Numerical Integration On Some Special Henstock-Kurzweil Integrals

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Abstract

In this paper, we describe how we can compute the integrals of a class of so called improper Riemann integrals and a class of non-absolute Henstock-Kurzweil integrals, which are highly oscillatory and are not Lebesgue integrable.

1 Introduction

It is known that the Riemann integration is useful for computation because of its even partition. However, we learn later that the family of Riemann integrable functions-denoted by $R(f)$, is only a subset of Lebesgue integrable functions-denoted by $L(f)$ and the family of Henstock-Kurzweil integrable functions-denoted by $HK(f)$-is an extension for Lebesgue integrable functions. In other words, we have

$$R(f) \subset L(f) \subset HK(f). \tag{1}$$

In this paper, we will describe how some functions in $HK(f)$ can be computed by introducing an uneven partition. We will take care of two types of functions: one is the family of monotonic functions with singularities, which normally are computed as 'improper' Riemann integrals, but now they are direct results from Henstock-Kurzweil definition. Second, we will take care of functions that are highly oscillatory. It is known that Fundamental Theorem of Calculus should be valid when a function F is differentiable on (a, b) and we have

$$\int_a^b F'(x)dx = F(b) - F(a). \tag{2}$$

However, the Lebesgue integration requires F' to be integrable over $[a, b]$. Thus we will see how to integrate certain class of functions that are not Lebesgue integrable but is Henstock-Kurzweil integrable.

In Sections 1.1, we describe some terminologies and an important outcome, Theorem 2, from the definition of Henstock-Kurzweil integral. In Section 2, we introduce open and closed types of quadratures in 1-D and how we can control the corresponding errors. In Section 3, we describe integration quadratures in 2-D and their corresponding errors.

1.1 Preliminary

Let $A = [a, b]$, we say $P = \{(A_1, x_1), ..., (A_n, x_n)\}$ is a partition of A if $A_1, ..., A_n$ are non-overlapping subintervals, $x_i \in A_i$, for $i = 1, 2, ..., n$, and $\cup_{i=1}^n A_i = A$.

Let δ be a positive function defined on A. A partition $P = \{(A_1, x_1), ..., (A_n, x_n)\}$ is called δ-fine if $A_i \subset (x_i - \delta(x_i), x_i + \delta(x_i))$, for $i = 1, 2, ...n$. We first give the definition of Henstock-Kurzweil integration on one dimension.

Definition 1 *A real-valued function f is said to be Henstock-Kurzweil integrable (or simply HK-integrable) with value I on $[a, b]$ if for every $\epsilon > 0$ there is a positive function δ on $[a, b]$ such that*

$$\left| \sum_{i=1}^n f(x_i)\,|A_i| - I \right| < \epsilon \tag{3}$$

for each δ-fine partition P of A, where $|A_i|$ denotes the length of $A_i, i = 1, 2, ..., n$. In such case, we write $\int_a^b f\ dx$ or simply $\int_a^b f$.

The next theorem says that there is no improper Henstock-Kurzweil integrals (see [4]).

Theorem 2 *Let f be a real-valued function defined on $[a, b]$,and let f be HK-integrable over $[c, b]$ for each $c \in (a, b]$. If the finite limit $\lim_{c \to a^+} \int_c^b f = I$. Then*

$$\int_a^b f = \lim_{c \to a^+} \int_c^b f = I. \tag{4}$$

The following is another way of stating the Theorem above.

Theorem 3 *Let f be HK-integrable over $[c, b]$ for each $c \in (a, b)$. If given $\epsilon > 0$ there exists $c \in (a, b)$ such that*

$$\left| \int_a^c f \right| < \epsilon, \tag{5}$$

then f is HK-integrable over $[a, b]$.

Proof. The condition states that f is HK-integrable over $[a,c]$, and since f is HK-integrable over $[c,b]$, this implies that f is integrable over $[a,b]$. ■

We shall see how this can be used for our numerical computations. First, we introduce uneven partition and integration quadratures.

2 One dimensional quadrature

Definition 4 *A matrix A with positive a_{nk} is called uniformly regular if the following conditions are satisfied:*
(i) $\lim_{n\to\infty} a_{nk} = 0$ *uniformly over* k.
(ii) $\sum_{k=1}^{n} a_{nk} = 1$.

