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<td>Source</td>
<td>Kyungpook Mathematical Journal, 54(2), 263-269</td>
</tr>
<tr>
<td>Published by</td>
<td>Department of Mathematics, Kyungpook National University</td>
</tr>
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Reflexive Index of a Family of Sets

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Abstract. As a further study on reflexive families of subsets, we introduce the reflexive
index for a family of subsets of a given set and show that the index of a finite family of
subsets of a finite or countably infinite set is always finite. The reflexive indices of some
special families are also considered.

Given a set $X$, let $\text{Sub}(X)$ denote the set of all subsets of $X$ and $\text{End}(X)$ denote
the set of all endomappings $f : X \rightarrow X$. For any $A \subseteq \text{Sub}(X)$ and $\mathcal{F} \subseteq \text{End}(X)$
define
\[
\text{Alg}(A) = \{ f \in \text{End}(X) : f(A) \subseteq A \text{ for all } A \in A \},
\]
\[
\text{Lat}(\mathcal{F}) = \{ A \in \text{Sub}(X) : f(A) \subseteq A \text{ for all } f \in \mathcal{F} \}.
\]
A family $A \subseteq \text{Sub}(X)$ is called reflexive if $A = \text{Lat}(\text{Alg}(A))$, or equivalently,$A = \text{Lat}(\mathcal{F})$ for some $\mathcal{F} \subseteq \text{End}(X)$.

As was shown in [9], $A \subseteq \text{Sub}(X)$ is reflexive iff it is closed under arbitrary
unions and intersections and contains the empty set and $X$. The reflexive families
$\mathcal{F} \subseteq \text{End}(X)$ were also introduced and characterized as those subsemigroups $\mathcal{L}$ of
$(\text{End}(X), \circ)$ such that $\mathcal{L}$ is a lower set and contains all existing suprema of subsets
of $\mathcal{L}$ with respect to a naturally defined partial order on $\text{End}(X)$. The similar work
in functional analysis is on the reflexive invariant subspace lattices and reflexive
operator algebras [1-6].

For any $A \subseteq \text{Sub}(X)$, let $\hat{A} = \text{Lat}(\text{Alg}(A))$. Then $\hat{A}$ is the smallest family
of subsets containing $A$ which is closed under arbitrary unions and intersections
containing empty set $\emptyset$ and $X$, and $\hat{A}$ is finite if $A$ is finite. Furthermore, $\hat{A} = \text{Lat}(\mathcal{F})$, where $\mathcal{F} = \text{Alg}(A)$.

It is, however, still not known whether for any finite family $A$ there is a finite
$\mathcal{F} \subseteq \text{End}(X)$ such that $\hat{A} = \text{Lat}(\mathcal{F})$.

In this short paper, we shall answer the above problem. It will be shown that
the answer is positive if and only if $X$ is a finite or countably infinite set. For

Received July 19, 2012; accepted April 24, 2013.
2010 Mathematics Subject Classification: 54C05, 54C60, 54B20, 06B99.
Key words and phrases: reflexive families, reflexive index, endomapping.
any family $\mathcal{A}$ of subsets of a set, we define a cardinal $\kappa(\mathcal{A})$, which, in a certain sense, reflects how the sets in $\mathcal{A}$ are interrelated. This cardinal will be called the reflexive index of $\mathcal{A}$. The reflexive indices of some special families are computed. For instance, we show that if $\mathcal{A}$ is a finite chain of subsets of $\mathbf{N}$ (the set of all natural numbers) with more than one member, then $\kappa(\mathcal{A}) = 2$.

**Definition 1.** Let $\mathcal{A}$ be a family of subsets of a set $X$. The reflexive index of $\mathcal{A}$ is defined as

$$
\kappa_X(\mathcal{A}) = \inf \{|\mathcal{F}| : \mathcal{F} \subseteq \text{End}(X), \mathring{\mathcal{A}} = \text{Lat}(\mathcal{F})\},
$$

where $|\mathcal{F}|$ is the cardinal of $\mathcal{F}$.

We shall write $\kappa(\mathcal{A})$ for $\kappa_X(\mathcal{A})$ if the set $X$ is clearly assumed.

If $C \subseteq X$ and $f \in \text{End}(X)$ such that $f(C) \subseteq C$, then we say that $C$ is invariant under $f$. Thus $\text{Lat}([f])$ is the set of all subsets which are invariant under $f$. Also note that $\text{Lat}([f]) = \bigcap\{\text{Lat}([f]) : f \in \mathcal{F}\}$ for any $\mathcal{F} \subseteq \text{End}(X)$.

