the conjecture is said to be justified. Mason et al. (1985) called these processes 'conjecturing' and 'justifying'. From the viewpoint of the students, they would not know whether they are justifying or refuting a conjecture, and so this process will be called 'testing of conjecture' instead. But from the perspective of mathematical thinking, refuting a conjecture does not lead to any new discovery; rather, it is the justification of a conjecture that may lead to generalisation or mathematisation (Wheeler, 1982). That is why Mason et al. (1985) included justification, and not testing of conjectures, as a core thinking process.

Although Mason et al. have applied these processes to solving closed problem-solving tasks, we observe that these are also the core cognitive processes in mathematical investigation.

Further support for this alternative view of mathematical investigation as a process can be found among certain writers although they did not openly define investigation in this manner. For example, Jaworski (1994) agreed with the many educators whom she cited that investigation must be open, but in the latter part of her book, she described the teaching of the three teachers whom she observed as "investigative in spirit, embodying questioning and inquiry" (p. 96) and "making and justifying conjectures was common to all three classrooms, as was seeking generality through exploration of special cases and simplified situations" (p. 171). So Jaworski seemed to view an investigative approach to mathematics teaching as involving conjecturing, justifying, specialising and generalising although she did not define it explicitly.

Figure 1 shows a model of how these processes interact with one another during an open investigative activity. There are five phases: entry, goal setting (or problem posing), attack (or problem solving), review and extension. It is similar to the problem-solving model of Mason et al. (1985): entry, attack and review (which includes extension), except that our model involves an addition phase of goal setting because it is for an open investigative activity. The arrows show the *logical* progression from one process to another although students can skip any of these processes, or they can just jump from one process to another process when they change their mind. For example, when trying to test a conjecture, a student may decide to pose another problem, formulate another conjecture, or extend the original task, etc. Since it is possible to jump from one process to any other process in this manner, it may not be feasible or even helpful for the model to keep track of all these ad hoc behaviours.

At the entry phase, the students should understand the task. As explained in Section 2, there are generally two approaches to goal setting: students may pose a specific problem or they may decide to search for any pattern. The latter approach usually leads to specialising, formulating and testing of conjectures, and generalising, in the attack phase. This is the process of investigation. Some educators (e.g., Mason et al., 1985; Stylianides, 2008) believe that it is good enough to test a conjecture using the underlying pattern or any non-proof argument, while others (e.g., Holding, 1991; Tall, 1991) advocate the use of a more rigorous deductive reasoning or formal proof.

Let us now look at the other branch in the attack phase. When students pose a specific problem to solve, they may also begin by examining specific examples and going down the investigation route; or they can use 'other heuristics' which in this article refer to any heuristics that do not involve specialising. In general, problem-solving heuristics can be divided into two broad categories. The first category involves any form of specialisation. For example, if students draw a diagram or use systematic listing to examine specific cases, then this is specialising and the students are engaged in investigation. The second category does not involve any specialisation. For example, if students use deductive reasoning directly, then this is not an investigation. But what about establishing a subgoal as a heuristic to solve a problem? By itself, this is not an investigation. The question is what happens after establishing a subgoal. If the students use deductive reasoning to achieve the subgoal, then this is not an investigation. However, if the students use some form of specialisation in order to attain the subgoal (refer to the arrow from 'use other heuristics' to 'specialisation' in Figure 1), then this is investigation. Therefore, using heuristics to solve problems are similar to solving problems by investigation or by 'other means'.