For example, we may use the finite sum formula, $\sum_{k=1}^{n} k^m$, $m = 1, 2, \ldots$, to form uniform regular matrices. If we define the matrix $a_{nk} = \frac{2(b-a)k}{n(n+1)}$, and choose $a = 0, b = 1$, the following rows shows the partition for the interval of $[0,1]$ when $n = 1, 2, \ldots, 10$.

$$\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{10} & \frac{1}{5} & \frac{3}{10} & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{15} & \frac{2}{15} & \frac{1}{5} & \frac{4}{15} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{21} & \frac{2}{21} & \frac{1}{7} & \frac{4}{21} & \frac{5}{21} & \frac{2}{7} & 0 & 0 & 0 & 0 \\
\frac{1}{28} & \frac{1}{14} & \frac{3}{28} & \frac{1}{7} & \frac{5}{28} & \frac{3}{14} & \frac{1}{4} & 0 & 0 & 0 \\
\frac{1}{36} & \frac{1}{18} & \frac{1}{12} & \frac{1}{9} & \frac{5}{36} & \frac{1}{6} & \frac{7}{36} & \frac{2}{9} & 0 & 0 \\
\frac{1}{45} & \frac{2}{45} & \frac{1}{15} & \frac{4}{45} & \frac{1}{9} & \frac{2}{15} & \frac{7}{45} & \frac{8}{45} & \frac{1}{5} & 0 \\
\frac{1}{55} & \frac{2}{55} & \frac{3}{55} & \frac{4}{55} & \frac{1}{11} & \frac{6}{55} & \frac{7}{55} & \frac{8}{55} & \frac{9}{55} & \frac{2}{11}
\end{array}$$

We introduce two quadratures in 1-D:

1. **The open type quadrature**

$$Q_n^1(f) = \sum_{k=2}^{n} \frac{a_{nk}}{2} \left(f(u_{n,k-1}) + f(u_{nk}) \right). \tag{6}$$

In this case we are dealing with a function having a singularity at $x = a$ in the interval $[a,b]$, our quadrature is to avoid the singularity at $x = a$.

2. **The closed type quadrature**

$$Q_n^2(f) = \sum_{k=1}^{n} \frac{a_{nk}}{2} \left(f(u_{n,k-1}) + f(u_{nk}) \right) \tag{7}$$

or

$$Q_n^2(f) = \frac{1}{2} a_{n1} f(u_{n1}) + \sum_{k=2}^{n} \frac{a_{nk}}{2} \left(f(u_{n,k-1}) + f(u_{nk}) \right) \tag{8}$$

where $u_{nk} = a + \sum_{i=1}^{k} a_{ni}$, and $u_{n,0} = a$.

Remark:

(1) By looking the $a_{nk} = \dfrac{2(b-a)k}{n(n+1)}$,we notice that for each n, we have $a_{n1} < a_{n2} < ... < a_{nn}$ and $\sum_{k=1}^{n} a_{nk} = b - a$, which is the basis of our choice of uneven partitions.
(2) Both quadratures are similar to the trapezoidal rule except we are using uneven partitions, which are the essence of the HK-integration.
(3) In the closed type quadrature, if the Eq. 7 contains the singularity at the end point at $x = a$, or $x = b$; subsequently, we set $f(a) = 0$ or $f(b) = 0$ in such case.
(4) In Eq. 8, we consider the integral value of f over the first interval $[a, u_{n1}]$. In other words, we ignore the singularity at $x = a$. We shall see we apply this quadrature for functions that are monotonic and have singularities near the end point.

2.1 Error in 1-D

We consider the function f in each subinterval $[u_{n,k-1}, u_{nk}]$, we quote a theorem in [6], which can be used to estimate the error in 1-D.

Theorem 5 *Let C be the simple curve, $\mathbf{w}(t) = (x(t), y(t))$, $t_1 \le t \le t_2$. Let R be the region bounded by C, by the line $y = mx + b$ (which does not intersect C) and by the perpendiculars to the line from $(x(t_1), y(t_1))$ and $(x(t_2), y(t_2))$. Then the area of R is given by*

$$\frac{1}{1+m^2}\int_{t_1}^{t_2} \left(-x(t)m + y(t) - b\right)\left(x'(t) + y'(t)m\right) dt. \tag{9}$$

We interpret the parametric curve in the function setting $(x, f(x))$, when $x \in (u_{n,k-1}, u_{nk})$. We write the slope and the y-intercept in the interval $(u_{n,k-1}, u_{nk})$ as M_{nk} and B_{nk} below respectively,

$$M_{nk} = \frac{f(u_{nk}) - f(u_{n,k-1})}{u_{nk} - u_{n,k-1}}, \text{ and} \tag{10}$$

$$B_{nk} = f(u_{n,k-1}) - M_{nk} \cdot u_{n,k-1}. \tag{11}$$

The error of the open type quadrature is

$$\frac{1}{1+M_{nk}^2}\int_{u_{n,k-1}}^{u_{nk}} (-xM_{nk} + f(x) - B_{nk})(1 + f'(x)M_{nk})dx. \tag{12}$$

Thus, the accumulative error of the closed type quadrature, $Q_n^2(f)$ of Eq. 7 over the entire interval $[a, b]$, when we partition it into n subintervals using the matrix a_{nk}, is

$$E_n^1(f) = \sum_{k=1}^{n}\left[\frac{1}{1+M_{nk}^2}\int_{u_{n,k-1}}^{u_{nk}} (-xM_{nk} + f(x) - B_{nk})(1 + f'(x)M_{nk})dx\right]. \tag{13}$$

The Formula 13 is useful since only the first derivative is needed for calculation and we can use a computational tool to evaluate $E_n^1(f)$.