In the following we shall use $\mathbf{N}$ to denote the set of all natural numbers.

**Remark 2.**

(1) For any $\mathcal{A} \subseteq \text{Sub}(X)$, $\mathring{\mathcal{A}} = \mathring{\mathcal{A}}$, thus $\kappa_X(\mathring{\mathcal{A}}) = \kappa_X(\mathcal{A})$.

(2) For any $g \in \text{End}(X)$, where $X$ is an infinite set, $\text{Lat}([g])$ is an infinite family. To see this, consider any $a \in X$. If $\{g^k(a) : k \in \mathbf{N}\}$ is an infinite set, then $g^k(a) \neq g^i(a)$ whenever $k \neq i$. In this case, $\{\{g^i(a) : i \geq k\} : k \in \mathbf{N}\}$ is an infinite subfamily of $\text{Lat}([g])$ ($g^k$ is the composition of $k$ copies of $g$). Now assume that for each $a \in X$, $\{g^k(a) : k \in \mathbf{N}\}$ is a finite set, then there are infinitely many sets of the form $\{g^k(a) : k \in \mathbf{N}\}$, each of them is a member of $\text{Lat}([g])$. Therefore $\text{Lat}([g])$ is infinite.

**Lemma 3.** Let $X$ be a nonempty set.

(1) If $X$ is a countably infinite set, then there are two mappings

$$
\mu_X^0, \mu_X^1 : X \longrightarrow X
$$

such that for any nonempty $B \subseteq X$, if $\mu_X^0(B) \subseteq B$ and $\mu_X^1(B) \subseteq B$ then $B = X$.

(2) If $X$ is a finite set, there is one mapping $\mu_X^0 : X \longrightarrow X$ such that for any nonempty $B \subseteq X$, $\mu_X^0(B) \subseteq B$ implies $B = X$.

**Proof.**

(1) If $X = \{a_1, a_2, \ldots\}$ is a countably infinite set, define $\mu_X^0(a_i) = a_{i+1}$ and $\mu_X^1(a_i) = a_{i+1}$ for each $i$. Then $\mu_X^0$ and $\mu_X^1$ satisfy the requirement.

(2) If $X = \{a_1, a_2, \ldots, a_n\}$ is a finite set, define $\mu_X^0(a_0) = a_{i+1}$ for $1 \leq i < n$ and $\mu_X^1(a_n) = a_0$. Then $B \subseteq X$ and $\mu_X^0(B) \subseteq B$ will imply $B = \emptyset$ or $B = X$. \qed

**Proposition 4.** Let $X$ be a nonempty set.

(1) If $X$ is countably infinite, $\kappa_X(\{\emptyset, X\}) = 2$.

(2) If $X$ is a finite set, $\kappa_X(\{\emptyset, X\}) = 1$.

**Proof.** First note that the family $\{\emptyset, X\}$ is closed under arbitrary intersections and unions, so it is reflexive, i.e. $\text{Lat}(\text{Alg}(\{\emptyset, X\})) = \{\emptyset, X\}$ (Theorem 1 of [9]).
The statement (2) clearly follows from Lemma 3(2).

To prove (1), by Lemma 3(1), we have \( \kappa_X(\{\emptyset, X\}) \leq 2 \). Also, by Remark 2 (2), for any \( f \in \text{End}(X) \) the set \( \text{Lat}(\{f\}) \) is infinite, implying \( \{\emptyset, X\} \neq \text{Lat}(\{f\}) \).

Hence \( \kappa_X(\{\emptyset, X\}) = 2 \). \( \square \)

**Proposition 5.** If \( X \) is a noncountable infinite set, then \( \kappa_X(\{\emptyset, X\}) = |X| \), where \( |X| \) is the cardinal of \( X \).

**Proof.** Let \( \emptyset \neq F \subseteq \text{End}(X) \) and \( |F| < |X| \). Take \( F^* \) to be the subsemigroup of \( (\text{End}(X), \circ) \) generated by \( F \), where \( \circ \) is the composition operation. If \( F \) is finite, then \( F^* \) is finite or countably infinite. Since \( X \) is uncountable it follows that \( |F^*| < |X| \). If \( F \) is infinite, then \( |F^*| = |F| < |X| \).

