Hindawi Publishing Corporation
The Scientific World Journal
Volume 2014, Article ID 607347, 7 pages
http://dx.doi.org/10.1155/2014/607347

Research Article

On Hardy-Pachpatte-Copson's Inequalities

Chang-Jian Zhao[1] and Wing-Sum Cheung[2]

[1] *Department of Mathematics, China Jiliang University, Hangzhou 310018, China*
[2] *Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong*

Correspondence should be addressed to Chang-Jian Zhao; chjzhao@aliyun.com

Received 6 December 2013; Accepted 19 January 2014; Published 18 March 2014

Academic Editors: B. Dragovich and D. Xu

We establish new inequalities similar to Hardy-Pachpatte-Copson's type inequalities. These results in special cases yield some of the recent results.

1. Introduction

The classical Hardy's integral inequality is as follows.

Theorem A. *If $p > 1$, $f(x) \geq 0$ for $0 < x < \infty$, and $F(x) = (1/x)\int_0^x f(t)dt$, then*

$$\int_0^\infty F(x)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx, \tag{1}$$

unless $f \equiv 0$. The constant is the best possible.

Theorem A was first proved by Hardy [1], in an attempt to give a simple proof of Hilbert's double series theorem (see [2]). One of the best known and interesting generalization of the inequality (1) given by Hardy [3] himself can be stated as follows.

Theorem B. *If $p > 1$, $m \neq 1$, $f(x) \geq 0$ for $0 < x < \infty$, and $F(x)$ is defined by*

$$\begin{aligned} F(x) &= \int_0^x f(t)\,dt, \quad m > 1; \\ F(x) &= \int_x^\infty f(t)\,dt, \quad m < 1, \end{aligned} \tag{2}$$

then

$$\int_0^\infty x^{-m} F(x)^p dx < \left(\frac{p}{|m-1|}\right)^p \int_0^\infty x^{p-m} f(x)^p dx, \tag{3}$$

unless $f \equiv 0$. The constant is the best possible.

Inequalities (1) and (3) which later went by the name of Hardy's inequalities led to a great many papers dealing with alternative proofs, various generalizations, and numerous variants and applications in analysis (see [4–15]). In particular, Pachpatte [4] established some generalizations of Hardy inequalities (1) and (3). Very recently, Leng and Feng [16] proved some new Hardy-type integral inequalities. In the present paper we establish new inequalities similar to Hardy's integral inequalities (1) and (3). These results provide some new estimates to these types of inequalities and in special cases yield some of the recent results.

2. Main Results

Our main results are given in the following theorems.

Theorem 1. *Let $a < b < R$, $c < d < R'$, $p > 1$, $q < 1$, and $\alpha > 0$ be constants. Let $w(x, y)$ be positive and locally absolutely continuous in $(a, b) \times (c, d)$. Let $h(x, y)$ be a positive continuous function and let $H(x, y) = \int_a^x \int_c^y h(s, t)ds\,dt$,* *for* $(x, y) \in (a,b) \times (c,d)$*. Let* $f(x, y)$ *be nonnegative and measurable on* $(a,b) \times (c,d)$*. If*

$$D(x,y) = 1 - \frac{1}{1-q}\frac{H(x,y)}{h(x,y)}\frac{1}{w(x,y)}\frac{\partial w(x,y)}{\partial x}\log\left(\frac{H(R,R')}{H(x,y)}\right) + \frac{p}{1-q}\frac{H(x,y)}{h(x,y)}\times\frac{1}{r(x,y)}\frac{\partial r(x,y)}{\partial x}\log\left(\frac{H(R,R')}{H(x,y)}\right) \geq \frac{1}{\alpha}, \tag{4}$$

for almost all $(x, y) \in (a,b) \times (c,d)$*, and if* $F(x, y)$ *is defined by*

$$F(x,y) = \frac{1}{r(x,y)}\int_a^x\int_c^y r(s,t)\,h(s,t)\,f(s,t)\,ds\,dt \tag{5}$$

for $(x, y) \in (a,b) \times (c,d)$*, then*

$$\int_c^d\int_a^b w(x,y)\,H(x,y)^{-1}\overline{h}(x,y) \times\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{-q}F(x,y)^p dx\,dy \leq \left[\alpha\left(\frac{p}{1-q}\right)\right]^p \times\int_c^d\int_a^b\left[\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{p-q} w(x,y)\,H(x,y)^{p-1} \times h(x,y)^{1-p}G(x,y)^p\right]dx\,dy, \tag{6}$$

where

$$\overline{h}(x,y) = \int_c^y h(x,t)\,dt, \qquad G(x,y) = \frac{1}{r(x,y)}\int_c^y r(x,t)\,h(x,t)\,f(x,t)\,dt. \tag{7}$$