Following the idea mentioned in [2], we can show that

Theorem 6 *If f is Riemann integrable over $[c, b]$ for each $c \in (a, b]$ and improper Riemann integrable over the interval $[a, b]$. Then f is HK-integrable over $[a, b]$, and we have*

$$\int_a^b f = \lim_{n\to\infty} Q_n^1(f) = \lim_{n\to\infty} \left(\sum_{k=2}^{n} \frac{a_{nk}}{2} \left(f(u_{n,k-1}) + f(u_{nk})\right) \right), \tag{14}$$

where $Q_n^1(f)$ is the quadrature applied on the interval $[a, b]$.
In particular, if function f is monotonic over $(a, b]$ and has a singularity at $x = a$, Eq. 14 can be replaced by

$$\int_a^b f = \lim_{n\to\infty} Q_n^2(f) = \lim_{n\to\infty} \left(\frac{1}{2} a_{n1} f(u_{n1}) + \sum_{k=2}^{n} \frac{a_{nk}}{2} \left(f(u_{n,k-1}) + f(u_{nk})\right) \right), \tag{15}$$

Proof. f is Riemann integrable over $[c, b]$. From the Theorem 20.1 in [2], we have

$$\int_c^b f = \lim_{m\to\infty} \sum_{k=1}^{m} b_{mk} f(x_{mk}) = \lim_{m\to\infty} \sum_{k=1}^{m} \frac{b_{mk}(f(v_{m,k-1}) + f(v_{mk}))}{2},$$

where $v_{m,k-1} = c + \sum_{i=0}^{k-1} b_{mk}$, $v_{mk} = c + \sum_{i=0}^{k} b_{mk}$, and $v_{m,k-1} \leq x_{mk} \leq v_{mk}$.

Since f is improper Riemann integrable on $[a, b]$, we have

$$\lim_{c\to a^+} \int_c^b f = \int_a^b f.$$

Let $c = u_{n1} = a + a_{n1}$, $b_{mk} = a_{n,k+1}$, $k = 1, ..., m$, so we have

$$\lim_{n\to\infty} \int_{u_{n1}}^b f = \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{a_{n,k+1}(f(u_{n,k}) + f(u_{n,k+1}))}{2} = \lim_{n\to\infty} \sum_{k=2}^{n} \frac{a_{n,k}(f(u_{n,k-1}) + f(u_{n,k}))}{2}.$$

When f is monotonic over $(a, b]$ and has a singularity at $x = a$, we note that $\lim_{n\to\infty} a_{n1} = 0$, and Eq. 15 follows. ■

Next, we describe how we can use Eq. 6 to approximate the integral for a function over the interval $[a, b]$, which has a singularity at the end point $x = a$. Assume the conditions of Theorem 6 are met, f is HK-integrable over $[a, b]$, then given $\epsilon > 0$ there exists a number $c \in (a, b]$ such that

$$\left| \int_a^c f \right| < \epsilon \tag{16}$$

1. If we write $\int_a^b f = \lim_{n\to\infty} \left(\sum_{k=2}^{n} \left[\frac{a_{nk}}{2} f(u_{n,k-1}) + f(u_{nk}) \right] \right)$. Given $\epsilon > 0$ there exists $c \in (a, b]$ and a positive integer N such that if $n \geq N$, we have $u_{n1} \in (a, c]$ satisfying

$$\left| \int_a^{u_{n1}} f \right| < \left| \int_a^c f \right| < \epsilon/2 \tag{17}$$

and

$$\left| \int_{u_{n1}}^b f - \sum_{k=2}^{n} \left[\frac{a_{nk}}{2} f(u_{n,k-1}) + f(u_{nk}) \right] \right| < \epsilon/2. \tag{18}$$

Thus, we can write

$$\left|\int_a^b f - Q_n^1(f)\right| \leq \left|\int_a^{u_{n1}} f\right| + \left|\int_{u_{n1}}^b f - \sum_{k=2}^n \left[\frac{a_{nk}}{2} f(u_{n,k-1}) + f(u_{nk})\right]\right| < \epsilon. \tag{19}$$

We can use Formula 13 to find the error for $\left|\int_{u_{n1}}^b f - \sum_{k=2}^n \left[\frac{a_{nk}}{2} f(u_{n,k-1}) + f(u_{nk})\right]\right|$, which can tell us how many partitions n is needed to take to achieve the required accuracy of $\epsilon/2$.

