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Concrete-Pictorial-Abstract:

Surveying its Origins and Charting its Future

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Abstract: The Concrete-Pictorial-Abstract (CPA) approach, based on Bruner's conception of the *enactive*, *iconic* and *symbolic* modes of representation, is a well-known instructional heuristic advocated by the Singapore Ministry of Education since early 1980's. Despite its ubiquity as a teaching strategy throughout the entire mathematics education community in Singapore, it is somewhat surprising to see a lack of an account of its theoretical roots. This paper is an attempt to contribute to this discussion on the CPA strategy and its potential role in continuing advancement of quality mathematics education.

Keywords: Concrete-Pictorial-Abstract, mathematics instruction, teacher professional development

Introduction

The theories of instruction proposed by Bruner in his 1966 book *Toward a Theory of Instruction* have undoubtedly bequeathed a rich legacy to generations of educators in the domain of learning and instruction. Amongst his voluminous contributions, one of the most well-known conception is that of "enactive-iconic-symbolic" modes of representation. This conception forms the foundation for a spectrum of instructional practices related to mathematics education, all bearing a conspicuous tripartite semblance to the Bruner's model.

One such adaptation of Bruner's model is the Concrete-Representation-Abstract (CRA) sequence. The CRA sequence has been shown to be particularly effective with students who have difficulties with mathematics (Jordan, Miller, & Mercer, 1998; Sousa, 2008).

The 'Concrete' segment of CRA, in particular, has been the theoretical basis for the use of manipulatives in learning mathematics (Reisman, 1982; Ross & Kurtz, 1993). The CRA approach has also been employed to aid students with learning disabilities to learn mathematics; CRA has been reported to be effective in remediating deficits in basic mathematics computation (Morin & Miller, 1998), in the teaching of place value (Peterson, Mercer, & O'Shea, 1998), fractions (Butler, Miller, Crehan, Babbit, & Pierce, 2003) and algebra (Maccinni & Ruhl, 2000; Witzel, Mercer, & Miller, 2003). With regards to mathematics students (first and third graders), Fuchs, Fuchs, and Hollenback (2007) also advocate the use of the CRA sequence to teach place value, geometry, and fractions.

In the practice of mathematics instruction in Singapore, Bruner's enactive-iconic-symbolic conception is at the heart of the Concrete-*Pictorial*-Abstract (CPA) approach. Since its inception in the early 1980's, the CPA approach has remained a key instructional strategy advocated by the Singapore Ministry of Education. This is attested by its regular mention in official curricular documents, including the latest syllabus for implementation in 2013:

> This [activity-based] approach is about learning by doing. It is particularly effective for teaching mathematical concepts and skills at primary and lower secondary levels, but is also effective at higher levels. Students engage in activities to explore and learn mathematical concepts and skills … . They could use manipulatives or other resources to construct meanings and understandings. From concrete manipulatives and experiences, students are guided to uncover abstract mathematical concepts or results. … During the activity, students communicate and share their understanding using concrete and pictorial representations. The role of the teacher is that of a facilitator who guides students through the *concrete, pictorial and abstract* levels of understanding by providing appropriate scaffolding and feedback. (Ministry of Education, 2012, p. 23, emphases added)

Although CPA is now well-known within (and even outside) the Singapore mathematics education community, it is surprisingly difficult to find scholarly works related to its theoretical roots and actual classroom implementation in the literature. This paper is an attempt to contribute to this discussion on the CPA strategy. In particular, we attempt to trace the origins of CPA and examine its influence over mathematics curriculum development and instruction in Singapore.

Theoretical considerations

First, this recommended instructional approach of starting with modes of learning that are more concrete to students and then gradually replacing the representations into forms that approximate formal mathematical symbols or language is not uncommon. For example, this is the instructional strategy that is recommended by Ketterlin-Geller, Chard, and Fien (2008): “a gradated instructional sequence that proceeds from concrete to representational to abstract (CRA) benefits struggling students” (p. 35).