Remark 2. Let $f(x, y), w(x, y), h(x, y)$, and $r(x, y)$ reduce to $f(x)$, $w(x)$, $h(x)$, and $r(x)$, respectively, and with suitable modifications in Theorem 1, (6) changes to the following result:

$$\int_a^b w(x)\,H(x)^{-1}h(x)\left(\log\left(\frac{H(R)}{H(x)}\right)\right)^{-q}F(x)^p dx \leq \left[\alpha\left(\frac{p}{1-q}\right)\right]^p \times\int_a^b\left[\left(\log\left(\frac{H(R)}{H(x)}\right)\right)^{p-q} \times w(x)\,H(x)^{p-1}h(x)\,f(x)^p\right]dx. \tag{8}$$

This is just a new inequality established by Pachpatte [4].

Moreover, we note that the inequality established in Theorem 1 is the further generalizations of the inequality established by Copson [17].

Taking for $w(x) = r(x) = 1$, $H(R) = R$, and $\alpha = 1$ in (8), (8) changes to the following result:

$$\int_a^b H(x)^{-1}h(x)\left(\log\left(\frac{R}{H(x)}\right)\right)^{-q}F(x)^p dx \leq \left(\frac{p}{1-q}\right)^p \times\int_a^b\left[\left(\log\left(\frac{R}{H(x)}\right)\right)^{p-q}H(x)^{p-1}h(x)\,f(x)^p\right]dx. \tag{9}$$

This is just a new inequality established by Love [7].

Let $h(x) = 1$, $a \to 0$, $b \to \infty$, and $\log(R/(x-a)) = 1$ in (9); then (9) changes to the following result:

$$\int_0^\infty x^{-1}F(x)^p dx \leq \left(\frac{p}{1-q}\right)^p\int_0^\infty x^{p-1}f(x)^p dx. \tag{10}$$

This result is obtained in (3) stated in the Introduction.

Theorem 3. *Let* $a < b < R, c < d < R', p > 1, q > 1$*, and* $\beta > 0$ *be constants. Let* $w(x, y)$ *be positive and locally absolutely continuous in* $(a,b)\times(c,d)$*. Let* $h(x, y)$ *be a positive continuous function and let* $H(x,y) = \int_a^x\int_c^y h(s,t)ds\,dt$*, for* $(x, y) \in (a,b) \times (c,d)$*. Let* $f(x, y)$ *be nonnegative and measurable on* $(a,b) \times (c,d)$*. Let*

$$E(x,y) = 1 - \frac{1}{q-1}\frac{H(x,y)}{h(x,y)}\frac{1}{w(x,y)}\frac{\partial w(x,y)}{\partial y}\log\left(\frac{H(R,R')}{H(x,y)}\right) + \frac{p}{q-1}\frac{H(x,y)}{h(x,y)}\times\frac{1}{r(x,y)}\frac{\partial r(x,y)}{\partial y}\log\left(\frac{H(R,R')}{H(x,y)}\right) \geq \frac{1}{\beta}, \tag{11}$$

for almost all $(x, y) \in (a, b) \times (c, d)$. *If* $F(x, y)$ *is defined by*

$$F(x, y) = \frac{1}{r(x, y)} \int_x^b \int_y^d r(s, t) h(s, t) f(s, t) \, ds \, dt \quad (12)$$

for $(x, y) \in (a, b) \times (c, d)$, *then*

$$\begin{aligned} &\int_c^d \int_a^b w(x, y) H(x, y)^{-1} \tilde{h}(x, y) \\ &\quad \times \left(\log \left(\frac{H(R, R')}{H(x, y)} \right) \right)^{-q} F(x, y)^p dx \, dy \\ &\leq \left[\beta \left(\frac{p}{q-1} \right) \right]^p \\ &\quad \times \int_c^d \int_a^b \left[\left(\log \left(\frac{H(R, R')}{H(x, y)} \right) \right)^{p-q} w(x, y) H(x, y)^{p-1} \right. \\ &\quad \left. \times h(x, y)^{1-p} L(x, y)^p \right] dx \, dy, \end{aligned} \quad (13)$$