Chose one element \( a \in X \) and let \( F^a = \{ f(a) : f \in F^* \} \), called the orbit of \( a \) under \( F \). Clearly \( F^a \) is a member of \( \text{Lat}(F) \).

Since \( |F^a| \leq |F^*| < |X| \), implying \( F^a \neq X \). Also as \( F^a \neq \emptyset \), so \( \text{Lat}(F) \neq \emptyset \).

Now consider \( K = \{ f_a : a \in X \} \subseteq \text{End}(X) \), where \( f_a : X \longrightarrow X \) is the constant mapping that sends every \( x \in X \) to \( a \). Then for any nonempty set \( B \subseteq X \), if \( f_a(B) \subseteq B \) for all \( a \in X \), then \( X = B \).

Therefore \( \text{Lat}(K) = \emptyset \) and so \( \kappa_X(\{\emptyset, X\}) \leq |K| = |X| \). All these show that \( \kappa_X(\{\emptyset, X\}) = |X| \). \( \square \)

Now we prove the main result of this paper.

**Theorem 6.** Let \( X \) be a finite or countably infinite set. Then for any finite family \( A \subseteq \text{Sub}(X) \), \( \kappa(A) \) is finite.

**Proof.** Since the conclusion is clearly true if \( X \) is a finite set, we only give the proof for countably infinite sets \( X \). To simplify the argument we take \( X = \mathbb{N} \) (the set of all natural numbers) and denote \( \kappa_N(F) \) simply by \( \kappa(F) \). Without lose of generality, we assume that \( N \in A \). By rearranging, if necessary, we can let \( A = \{A_1, A_2, \cdots, A_m\} \) such that \( A_1 = \mathbb{N} \) and \( j > i \) if \( A_j \subset A_i \) (however, \( j > i \) need not imply \( A_j \subset A_i \)). Let \( \Theta = \{(i_1, i_2, \cdots, i_k) : 1 \leq i_1 < i_2 < \cdots < i_k \leq m, 1 \leq k \leq m\} \). For each \( \sigma = (i_1, i_2, \cdots, i_k) \in \Theta \), define \( X_{i_1} = \bigcap_{t=1}^{k} A_{i_t} \setminus \bigcup_{s \neq i_t, t = 1, 2, \cdots, k} A_s \).

It’s easy to see that the following statements are true:

(a) \( X_{i_1} \) and \( X_{i_2} \) are disjoint if \( \sigma \neq \beta \);
(b) for each \( A_i \in A, A_i = \bigcup \{X_\sigma : i \text{ is a component of } \sigma\} \);
(c) for each \( \sigma = (i_1, i_2, \cdots, i_k) \in \Theta \),

\[ \bigcap \{A_{i_t} : t = 1, 2, \cdots, k\} = \bigcup \{X_\beta : \beta \in \Theta, \beta \leq \sigma\} \]

Now let \( f^0 : \mathbb{N} \longrightarrow \mathbb{N} \) be a mapping such that \( f^0|_{X_\sigma} = \mu_{X_\sigma}^0 \) as constructed in the proof of Lemma 3 for each set \( X_\sigma \) (note that \( X_{\sigma} \)'s are disjoint sets and their
union is \( D \). Let \( f^1 : D \to D \) be the mapping such that \( f^1(x) = a^2_{\sigma} \) for each \( x \in X_{\sigma} \).

For any \( \sigma = (s_1, s_2, \cdots, s_k) \mid \beta = (t_1, t_2, \cdots, t_m) \) define \( f_{\sigma, \beta} : D \to D \), if \( X_{\sigma} \) and \( X_{\beta} \) are nonempty, as follows:

\[
f_{\sigma, \beta}(x) = \begin{cases} 
  x, & \text{if } x \notin X_{\sigma}, \\
  a^2_{\sigma}, & \text{if } x \in X_{\sigma} - \{a^2_{\sigma}\}, \\
  a^1_{\beta}, & \text{if } x = a^1_{\beta}.
\end{cases}
\]

Now consider the finite family \( \mathcal{F} = \{f^0, f^1\} \cup \{f_{\sigma, \beta} : \sigma, \beta \in \Theta, \sigma > \beta, X_{\sigma} \neq \emptyset, X_{\beta} \neq \emptyset\} \).