In a recent study on using the CRA instruction sequence in teaching subtraction with regrouping to some low-achieving Grade 3 mathematics students, Flores (2010) reported that the students show improvement in fluency and confidence in doing arithmetic computations involving subtractions. In addition, a number of other studies have provided evidence of positive effect of using CRA on low achievers in the area of fractions (Butler et al., 2003), word problems (Maccini & Hughes, 2000), simple linear functions (Witzel, 2003), and advanced linear functions (Witzel, Mercer, & Miller, 2003). Indeed, the use of CRA approach to teaching mathematics concepts, especially at the elementary level has been proven to be effective.

Despite the commonalities to these other ways of labelling the instructional sequence, it appears that certain features of CPA are somewhat unique to Singapore Mathematics education. Its uniqueness is not restricted to the C-P-A as labels for the respective modes; it is also in its ubiquity throughout the entire mathematics education community in Singapore. CPA is a teaching strategy that is advocated by the Ministry of Education (Ministry of Education, 2012), embedded in textbooks used by schools (Fan, 2012), and taught in pre-service courses of mathematics teachers (e.g., Chua, 2010; Edge, 2006).

A number of writers (e.g., Edge, 2006, Wong, 2010) attributed the theoretical roots of the CPA to Bruner’s (1966) “enactive”, “iconic”, and "symbolic". To answer the question about the "Singapore CPA" and "Bruner" link, we directed our inquiry to a source within the Ministry of Education. Kho Tek Hong is a consultant to the Mathematics Unit, Curriculum, Planning and Development Division. He oversaw the school mathematics syllabus formulations since the late 1970s and remains involved in an advisory role in recent syllabus revisions. In response to our enquiry, Kho (personal communication, 2012) replied and part of his response is reproduced here:

> Truly the CPA Approach was idealised from Bruner's … . [It was] researched and adopted by me at the initiation of the Primary Mathematics Project (PMP) in 1979 and 1980, and the CPA Approach was highlighted in the First Edition of the PMP's Primary Mathematics instructional materials first published in 1981. The approach was reported in Ministry of Education internal documents (not for circulation).

Since CPA is acknowledged to be based on Bruner's conceptions, we now turn to the latter for theoretical foundations.

Examining Bruner's Enactive-Iconic-Symbolic

Perhaps a good point to start is to note that Bruner's original project was far more ambitious than the enactive-iconic-symbolic that he is now more known for. He set forth to craft a "theory of instruction"—as revealed in the title of his 1966 book, and he began by making explicit the parameters which such a theory must address: (1) specify ways to help students develop a "predisposition towards learning" (p. 40), (2) specify ways to structure an intended body of knowledge for learners, (3) specify the most effective sequences to present teaching materials, and (4) specify the involvement of rewards and punishments. Enactive-iconic-symbolic played some parts (but not the whole) in (2) and (3) but none at all in (1) and (4). Not wanting to discuss Bruner's vision of a theory of instruction here, it suffices to remark that (4) fell out of fashion in recent times together with behaviourist theories and (1) is closely related to affect and remains a challenging area of intense research. That (2) and (3) survive in various forms (including CPA) may have to do with its relative simplicity—at least in the forms that are perpetrated—and thus suitability for dissemination in teacher professional development.

It is under point (2) on the structure of knowledge that Bruner (1966) introduced the enactive-iconic-symbolic as “modes of representation”:

> Any set of knowledge … can be represented in three ways: by a set of actions appropriate for achieving a certain result (enactive representation); by a set of summary images or graphics that stand for a concept without defining it fully (iconic representation); and a set of symbolic or logical propositions drawn from a symbolic system that is governed by rules or laws forming and transforming propositions (symbolic representation) (pp. 44-45).

Quite clearly, using modern parlance, Bruner was not referring to representations as conceived in internal mental states; rather, he was interested in external representations of knowledge for the purpose of public discourse, and more particularly, in instructional settings. He was asserting that knowledge (including and especially for educative purposes) can be embodied in any one of these forms: action, visual image, or language-symbolic. Far be it that Bruner was advocating that a unique representation of a concept exists under each of the modes. In fact, contrary to this, he noted that “[m]any subjects, such as mathematics, have alternative modes of representation” (p. 45).