where

$$\tilde{h}(x, y) = \int_a^x h(s, y) \, ds, \quad (14)$$

Remark 4. Let $f(x, y)$, $w(x, y)$, $h(x, y)$, and $r(x, y)$ reduce to $f(x)$, $w(x)$, $h(x)$, and $r(x)$, respectively, and with suitable modifications in Theorem 3, (13) changes to the following result:

$$\begin{aligned} &\int_a^b w(x) H(x)^{-1} h(x) \left(\log \left(\frac{H(R)}{H(x)} \right) \right)^{-q} F(x)^p dx \\ &\quad \leq \left[\beta \left(\frac{p}{q-1} \right) \right]^p \int_a^b \left[\left(\log \left(\frac{H(R)}{H(x)} \right) \right)^{p-q} \right. \\ &\quad \left. \times w(x) H(x)^{p-1} h(x) f(x)^p \right] dx. \end{aligned} \quad (15)$$

This is just a new inequality established by Pachpatte [4].

On the other hand, we note that the inequality established in Theorem 3 is the further generalizations of the inequality established by Copson [17].

Taking for $w(x) = r(x) = 1$, $H(R) = R$, and $\beta = 1$ in (15), (15) changes to the following result:

$$\begin{aligned} &\int_a^b H(x)^{-1} h(x) \left(\log \left(\frac{R}{H(x)} \right) \right)^{-q} F(x)^p dx \\ &\quad \leq \left(\frac{p}{q-1} \right)^p \int_a^b \left[\left(\log \left(\frac{R}{H(x)} \right) \right)^{p-q} \right. \\ &\quad \left. \times H(x)^{p-1} h(x) f(x)^p \right] dx. \end{aligned} \quad (16)$$

This is just a new inequality established by Love [7].

3. Proof of Theorems

Proof of Theorem 1. If we let $u(x, y) = w(x, y) F(x, y)^p$ and in view of

$$F(x, y) = \frac{1}{r(x, y)} \int_a^x \int_c^y r(s, t) h(s, t) f(s, t) \, ds \, dt \quad (17)$$

for $(x, y) \in (a, b) \times (c, d)$, then

$$\begin{aligned} &\frac{\partial u(x, y)}{\partial x} \\ &\quad = \frac{\partial w(x, y)}{\partial x} F(x, y)^p + w(x, y) p F(x, y)^{p-1} \\ &\quad \times \left(\frac{1}{r(x, y)} \int_c^y r(x, t) h(x, t) f(x, t) \, dt \right. \\ &\quad - \frac{\partial r(x, y) / \partial x}{r^2(x, y)} \\ &\quad \left. \times \int_a^x \int_c^y r(s, t) h(s, t) f(s, t) \, ds \, dt \right). \end{aligned} \quad (18)$$

Let

$$\frac{\partial v(x, y)}{\partial x} = H(x, y)^{-1} \overline{h}(x, y) \left[\log \left(\frac{H(R, R')}{H(x, y)} \right) \right]^{-q}, \quad (19)$$

where $\overline{h}(x, y) = \int_c^y h(x, t) dt$ and in view of $H(x, y) = \int_a^x \int_c^y h(s, t) ds \, dt$, for $(x, y) \in (a, b) \times (c, d)$, then

$$v(x, y) = -\frac{\left[\log \left(H(R, R') / H(x, y) \right) \right]^{-q+1}}{(-q+1)}. \quad (20)$$