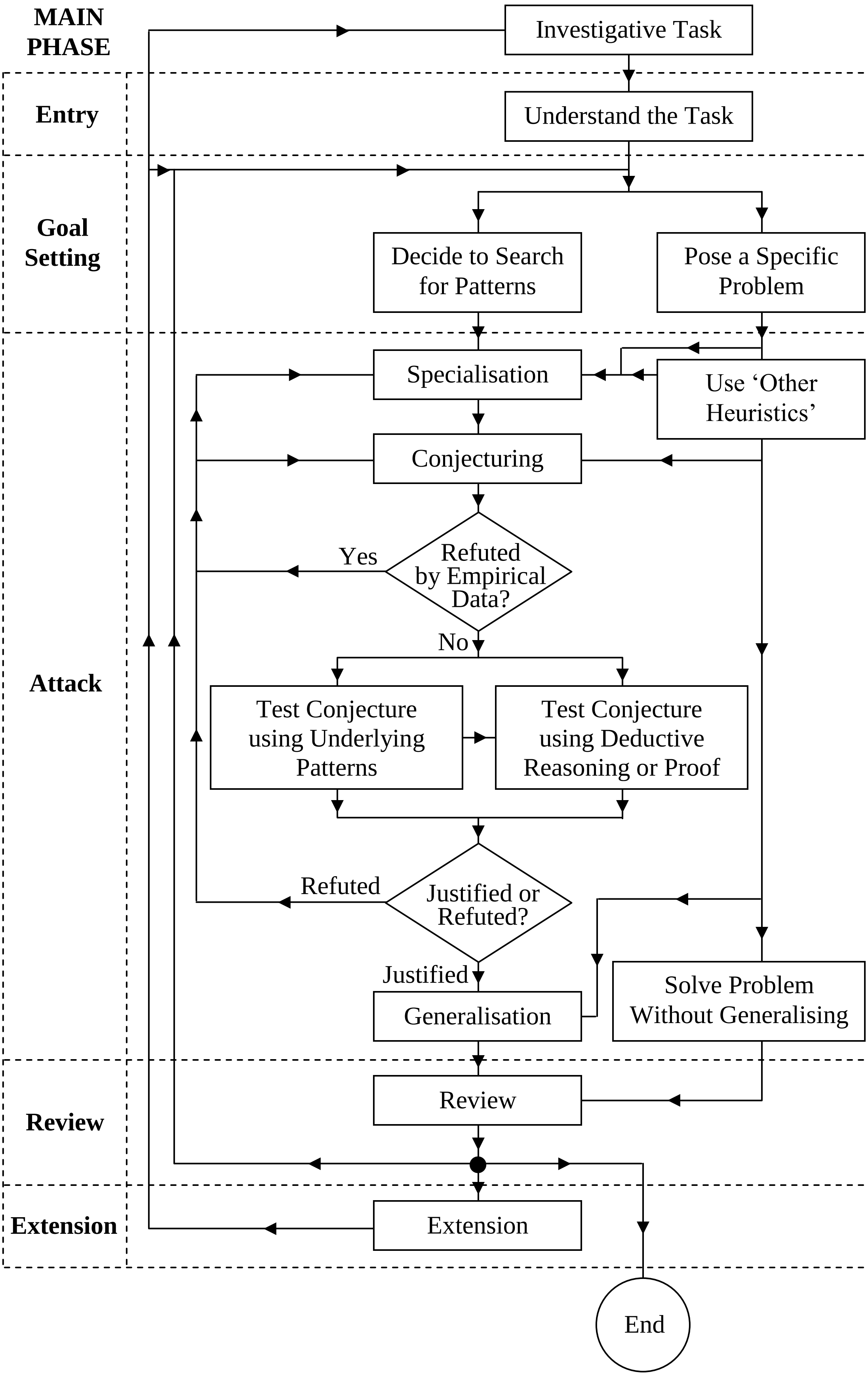

Figure 1. Model for Open Investigative Activity: Interaction of Cognitive Processes

When students use ‘other heuristics’ to solve the specific problem, it is still possible to formulate and test conjectures (refer to the corresponding arrow in Figure 1); or they can solve the problem with or without generalising. In other words, the students can generalise without specialising. It is also important to observe that the two branches in the attack phase are not mutually exclusive: it is possible to go from one branch to the other, and vice versa. Moreover, deductive reasoning is not restricted to the branch containing the use of ‘other heuristics’ since it can also occur when testing conjectures in the investigation route.

The fourth phase is the review phase. Although review may also include extension, we have decided to separate the two phases of review and extension because it is possible to pose more problems to solve, or to end the investigative activity, without extending the task (refer to the corresponding arrows in Figure 1). Most models for open investigation either show a linear pathway (e.g., Frobisher, 1994), or a cycle (e.g., Lakatos, 1976), but a model for investigation should include both (e.g., Height, 1989): although investigation is cyclic, it has to stop somewhere.

If you look closely at the model in Figure 1, you may observe that the attack phase is actually a combination of the second and third phases of Pólya’s (1957) problem-solving model: devise a plan and carry out the plan, i.e., the attack phase is simply the process of problem solving. It seems natural that after the problem-posing phase, the next phase should be called the problem-solving phase (e.g., see Cifarelli & Cai, 2004). But what is problem solving doing inside a model for an open investigative activity? Orton and Frobisher (1996) asked rhetorically, “Do you ask your students to solve an investigation? When was the last time you asked your students to explore [or investigate] a problem?” (p. 32) We will answer these questions in the next section when we examine how investigation is related to problem solving.

4. RELATIONSHIP BETWEEN INVESTIGATION AND PROBLEM SOLING

In Section 2, we have decoupled problem posing from the process of investigation. In other words, an open investigative activity involves both problem posing and problem solving, i.e., problem posing is a subset of an open investigative activity and not a subset of the process of investigation. In Section 3, we have seen how the attack phase in the model for an open investigative activity is actually the problem-solving phase, and how problem-solving heuristics can be classified into two broad categories: those that involve some form of specialisation and thus the process of investigation; and those that do not involve specialising and investigating. In other words, problem solving involves either the process of investigation and/or the process of solving by using ‘other heuristics’, i.e., the process of investigation is a subset of problem solving.