As part of his theory of instruction, the choice among multiple modes is dependent on other features of representation: *economy* and *power*. The former has to do with the amount of information needed to process the representation in order to comprehend the underlying knowledge; the latter refers to the potential of the representation in helping the learner go beyond what is couched on the surface of the representation to connect to deeper or related ideas. Bringing it closer to mathematics classroom instruction, Bruner’s theory of representations is part of his broader theory of instruction. In other words, he was positing modes of representations of mathematical ideas that teachers can bring into the classroom and how they can decide, based on notions such as economy and power, on the actual forms these modes can take for students’ learning.

As to the more popular interpretation of Bruner’s method as moving from enactive to iconic to symbolic, he dealt with this matter under (3) on effective sequences: “If it is true that the usual course of intellectual development moves from enactive through iconic to symbolic representation of the world, it is likely that an optimum sequence will progress in the same direction" (p. 49).

Before proceeding, we propose to make a distinction here between the language of "mode" and the language of "stage". Bruner uses these two terms and sometimes interchangeably to refer to each of the three constructs. For clarity of the readers, we use "mode" to refer to the representational form and "stage" to refer to the predominant mode used during the time sequence of instruction.

Bruner moved closer to the specific domain of mathematics instruction in his illustration within the same book of how the sequence can be carried out in the teaching of solving quadratic equation. He described at length how the "enactive" stage could be carried out by getting students to work on algebra blocks (a three-dimensional form of algebra tiles) and then gradually guiding them "to an iconic representation … . *Along the way*, notation was developed and … converted into a properly symbolic system" (pp. 64-65, emphases added). Bruner's "along the way" debunked a myth that seems commonly held within some sectors of the mathematics education community: that Bruner's modes are distinct and separated chronologically. In his conception, elements of the "symbolic" mode, such as algebraic notations, are developed alongside the primarily enactive and iconic stages of instruction, leading towards a proficiency of operation within the symbolic system. The moving through the stages provides an overarching broad instructional flow, with careful attention given to developing notations of the symbolic system gradually across the changing stages and overlapping modes.

Also, the goal of starting with "enactive" stage is not to remain merely at that mode; it is ultimately to get the students to fluency in the "Symbolic" mode. In the process, teachers are to help students "wean themselves from the perceptual embodiment to the symbolic notation" (p. 63). This point, to us, is significant as it avoids two extremes: being complacent merely with students' comfort in enactive or iconic modes on one end; and, proceeding quickly through (or not at all) through the earlier stages to get to symbolic mode on the other. The former is perhaps more common among teachers working with mathematics 'low-achievers'. The possible defence is that they are incapable of symbolic manipulations and thus working with less formal representations can be considered, for them, as success. Bruner (1966), however, did not share this view: “For if a child is to deal with mathematical properties he will have to deal with symbols per se, else he will be limited to the narrow and rather trivial range of symbolism that can be given direct … visual embodiment” (p. 63). We stand with Bruner on this, and add that stopping short of working primarily in the symbolic mode denies students of a wide range of rich and rewarding mathematical experiences. In fact, if students work purely in enactive and iconic modes over the long term, they can hardly be said to be doing mathematics; we contend, with others (e.g., Ma, 1999), that fluent operation within the symbolic domain is at the heart of the mathematics discipline. The other temptation for teachers to skip or move quickly through the earlier stages to ‘get to’ symbolic representations is also real. And the argument for it may seem compelling: if the final goal is to get students proficient with working purely in the symbolic system, why waste (so much) time teaching the other modes? Again, Bruner (1966) provided a balanced answer:

> For when the learner has a well-developed symbolic system, it may be possible to by-pass the first two stages. But one does so with the risk that the learner may not possess the imagery to fall back on when his symbolic transformations fail to achieve a goal in problem solving (p. 49).

One inference from this is that, to Bruner, (i) while it is important for students to be able to work solely in the symbolic system, the symbolic mode of representation is not necessarily a ‘superior’ mode to, say, the iconic mode across all mathematical situations. He gave the context of problem solving as one example in which the *imagery* of a concept may provide a good alternative in attacking the problem. This implies that one’s inability to switch to a different mode limits one’s problem solving ability; (ii) familiarity with other modes allows students to “fall back”. We understand this to mean that if a student learns purely in the symbolic mode, in the event that he cannot recall its workings, he can then have something to “fall back” to in order to recover the meanings of the notations in the symbolic mode. There is no way to “fall back” if there is no history of meaningful learning in other modes of representation; (iii) the moving through the three modes in instruction theoretically mirrors the “usual course of intellectual development” (p. 49).