(1) Let \( A_i \in A \). By above (a) and (b), \( A_i \) is a disjoint union of some \( X'_{\sigma} \)'s. Since \( f^0(X_{\sigma}) \subseteq X_{\sigma}, f^1(X_{\sigma}) \subseteq X_{\sigma}, \) thus \( f^0(A_i) \subseteq A_i \) and \( f^1(A_i) \subseteq A_i \).

Now let \( \sigma, \beta \in \Theta \) such that \( \sigma > \beta \). If \( x \in A_i \) and \( x \notin X_{\sigma} \) then \( f_{\sigma, \beta}(x) = x \in A_i \). If \( x \in X_{\sigma} \), then \( i \) is a component of \( \sigma \), so \( i \) is also a component of \( \beta \). Now \( f_{\sigma, \beta}(x) \in X_{\sigma} \) or \( f_{\sigma, \beta}(x) \in X_{\beta} \). But \( X_{\sigma}, X_{\beta} \subseteq A_i \), so \( f_{\sigma, \beta}(x) \in A_i \), therefore \( f_{\sigma, \beta}(A_i) \subseteq A_i \). It follows that \( A \subseteq \text{Lat}(\mathcal{F}) \). Then \( \text{Alg}(A) \supseteq \text{Alg}(\text{Lat}(\mathcal{F})) \) and so \( \hat{A} = \text{Lat}(\text{Alg}(A)) \subseteq \text{Lat}(\text{Alg}(\text{Lat}(\mathcal{F}))) = \text{Lat}(\mathcal{F}) \), the last equation holds for any \( \mathcal{F} \) (see Lemma 1(3) of [9]).

(2) Given any \( C \subseteq D \) such that \( C \in \text{Lat}(\mathcal{F}) \), we show that \( C \in \hat{A} = \text{Lat}(\text{Alg}(A)) \). First, if \( C \cap X_{\sigma} \neq \emptyset \), then there is a point \( x \in C \cap X_{\sigma} \), so \( f^1 \in \mathcal{F} \), \( f^1(x) = a^2_{\sigma} \in C \). Hence \( a^2_{\sigma} = f^0(a^2_{\sigma}) \in C \), \( a^1_{\beta} = f^1(a^1_{\beta}) \in C \), etc. It then follows that \( X_{\sigma} \subseteq C \). Now if \( \beta < \sigma \), then \( a^1_{\beta} = f_{\sigma, \beta}(a^2_{\beta}) \in C \), so \( X_{\beta} \cap C \neq \emptyset \), therefore we also can deduce that \( X_{\beta} \subseteq C \). For any element \( x \in C \), there exists \( \gamma = (i_1, i_2, \cdots, i_k) \) such that \( x \in A_{i_t} \) for each \( t = 1, 2, \cdots, k \) and \( x \notin A_j \) for all \( j \notin \{i_1, i_2, \cdots, i_k\} \). Then \( x \in X_{\gamma} \). In addition, \( X_{\gamma} \subseteq C \) because \( x \in X_{\gamma} \cap C \) which implies \( X_{\gamma} \cap C \neq \emptyset \). By property (c), \( \bigcap \{A_{i_t} : t = 1, 2, \cdots, k\} = \bigcup \{X_{\beta} : \beta \leq \gamma\} \subseteq C \). In addition, \( x \in \bigcap \{A_{i_t} : t = 1, 2, \cdots, k\} \in \hat{A} \) (\( \hat{A} \) is closed under arbitrary intersections and each \( A_i \in A \)). All these show that \( C \) is a union of members of \( A \), thus \( C \in A \) because \( A \) is closed under arbitrary unions. Hence \( \text{Lat}(\mathcal{F}) \subseteq \hat{A} \).

The combination of (1) and (2) implies that \( \text{Lat}(\mathcal{F}) = \hat{A} \). Since \( |\mathcal{F}| \) is finite, the proof is completed.

Now we consider \( \kappa(A) \) for some special families \( A \) of subsets of \( D \).

**Example 7.** Let \( A = \{2N, 3N, 5N\} \). We show that \( \kappa(A) \leq 4 \).

Let \( D = 2N \cup 3N \cup 5N \) = \( \{a_k : k = 1, 2, \cdots\} \). Then \( 2N = \{a_k : k = 1, 2, \cdots\} \), \( 3N = \{b_k : k = 1, 2, \cdots\} \), \( 5N = \{c_k : k = 1, 2, \cdots\} \), \( 10N = \{d_k : k = 1, 2, \cdots\} \), \( 15N = \{e_k : k = 1, 2, \cdots\} \), and \( 30N = \{f_k : k = 1, 2, \cdots\} \).