We can now answer the rhetorical questions posed by Orton and Frobisher (1996) in Section 3. “Do you ask your students to solve an investigation?” Yes, if this means solving a specific problem posed during an open investigative activity. “When was the last time you asked your students to explore [or investigate] a problem?” Yes, it is possible to investigate a problem if this means that the students pose a specific problem and then try to solve it by using the process of investigation.

This discussion will not be complete unless we look at closed problem-solving tasks. Similar to the distinction between the task, process and activity in mathematical investigation, we are going to differentiate between the task, process and activity of problem solving. An example of a closed problem-solving task is:

> **Task 2: Handshakes (Problem-Solving Task)**
> At a workshop, each of the 100 participants shakes hand once with each of the other participants. Find the total number of handshakes.

This type of tasks is closed in its goal and in its answer although it can be extended and thus opened up (Frobisher, 1994). For students who know how to solve this immediately, this is not a problem to them (Lester, 1980). For those who are stuck, this is a problem to them and since this is a problem-solving task, the students will be engaged in a problem-solving *activity*, i.e., we distinguish between the task and the activity (Christiansen & Walther, 1986). As explained in Section 2, in Pólya's (1957) problem-solving model for a problem-solving activity, the actual *process* of problem solving occurs in the second and third phases, thus implying that there is a difference between the whole problem-solving activity and the actual problem-solving process.

There are generally two approaches to the problem-solving process. Some students may specialise by examining a smaller number of participants, with or without drawing a diagram, to see if there is any pattern. They may formulate and test conjectures in order to generalise to 100 participants. This is the process of investigation. Other students may use 'other heuristics' such as a deductive argument: the first participant shakes hand with the other 99 participants, the second participant shakes hand with the remaining 98 participants, etc., and so the total number of handshakes is 99 + 98 + 97 + … + 1. Therefore, it is possible for the process of investigation to occur within the process of problem solving, i.e., the investigation process is a subset of the problem-solving process.

Bloom, Engelhart, Furst, Hill and Krathwohl (1956) classified the "ability to integrate the results of an investigation into an effective plan or solution to solve a problem" in the *synthesis* class in Bloom's taxonomy of educational objectives in the cognitive domain, thus implying that investigation is a method to solve mathematical problems. The *Curriculum and Evaluation Standards for School Mathematics* states that "our ideas about problem situations and learning are reflected in the verbs we use to describe student actions (e.g. to investigate, to formulate, to find, to verify) throughout the Standards" (NCTM, 1989, p. 10). Therefore, the Standards also recognise investigation as a means of dealing with problem situations.

Figure 2 shows a model of how these processes interact with one another during a problem-solving activity. It is essentially the same as the model for an open investigative activity except that there is no goal setting or problem posing phase. Notice that the attack phase (which is essentially the process of problem solving) involves solving the problem by investigation and/or by using 'other heuristics'. Another difference is after the review phase: students can choose to end the problem-solving activity or to extend the problem; they cannot pose more problems to solve without extending the problem-solving task, unlike an open investigative activity where they can pose more problems to solve without extending the open investigative task.