Nevertheless, Bruner's point that "it may be possible to by-pass the first two stages" is not an insignificant statement. It is a reminder that students who are adept at the symbolic system should not be compelled to *always* start their learning at the earlier stages. Where the tools learnt in the other modes are not potentially powerful problem solving resources, these students should not be 'forced' to go through the full works of the three stages. Rather, an alternative learning trajectory that bypasses the earlier modes should be catered to these students.

Today, stadial theories have gone out of fashion in favour of more complex views of instruction that take into account socio-cultural factors (Merttens, 2012). Bruner's enactive-iconic-symbolic is not spared from this effect. We recognize that teaching is a complex cultural activity that is not easily reducible to a context-free universal three-step method. But the aspects of Bruner's modes and sequence that we reviewed does not claim to do that; it sets out what is to us a reasonable theory of external representations for use in teaching, without over-complexifying it to a point that renders it unworkable for dissemination; it recognises differences among students but sets up a broad sequence of stages that serves as a good rule of thumb for structuring instruction when introducing mathematics ideas/methods. Insofar as its theoretical assumptions are sound and can be easily translatable to classroom use, we think it remains a useful teaching heuristic for teachers' reference.

From Bruner to Singapore's CPA

There is quite clearly a one-to-one correspondence between Singapore's Concrete-Pictorial-Abstract to Bruner's Enactive-Iconic-Symbolic. The change in labels of each of the modes appears more an attempt at language simplification rather than conscious theory revision. The extract from the Ministry of Education syllabus document (Ministry of Education, 2012) at the beginning of this paper makes clear the official interpretation of "Concrete" as not restricted to "concrete manipulatives", but also "concrete experiences". The latter was further explained as comprising *activities* with suitable manipulatives. This view of "Concrete" is thus very much in line with Bruner's "Enactive" which is also about mathematical knowledge as embodied in actions. Within the same extract, relatively little is mentioned about "Pictorial" and "Abstract". There is, nevertheless, a reference to "Pictorial" as "representations", which aligns closely to Bruner's "Iconic"; and the language of "guiding through" is in line with the sequential order of the three modes. We can then infer that "Abstract" is conceptually not far off from the language-symbolic emphasis of Bruner's "Symbolic".

Another source of reference about what the official take on these terms are can be found in Singapore mathematics textbooks commissioned by the Ministry of Education, since CPA is said to be incorporated in these textbooks:

> The Primary Mathematics Project (PMP), led by Dr Kho Tek Hong, was tasked to produce instructional materials for the teaching and learning of primary mathematics with effective teaching approaches and professional development of teachers. The PMP instructional materials advocated the Concrete-Pictorial-Abstract Approach (Kho, Yeo, & Lee, 2009, p. 2).

We reviewed a number of textbooks that arose from the work of the PMP (e.g. Curriculum Development Institute of Singapore, 1982; 1983). A typical chapter introduction of these textbooks follows this order: a 'real-life' setting that provides a context for a noteworthy situation or problem (e.g., a pie-division problem), a visual representation of the situation or other related problems (e.g., representing pies by circles), and abstracting from visual forms to a symbolic form (e.g., working with numeric fractions). There is thus a sequence that mirrors closely the CPA stages. However, the "Concrete" as is presented in the textbooks appeared to have deviated from Bruner's original conception of activity to taking the form of a mere *description* of an activity. In other words, the textbook writers appeared to have taken the liberty to broaden "Concrete" to include not just activity, but also a reading (or teacher-talk) about the activity.

There appears then a difference between the notion of "Concrete" as used by the PMP writers in the early 1980s and the writers of the most recent syllabus document for implementation in 2013. If so, we surmised that, within the Ministry of Education, there could have been a change in thinking or emphasis about "Concrete" over the years (and recently converging to Bruner's original conception). As it turned out, Kho confirmed this hunch (personal communication, 2012):

> The significant change was the shift of emphasis from teaching to learning in the 1990s. Besides teaching aids, a wide range of learning manipulatives were introduced. The teacher's role is to provide appropriate learning experiences, including concrete experiences … to facilitate learning. Concrete experiences can take the form of activities, real-life context, or use of manipulatives.