Define the mappings \( f, g_1, g_2, g_3 \) in \( \text{End}(D) \) as follows:
Let $\mathcal{F} = \{f, g_1, g_2, g_3\}$ and $A \in \operatorname{Lat}(\mathcal{F})$.

(i) Each of $2N, 3N$ and $5N$ is invariant under every mapping in $\mathcal{F}$. Thus $\hat{A} \subseteq \operatorname{Lat}(\mathcal{F})$.

(ii) If $A \cap (N - (2N \cup 3N \cup 5N)) \neq \emptyset$, then, as $g_1(A) \subseteq A$, it follows that $a_1 \in A$. Then, each $a_{k+1}, k \geq 1$ is in $A$ because $f(A) \subseteq A$. Since $g_1(A) \subseteq A$ it follows that $b_1^k \in A (i = 1, 2, 3)$. Again, as $f(A) \subseteq A$, we deduce that $A$ contains each of $2N - (3N \cup 5N), 3N - (2N \cup 5N)$ and $5N - (2N \cup 3N)$. Now $A$ contains each of $c_1^i (i = 1, 2, 3)$. With a similar argument we deduce that $A$ contains each of $6N - 5N, 10N - 3N, 15N - 2N$ and $30N$. Hence $A = N \in \operatorname{Lat}(\mathcal{F})$.

In a similar way we can show the following statements are true:

(iii) If $A \cap (2N - (3N \cup 5N)) \neq \emptyset$, then $A$ contains $2N$. If $A \cap (3N - (2N \cup 5N)) \neq \emptyset$, then $A$ contains $3N$. If $A \cap (5N - (2N \cup 3N)) \neq \emptyset$, then $A$ contains $5N$.

(iv) If $A \cap (6N - 5N) \neq \emptyset$, respectively, $A \cap (10N - 3N) \neq \emptyset$, $A \cap (15N - 2N) \neq \emptyset$, then $A \supseteq 6N$, respectively, $A \supseteq 10N$, $A \supseteq 15N$.

(v) If $A \cap 30N \neq \emptyset$, then $A \supseteq 30N$.

From (i)-(v), it follows that $A$ either equals $N$ or is a union of intersections of $2N, 3N, 5N, 6N, 10N, 15N, 30N$, that is $A \in \hat{A}$ and so $\hat{A} = \operatorname{Lat}(\mathcal{F})$.

Thus $\operatorname{Lat}(\mathcal{F}) = \hat{A}$, so $\kappa(A) \leq 4$. 

\[
f(x) = \begin{cases} 
  a_{k+1}, & \text{if } x = a_k (k \geq 1), \\
  b_{k+1}^i, & \text{if } x = b_k^i (i = 1, 2, 3, \text{ and } k \geq 1), \\
  c_{k+1}^i, & \text{if } x = c_k^i (i = 1, 2, 3, \text{ and } k \geq 1), \\
  d_{k+1}, & \text{if } x = d_k (k \geq 1).
\end{cases}
\]

\[
g_1(x) = \begin{cases} 
  a_1, & \text{if } x = a_k + 1 (k \geq 1), \\
  b_1^i, & \text{if } x = a_1, \\
  c_1^i, & \text{if } x = b_1^i (i = 1, 3), \\
  c_1^i, & \text{if } x = b_1^i,
\end{cases}
\]

\[
g_2(x) = \begin{cases} 
  a_1, & \text{if } x = a_k + 1 (k \geq 1), \\
  b_1, & \text{if } x = b_1^i (i = 1, 2), \\
  c_1^i, & \text{if } x = b_1^i,
\end{cases}
\]

\[
g_3(x) = \begin{cases} 
  b_1^i, & \text{if } x = a_1, \\
  x, & \text{otherwise}.
\end{cases}
\]
Remark 8. From the proof in the above example, we can see that a more general conclusion is true: if $p_1, p_2, \cdots, p_m$ are distinct primes, then $\kappa(\{p_i\N : i = 1, 2, \cdots, m\}) \leq m + 1$.

Proposition 9. If $\mathcal{A} = \{A_1, A_2, \cdots, A_m\}$ is a finite chain of distinct subsets of $\N$ with $m \geq 2$, then $\kappa(\mathcal{A}) = 2$.