From (18), (20), and integrating by parts for x, we have

$$\begin{aligned}
&\int_c^d \int_a^b w(x,y) H(x,y)^{-1} \overline{h}(x,y) \\
&\quad \times \left(\log\left(\frac{H(R,R')}{H(x,y)} \right) \right)^{-q} F(x,y)^p \,dx\,dy \\
&= -\int_c^d \Bigg\{ w(x,y) F(x,y)^p \\
&\quad \times \left. \frac{\left[\log\left(H(R,R')/H(x,y)\right)\right]^{-q+1}}{-q+1} \right|_{x=a}^{x=b} \\
&\quad - \int_a^b \frac{\left[\log\left(H(R,R')/H(x,y)\right)\right]^{-q+1}}{-q+1} \\
&\quad \times \Bigg[\frac{\partial w(x,y)}{\partial x} F(x,y)^p \\
&\quad + w(x,y)\, pF(x,y)^{p-1} \\
&\quad \times \Bigg(G(x,y) - \frac{\partial r(x,y)/\partial x}{r^2(x,y)} \\
&\quad \times \int_a^x \int_c^y r(s,t) h(s,t) \\
&\quad \times f(s,t) \,ds\,dt \Bigg) \Bigg] dx \Bigg\} dy,
\end{aligned} \tag{21}$$

where

$$G(x,y) = \frac{1}{r(x,y)} \int_c^y r(x,t) h(x,t) f(x,t) \,dt. \tag{22}$$

If $q < 1$, then we observe that

$$\begin{aligned}
&\int_c^d \int_a^b D(x,y) w(x,y) H(x,y)^{-1} \overline{h}(x,y) \\
&\quad \times \left(\log\left(\frac{H(R,R')}{H(x,y)} \right) \right)^{-q} F(x,y)^p \,dx\,dy \\
&\le \frac{p}{1-q} \int_c^d \int_a^b \Bigg[w(x,y) \left(\log\left(\frac{H(R,R')}{H(x,y)} \right) \right)^{-q+1} \\
&\quad \times G(x,y) F(x,y)^{p-1} \Bigg] dx\,dy \\
&= \frac{p}{1-q} \\
&\quad \times \int_c^d \int_a^b \Bigg[\left\{ \left(\log\left(\frac{H(R,R')}{H(x,y)} \right) \right)^{-q} \right\}^{1/p} \\
&\quad \times \log\left(\frac{H(R,R')}{H(x,y)} \right) w(x,y)^{1/p} \\
&\quad \times H(x,y)^{(p-1)/p} \\
&\quad \times h(x,y)^{-(p-1)/p} G(x,y) \Bigg] \\
&\quad \times \Bigg[\left\{ \left(\log\left(\frac{H(R,R')}{H(x,y)} \right) \right)^{-q} \right\}^{(p-1)/p} \\
&\quad \times w(x,y)^{(p-1)/p} \\
&\quad \times \left[H(x,y)^{-1} \times h(x,y) \right]^{(p-1)/p} \\
&\quad \times F(x,y)^{p-1} \Bigg] dx\,dy.
\end{aligned} \tag{23}$$

By applying Hölder's inequality with indices p, $p/(p-1)$ on the right side of (23), we obtain

$$\begin{aligned}
&\int_c^d \int_a^b w(x,y) H(x,y)^{-1} \overline{h}(x,y) \\
&\quad \times \left(\log\left(\frac{H(R,R')}{H(x,y)} \right) \right)^{-q} F(x,y)^p \,dx\,dy \\
&\le \alpha \left(\frac{p}{1-q} \right) \\
&\quad \times \Bigg\{ \int_c^d \int_a^b \Bigg[\left(\log\left(\frac{H(R,R')}{H(x,y)} \right) \right)^{-q} \\
&\quad \times \left(\log\left(\frac{H(R,R')}{H(x,y)} \right) \right)^{p} \\
&\quad \times w(x,y) H(x,y)^{p-1} \\
&\quad \times h(x,y)^{-(p-1)} G(x,y)^p \Bigg] dx\,dy \Bigg\}^{1/p}
\end{aligned}$$

$$\times\left\{\int_c^d\int_a^b\left[\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{-q}w(x,y)H(x,y)^{-1}\times h(x,y)F(x,y)^p\right]dx\,dy\right\}^{(p-1)/p}. \tag{24}$$

Dividing both sides of (24) by the second integral factor on the right side of (24) and raising both sides to the pth power, we obtain

$$\begin{aligned}\int_c^d\int_a^b & w(x,y)H(x,y)^{-1}\tilde{h}(x,y)\times\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{-q}F(x,y)^p dx\,dy\\ &\le\left[\alpha\left(\frac{p}{1-q}\right)\right]^p\\ &\times\int_c^d\int_a^b\left[\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{p-q}\times w(x,y)H(x,y)^{p-1}\times h(x,y)^{-(p-1)}G(x,y)^p\right]dx\,dy.\end{aligned} \tag{25}$$