In tracing the Ministry of Education’s documents over the last three decades on CPA, we also notice that CPA as an instructional strategy was first introduced only to the Primary levels through the outcomes of the PMP in the early 1980s. The oldest source available dates back to 1990 (Ministry of Education, 1990b) in which CPA was also officially endorsed as a recommended teaching approach to the Lower Secondary levels.

There is, however, one feature that appears to be different from Bruner: concrete, pictorial, and abstract are depicted in the Ministry of Education documents as “levels of understanding” (Ministry of Education, 1990a, p. 10; Ministry of Education, 2012, p. 23). Bruner does not use the language of “levels”. His use of “stages” has more to do with instructional sequencing with respect to time rather than “levels of understanding”. The latter appears to go beyond the external representational forms into psychological modes of operation within learners. We think this association (intended or otherwise) to internal states of competence of students is perhaps a step too far. Psychological workings in, say, a problem solving situation involve complex working flexibly across different modes of representations, and not merely mental operation at a *single* level. Nevertheless, we agree (with Bruner too) that students capable of operating fluently at the Abstract mode possess mental tools that enable them to handle more sophisticated mathematical tasks.

A final point about CPA is the potential ambiguity with the terms “Concrete” and “Abstract”. Part of the ambiguity has to do with the different definitions of these terms from different theoretical traditions. In some schools of thought (e.g., of the Piagetian tradition), “Concrete” tends to be associated more with objects rather than actions. Similarly, “Abstract” can be defined as the end result of the process of abstracting by comparing similarities (e.g., Skemp, 1986) rather than the Brunerian conception of operating in a symbolic system. A further complication is in the subjective nature of what is viewed as concrete or abstract. For example, a mathematician will consider working with algebraic symbols in the quadratics “concrete” while the same activity would be seen as “abstract” to students not accustomed to the task. One implication of these ambiguities and subjectivities to mathematics instruction is that what is considered “concrete”, “pictorial”, and “abstract” for a particular body of mathematical knowledge is not a fixed universal; rather, teachers will need to calibrate the modes to suit the needs of their students. For this, the features of economy and power of representations that Bruner purported remain useful guidelines.

So far, we have discussed Singapore CPA in its general characteristics. In the next section, we go into the specifics of how CPA can be utilised as a guiding heuristic in actual classroom mathematical instruction. In particular, we provide a description of our interpretation of CPA as it was applied in the design of a lesson on quadratic factorisation.

CPA applied in actual lessons: An example

The mathematics lesson was part of a project led by the first author that involved the application of CPA—including the principles of instruction and representations discussed in the earlier sections—in the context of teaching a Year Eight class in a Singapore secondary school (henceforth referred to as the project school). The details of the project are reported in Leong, Yap, Teo, Thilagam, Karen, Quek, & Tan (2010). As the focus of this paper is not on that project, only a brief description is provided here.

Discussions among teachers prior to the project were over the difficulty that students—particularly students who were mathematically-challenged—in the school faced when confronted with algebraic manipulation. The teachers reported that students made many mistakes in symbolic manipulations and it seemed that the students could not make sense of the basic rules and symbols of algebra. The goal of the project was thus to design lessons that will help students make sense of the algebra they do. Seen through the lens of CPA, the project aimed to help students start with concrete representations of algebra and then connect it gradually to the symbolic form of formal algebra over the course of the lessons. The topic selected for the study was “factorisation involving quadratic polynomials” as this was agreed among the teachers to be the most challenging topic for their students.

Sixty students from the Secondary Two Normal (Academic) class were taught using the CPA approach. In Singapore, pupils who have completed primary education are streamed into three ability streams, according to their performance in the national examination. The streams are known as Express, Normal (Academic), Normal (Technical) and the percentage of students in each of these streams are roughly 60, 25, and 15 respectively. These selected students were judged by their resident teachers to be among those who had the most difficulty with algebra.