Proof. Without lose of generality, we assume that $A_1 \subset A_2 \subset \cdots \subset A_m$ and $A_1 \neq \emptyset$ and $A_m = \N$. Let $A_1 = \{a_1^1, a_2^1, \cdots\}, A_2 - A_1 = \{a_1^2, a_2^2, \cdots\}, \ldots, A_m - A_{m-1} = \{a_1^m, a_2^m, \cdots\}$. Define $f, g \in \text{End}(\N)$ as follows: for $i = 1, 2, \cdots, m$, and $k \in \N$,

$$f(a_k^i) = \begin{cases} a_{k+1}^i, & \text{if } a_k^i \text{ is not the last element in } A_i - A_{i-1}, \\ a_k^i, & \text{if } a_k^i \text{ is the last element in } A_i - A_{i-1}. \end{cases}$$

$$g(x) = \begin{cases} a_1^i, & \text{if } x = a_{k+1}^i, \\ a_1^i, & \text{if } x = a_k^{i+1}. \end{cases}$$

Since $\hat{\mathcal{A}}$ is the smallest family containing $\mathcal{A}$ which is closed under arbitrary unions and intersections, $\hat{\mathcal{A}} = \mathcal{A} \cup \{\emptyset\}$. Furthermore, $\mathcal{A} \cup \{\emptyset\} = \text{Lat}(\{f, g\})$. Thus $\kappa(\mathcal{A}) \leq 2$. By Remark 2(2), for any $h \in \text{End}(\N)$, $\text{Lat}(\{h\})$ is an infinite family, so $\kappa(\mathcal{A}) \neq 1$, therefore $\kappa(\mathcal{A}) = 2$. \hfill \Box

Remark 10.

(1) The reader may wonder whether there is a set family whose reflex index is 1. Consider $\mathcal{A} = \{\emptyset, \N\} \cup \{C_n : n = 1, 2, \cdots\}$, where $C_n = \{n, n + 1, \ldots\}$. Then $\mathcal{A} = \hat{\mathcal{A}} = \text{Lat}(\{f\})$, where $f$ is defined by

$$f(m) = \begin{cases} 1, & \text{if } m = 1, \\ m - 1, & \text{if } m > 1. \end{cases}$$

(2) The following is a chain of subsets of $\N$ whose reflexive index is not finite. Put $\mathcal{B} = \{\emptyset, \N, \{k : k \in \N, k \geq 2\}\} \cup \{D_n : n \in \N, n > 1\}$, where for each $n > 1$, $D_n = \{2, 3, \cdots, n\}$. Clearly $\mathcal{B} = \mathcal{B}$. Let $\mathcal{F} \subseteq \text{End}(\N)$ be any finite family of endomappings on $\N$ satisfying $\mathcal{B} \subseteq \text{Lat}(\mathcal{F})$. If $f(1) = 1$ for all $f \in \mathcal{F}$, then $\mathcal{F} = \text{Lat}(\mathcal{F}) - \mathcal{B}$. If there is $f \in \mathcal{F}$ with $f(1) \neq 1$, let $l = \max \{f(1) : f \in \mathcal{F}\}$, then $l \geq 2$ and the subset $\{1\} \cup D_l$ is in $\text{Lat}(\mathcal{F}) - \mathcal{B}$. Thus for any finite $\mathcal{F} \subseteq \text{End}(\N)$, $\text{Lat}(\mathcal{F}) \neq \mathcal{B} = \hat{\mathcal{B}}$, therefore $\kappa(\mathcal{B})$ is not finite.

Remark 11.

(1) It is possible and necessary to identify the exact values of the reflexive indices of more concrete families (such as $\mathcal{A} = \{p_i\N : i = 1, 2, \cdots, n\}$ for any distinct prime numbers $p_1, p_2, \cdots, p_n$). We leave this to interested readers to try.

(2) In [7][8], the reflexive families of closed subsets of a topological space are studied. We can also define the reflexive index for a family of closed sets. One of the natural problems would be: for which spaces, does every finite family of closed
sets have a finite reflexive index? Furthermore, one can introduce and consider the reflexive index of a family of closed subspaces of a Hilbert space.

Acknowledgements. I would like to thank Professor Carsten Thomassen for a very helpful conversation with him while he was visiting our department in 2009, which led me to prove the main result in this paper. I also must thank the referees for identifying some errors in the earlier draft and giving me valuable comments and suggestions for improvement.

References