□

Proof of Theorem 3. If we let $u(x,y)=w(x,y)F(x,y)^p$ and in view of

$$F(x,y)=\frac{1}{r(x,y)}\int_x^b\int_y^d r(s,t)h(s,t)f(s,t)\,ds\,dt \tag{26}$$

for $(x,y)\in(a,b)\times(c,d)$, then

$$\begin{aligned}\frac{\partial u(x,y)}{\partial y} &=\frac{\partial w(x,y)}{\partial y}F(x,y)^p+w(x,y)\,pF(x,y)^{p-1}\\ &\times\left(-\frac{1}{r(x,y)}\int_y^d r(x,t)h(x,t)f(x,t)\,dt-\frac{\partial r(x,y)/\partial y}{r^2(x,y)}\int_x^b\int_y^d r(s,t)h(s,t)f(s,t)\,ds\,dt\right).\end{aligned} \tag{27}$$

Let

$$\frac{\partial v(x,y)}{\partial y}=H(x,y)^{-1}\tilde{h}(x,y)\left[\log\left(\frac{H(R,R')}{H(x,y)}\right)\right]^{-q}, \tag{28}$$

where $\tilde{h}(x,y)=\int_a^x h(s,y)ds$ and in view of $H(x,y)=\int_a^x\int_c^y h(s,t)ds\,dt$, for $(x,y)\in(a,b)\times(c,d)$, then

$$v(x,y)=-\frac{\left[\log\left(H(R,R')/H(x,y)\right)\right]^{-q+1}}{(-q+1)}. \tag{29}$$

From (27), (29), and integrating by parts for y, we have

$$\begin{aligned}\int_c^d\int_a^b & w(x,y)H(x,y)^{-1}\tilde{h}(x,y)\times\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{-q}F(x,y)^p dx\,dy\\ &=-\int_a^b\Bigg\{w(x,y)F(x,y)^p\times\frac{\left[\log\left(H(R,R')/H(x,y)\right)\right]^{-q+1}}{-q+1}\Bigg|_{y=c}^{y=d}\\ &\quad-\int_c^d\frac{\left[\log\left(H(R,R')/H(x,y)\right)\right]^{-q+1}}{-q+1}\times\Bigg[\frac{\partial w(x,y)}{\partial y}F(x,y)^p+w(x,y)\,pF(x,y)^{p-1}\\ &\quad\times\left(L(x,y)-\frac{\partial r(x,y)\,\partial y}{r^2(x,y)}\times\int_x^b\int_y^d r(s,t)h(s,t)\times f(s,t)ds\,dt\right)\Bigg]dy\Bigg\}dx,\end{aligned} \tag{30}$$

where

$$L(x,y)=-\frac{1}{r(x,y)}\int_x^b r(s,y)h(s,y)f(s,y)\,ds. \tag{31}$$

If $q>1$, then we observe that

$$\int_c^d\int_a^b E(x,y)w(x,y)H(x,y)^{-1}\tilde{h}(x,y)\times\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{-q}F(x,y)^p dx\,dy$$

$$\leq \frac{p}{q-1}\int_c^d\int_a^b\left[w(x,y)\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{-q+1}\times L(x,y)F(x,y)^{p-1}\right]dx\,dy$$

$$= \frac{p}{q-1}\times\int_c^d\int_a^b\left[\left\{\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{-q}\right\}^{1/p}\times\log\left(\frac{H(R,R')}{H(x,y)}\right)w(x,y)^{1/p}H(x,y)^{(p-1)/p}\times h(x,y)^{-(p-1)/p}L(x,y)\right]$$

$$\times\left[\left\{\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{-q}\right\}^{(p-1)/p}\times w(x,y)^{(p-1)/p}\times\left[H(x,y)^{-1}\times h(x,y)\right]^{(p-1)/p}\times F(x,y)^{p-1}\right]dx\,dy. \tag{32}$$