Guided by CPA, the connection between the concrete mode of representing factorisation to its more abstract algebraic form was carried out. We used the geometric analogy of factorisation as finding length/breadth of the rectangle given the area. To concretise this "forming of rectangle" stage, we introduced *AlgeCards* for a start. *AlgeCards* are similar to Algebra tiles with the difference that "x^2", "x", and "1" are imprinted on the cards to help students make clearer visual connections between the concrete and the symbolic modes. The purpose of using the *AlgeCards* is to help students actively carry out the "forming rectangle" as an essential part of factorisation in a concrete way. In line with the foregoing discussion in the earlier sections, we were mindful that students ought not to stay too comfortable with *AlgeCards*; rather, we wanted students to make entrance into factorisation using a representation that made sense to them but would subsequently progress to a method that approximates algebraic dexterity. Figure 1 shows the links among these modes of representation. "Rectangle Diagram" is a pictorial simplification of the concrete *AlgeCards*; and unlike the latter, it can be easily drawn and thus portable as a useable tool in paper-and-pencil contexts, including paper-and-pencil test situations.

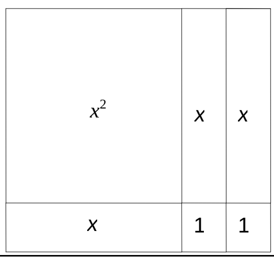
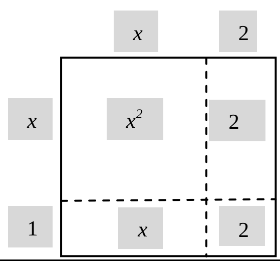
Factorise	***AlgeCards* Diagram**	**Rectangle Diagram**
x^2+3x+2 $= (x+1)(x+2)$		

Figure 1. Linking AlgeCards to rectangle diagram and to the algebraic factorization.

The design of *AlgeCards* is similar to those employed by Bruner himself (Bruner, 1966, pp. 60-62). The co-existence of the written algebraic symbols with the concrete manipulatives provides a gradual build-up of the algebraic notations through the modes of representation and creates a physical passage to allow a gradual transition from the "Concrete" to the "Abstract", with an explicit feature of "decontextualizing" or "fading away". In our design, this is characterized by the following sequence of representations: *AlgeCards* (Concrete pieces suitable for manipulation) → Rectangle Diagram (Pictorial representation) → Quadratic expressions (Abstract symbols). Such a gradual fading process has been hinted by Bruner (1966):

> … by giving a child multiple embodiments of the same general idea expressed in a common notation we lead him to "empty" the concept of specific sensory properties until he is able to grasp its abstract properties (p. 65).

Many proponents of concrete manipulatives have also made mention of this fading process; for instance, Goldstone & Son, 2005; Gravemeijer, 2002; Lehrer & Schauble, 2002; Lesh, 1979. More recently, McNeil & Fyfe (2012) reported on the positive effects of fading on the transfer performance for a sample of undergraduates learning group theory.

The duration of the fading process varied across students. Subsequent lessons were designed in such a way as to allow students to transit to using the Pictorial as the predominant mode gradually by introducing larger positive coefficients as a motivation. When students could operate comfortably with the Rectangle Diagram without first starting with the *AlgeCards*, then expressions with negative coefficients, such as $x^2 - 3x + 2$, were given. For these items, the students were encouraged to use the Rectangle Diagram as a template to work out the factorisation instead of still thinking of the components as 'negative areas'.

Alongside the goal of helping students learn factorisation through the CPA heuristic, this project was also conceived as a *Lesson Study* enterprise. The school invited the first author into the group as a Knowledgeable Other. Led by the stages advocated by Lewis (2002), and Stepanek, Appel, Leong, Mangan, and Mitchell (2007), we used the common features of their *Lesson Study* model to guide the entire process: We met to discuss the difficulties students faced and identified the goals of the project; that was followed with more discussion meetings on the design of the module—as mentioned in the earlier paragraphs; subsequently one teacher in the team carried out the teaching of the lessons. Other teachers in the team sat in for the lessons, made disciplined observations, and shared in post-lesson meetings after each lesson.