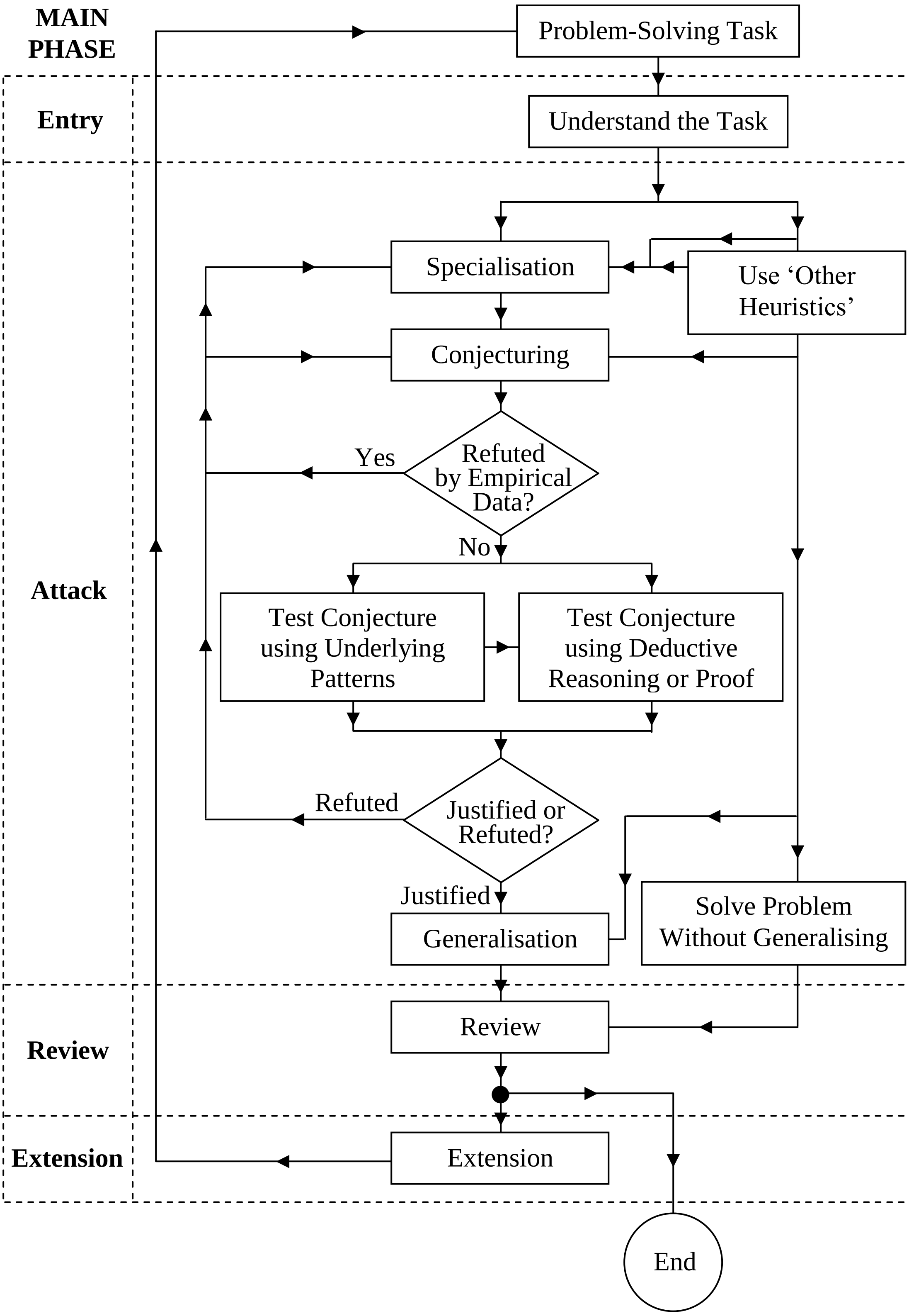

Figure 2. Model for Problem-Solving Activity: Interaction of Cognitive Processes

5. CONCLUSION AND IMPLICATIONS

A lot of confusion can arise because we often use the term ‘mathematical investigation’ to mean different things. This article recommends distinguishing between open investigative tasks, investigation as a process, and investigation as an activity involving open investigative tasks. With such a differentiation, the relationship between problem solving and investigation becomes clearer: the process of problem solving involves solving by using the process of investigation and/or solving by using ‘other heuristics’, while an open investigative activity includes both problem posing and problem solving as a process. Thus investigation should not be restricted to open investigative tasks only, but it can also occur in closed problem-solving tasks because investigation is *primarily* a process (Ernest, 1991) involving specialising, conjecturing, justifying and generalising. Hence, the characterisation of mathematical investigation does not lie in the open goal of the investigative task itself, but in what it entails, i.e., the four core cognitive processes.

The first implication is that the characterisation of the process of mathematical investigation, and the clarification of the relationship between investigation and problem solving, may help to inform teachers on how and what they teach their students. If teachers are confused what mathematical investigation involves, then they may not teach their students effectively (Frobisher, 1994). Knowing that investigation has nothing to do with the openness of open investigative tasks, teachers can now use closed problem-solving tasks and focus on developing the cognitive processes of mathematical investigation instead of having to sidetrack into teaching problem posing in open investigative activities. Teachers can also explain more clearly and confidently to their students what it means to investigate a problem, or to engage in problem solving during an open investigative activity. This, in turn, may help students to learn more effectively if they are not so confused (Orton & Frobisher, 1996).

The second implication is to inform research. As explained in Section 1, there is a research gap in studying the thinking processes of students when they engage in mathematical investigation, partly due to the difficulty in defining clearly the cognitive processes in an investigation, and especially when the investigation process is usually confused with the openness of open investigative activities. The characterisation of mathematical investigation as a process involving the four core thinking processes, and the development of a model for an open investigative activity showing the interaction of these processes, may help researchers to study these processes more effectively. Their research, in turn, may help to inform and refine the model. A proper understanding of how these processes interact with one another during an investigation may be useful to help teachers develop these processes in their students.

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