By applying Hölder's inequality with indices p, $p/(p-1)$ on the right side of (32), we obtain

$$\int_c^d\int_a^b w(x,y)H(x,y)^{-1}\tilde{h}(x,y)\times\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{-q}F(x,y)^p dx\,dy$$

$$\leq\beta\left(\frac{p}{q-1}\right)\times\left\{\int_c^d\int_a^b\left[\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{-q}\times\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{p}w(x,y)H(x,y)^{p-1}\times h(x,y)^{-(p-1)}L(x,y)^p\right]dx\,dy\right\}^{1/p}$$

$$\times\left\{\int_c^d\int_a^b\left[\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{-q}\times w(x,y)H(x,y)^{-1}\times h(x,y)F(x,y)^p\right]dx\,dy\right\}^{(p-1)/p}. \tag{33}$$

Dividing both sides of (33) by the second integral factor on the right side of (33) and raising both sides to the pth power, we obtain

$$\int_c^d\int_a^b w(x,y)H(x,y)^{-1}\tilde{h}(x,y)\times\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{-q}\times F(x,y)^p dx\,dy$$

$$\leq\left[\beta\left(\frac{p}{q-1}\right)\right]^p\times\int_c^d\int_a^b\left[\left(\log\left(\frac{H(R,R')}{H(x,y)}\right)\right)^{p-q}w(x,y)H(x,y)^{p-1}\times h(x,y)^{-(p-1)}L(x,y)^p\right]dx\,dy. \tag{34}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

Chang-Jian Zhao's research is supported by National Natural Science Foundation of China (11371334). Wing-sum Cheung's research is partially supported by HKU URG grant.

References

[1] G. H. Hardy, "Note on a theorem of Hilbert," *Mathematische Zeitschrift*, vol. 6, no. 3-4, pp. 314–317, 1920.

[2] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, UK, 1934.

[3] G. H. Hardy, "Notes on some points in the integral calculus," *Messenger of Mathematics*, vol. 57, pp. 12–16, 1928.

[4] B. G. Pachpatte, "On some generalizations of Hardy's integral inequality," *Journal of Mathematical Analysis and Applications*, vol. 234, no. 1, pp. 15–30, 1999.

[5] B. G. Pachpatte, "On Hardy type integral inequalities for functions of two variables," *Demonstratio Mathematica*, vol. 28, no. 2, pp. 239–244, 1995.

[6] J. E. Pecaric and E. R. Love, "Still more generalizations of Hardy's inequality," *Journal of the Australian Mathematical Society*, vol. 58, pp. 1–11, 1995.

[7] E. R. Love, "Generalizations of Hardys inequality," *Proceedings of the Royal Society of Edinburgh*, vol. 100, pp. 237–262, 1985.

[8] B. C. Yang, I. Brnetić, M. Krnić, and J. Peĉarić, "Generalization of Hilbert and Hardy-Hilbert integral inequalities," *Mathematical Inequalities & Applications*, vol. 8, pp. 259–272, 2005.

[9] B. Yang and L. Debnath, "On the extended Hardy-Hilbert's inequality," *Journal of Mathematical Analysis and Applications*, vol. 272, no. 1, pp. 187–199, 2002.

[10] K. Jichang and L. Debnath, "On new generalizations of Hilbert's inequality and their applications," *Journal of Mathematical Analysis and Applications*, vol. 245, no. 1, pp. 248–265, 2000.

[11] M. Z. Sarikaya and H. Yildirim, "Some Hardy type integral inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 5, article 178, 2006.

[12] C. J. Zhao and L. Debnath, "Some new inverse type Hilbert integral inequalities," *Journal of Mathematical Analysis and Applications*, vol. 262, pp. 411–418, 2001.

[13] V. M. Miklyukov and M. K. Vuorinen, "Hardy's inequality for $w_0^{1,p}$-functions on riemannian manifolds," *Proceedings of the American Mathematical Society*, vol. 127, no. 9, pp. 2745–2754, 1999.

[14] B. Opic and A. Kufner, *Hardy-Type Inequalities*, Longman, Essex, UK, 1990.

[15] A. Kufner and L. E. Persson, *Weighted Inequalities of Hardy Type*, World Scientific, 2003.

[16] T. Leng and Y. Feng, "On Hardy-type integral inequalities," *Applied Mathematics and Mechanics*, vol. 34, no. 10, pp. 1297–1304, 2013.

[17] E. T. Copson, "Some integral inequalities," *Proceedings of the Royal Society of Edinburgh*, vol. 75, pp. 157–164, 1976.