As the focus of this paper is a study on the roots and realisations of CPA in the Singapore mathematics curriculum, rigorous investigations into the effect of this CPA-based innovation in the participating classes will be the subject of another study. Suffice to mention here that the teachers who participated in the team shared that they learnt much both from the approach taken in the module as well as the observation of students' work in class. Since this method of teaching the topic was first designed and carried out in 2009, it has been embedded as standard instructional practice in the project school: the teachers have continued to employ the CPA approach and the instructional materials—the full works of *AlgeCards* and the accompanying worksheets—for every Secondary Two Normal (Academic) cohorts since then.

Regularising the CPA strategy in classrooms

Despite strong emphasis at the policy level and actual examples of alignment to CPA in classroom teaching, there are anecdotal evidences that suggest that the common instructional approach in Singapore mathematics classrooms is mainly that of direct teaching of the "Abstract"—the rules, the symbols, and the question-specific techniques—without building up from "Concrete" and/or "Pictorial". In this last section of the paper, we discuss the challenges of regularising CPA in Singapore classrooms. While we mention Singapore as the arena of research, insofar as some of these challenges are similarly encountered in other jurisdictions, we think the discussion would be of interest to a wider readership.

It is well-known that teachers experience time pressure in attempting to complete the assigned syllabus within the constraints of the allocated time as stipulated in given teaching schedules (e.g., Assude, 2005; Leong & Chick, 2011). Working regularly under such conditions of limited time to 'cover topics', there is a natural tendency to get to the main skills to teach for each topic in the most time-efficient way. This result in a quick convergence to the rules and formulas that students are expected to learn, which means the bypassing of other modes of representation and proceeding straight to the ‘Abstract’. Doing so, however, often also means a bypassing of students’ sense-making and hence the basis for fall back as discussed in the earlier sections.

Despite realising that direct teaching of arbitrary rules compromises the strength in which the students grasp the underlying mathematical concepts, the time pressure challenge is so strong that teachers are unlikely to buy-in to CPA so long as it is perceived as (i) taking up an unrealistic amount of classroom time; and/or (ii) not of direct benefit to students in terms of test scores for the topic. We think that any genuine attempt to regularise CPA needs to take into consideration these concerns.

We propose that a feasible way forward is to begin with designing exemplary instructional units where CPA can be easily trialled and where the results can be observed immediately. Doing so is an acknowledgement that (a) it is far too ambitious to implement CPA throughout the whole span of the syllabus at one go; (b) a “unit”—of about 4 to 6 hours of lesson time—is reasonable temporal space to test the time-workability of the CPA sequence and to observe its outcomes on students’ learning; and (c) CPA may not be appropriate for the teaching of some mathematics topics. As such, the focus should be on those mathematics units whose instructional development lends itself well to a CPA progression.

Developing the CPA strategy over a unit of lessons also allows the gradual transition between stages to take place more seamlessly. Moreover, given the diversity among learners, the duration of 4-6 lessons provide the temporal latitude for different students to transit to the next predominant mode of representation at points they feel they are ready to. In contrast, ‘forcing’ the entire CPA progression within, say, a single lesson would not allow each mode to be developed to a point in which the sense-making takes root and where intermodal links can be meaningfully established, thus heightening the sense of failure of the CPA innovation; on the other hand, stretching the CPA development over a much longer time period beyond a “unit” would render it unrealistic from the point of view of keeping to the time allocations of the teaching schedule.

Beginning with unit design is also advantageous from the standpoint of teacher development. Where feasible, teachers can be involved in the design process. As such, they do not see themselves as mere ‘end-user’ of the CPA-based design; rather, through active participation in the crafting of the unit sequence and instructional materials, they are not only given an opportunity to develop a more refined interpretation of CPA; they are also able to contribute to concretising its use in actual classroom instruction.

Over time, if the unit design strategy can be sustained, we would have a collection of a number of CPA-based instructional units that are trialled, refined, and avowed by teachers to be workable in actual mathematics classrooms. It can then serve as a repository for other teachers who want to learn about the usefulness of the CPA heuristic to start their inquiry and subsequent adaptations for use in their classrooms.

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