2. On the other hand, in case f is monotonic and has a singularity at $x = a$, we write

$$\int_a^b f = \lim_{n\to\infty} Q_n^2(f) = \lim_{n\to\infty} \left(\frac{1}{2} a_{n1} f(u_{n1}) + \sum_{k=2}^n \frac{a_{nk}}{2} \left(f(u_{n,k-1}) + f(u_{nk})\right)\right).$$

Given $\epsilon > 0$ there exists $c \in (a, b]$ and a positive integer N such that if $n \geq N$, we find $u_{n1} \in (a, c]$ such that

$$\left|\int_a^{u_{n1}} f\right| < \left|\int_a^{a+c} f\right| < \epsilon/4, \tag{20}$$

$$a_{n1} \left|f(u_{n1})\right| < \epsilon/2 \text{ and} \tag{21}$$

$$\left|\int_{u_{n1}}^b f - \sum_{k=2}^n \frac{a_{nk}}{2} \left(f(u_{n,k-1}) + f(u_{nk})\right)\right| < \epsilon/2. \tag{22}$$

Thus, we can write

$$\begin{aligned}\left|\int_a^b f - Q_n^1(f)\right| &\leq \left|\int_a^{u_{n1}} f - \frac{a_{n1} f(u_{n1})}{2}\right| \\ &\quad + \left|\int_{u_{n1}}^b f - \sum_{k=2}^n \frac{a_{nk}}{2} \left(f(u_{n,k-1}) + f(u_{nk})\right)\right| \\ &< \left|\int_a^{u_{n1}} f\right| + \left|\frac{a_{n1} f(u_{n1})}{2}\right| \\ &\quad + \left|\int_{u_{n1}}^b f - \sum_{k=2}^n \frac{a_{nk}}{2} \left(f(u_{n,k-1}) + f(u_{nk})\right)\right| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{23}$$

Remark:

(1) When we do not know the exact numeric value for $\int_a^b f$, and use the open type quadrature $Q_n^1(f)$ to approximate $\int_a^b f$, we may use Eq. 18 to see how many partitions are needed or how we can choose u_{n1} in the interval $[u_{n1}, b]$. However, if we do not know the exact value of $\int_a^{u_{n1}} f$, we do not know the smallest positive integer n is needed for satisfying Eq. 19. Similar conclusion can be drawn when f is monotonic and has a singularity at $x = a$.

(2) For a monotone function f that has a singularity at $x = a$, it is clear that the closed type quadrature $Q_n^2(f) = \dfrac{1}{2}a_{n1}f(u_{n1}) + \sum_{k=2}^{n} \frac{a_{nk}}{2}\left(f(u_{n,k-1}) + f(u_{nk})\right)$ is better than the open type quadrature $Q_n^1(f)$.

The following example shows how we use $Q_n^1(f)$ and $Q_n^2(f)$ respectively to approximate an improper integral.

Example 7 *We define* $f(x) = \dfrac{1}{\sqrt{x}}$ *if* $x \neq 0$, *and* $f(0) = 0$. *It is easy to prove that* f *is HK-integrable over* $[0, 1]$ *and we note the followings.*
We choose $a_{nk}=\dfrac{2k}{n(n+1)}$. *With the help of Maple, we can compute directly that*

$$\lim_{n\to\infty} Q_n^1(f) = \lim_{n\to\infty}\left(\sum_{k=2}^{n}\frac{a_{nk}}{2}\left(f(u_{n,k-1}) + f(u_{nk})\right)\right) = 2, \ \textit{and}$$
$$\lim_{n\to\infty} Q_n^2(f) = \lim_{n\to\infty}\left(\frac{1}{2}a_{n1}f(u_{n1}) + \sum_{k=2}^{n}\frac{a_{nk}}{2}\left(f(u_{n,k-1}) + f(u_{nk})\right)\right) = 2.$$

Since f *is monotone, we see from the following Matlab computations that the closed quadrature (see Matlab Example 7 closed1d.m) gives a better estimate than the open quadrature (see Matlab Example 7 open1d.m):*

$$\begin{array}{ll}
Q_{1000}^2(f) = 1.998246073 & Q_{1000}^1(f) = 1.997539274\\
Q_{1100}^2(f) = 1.998405479 & Q_{1100}^1(f) = 1.997762942\\
Q_{1200}^2(f) = 1.998538346 & Q_{1200}^1(f) = 1.997949351\\
Q_{1300}^2(f) = 1.998650698 & Q_{1300}^1(f) = 1.998106978\\
Q_{1400}^2(f) = 1.998747175 & Q_{1400}^1(f) = 1.998242279\\
Q_{1500}^2(f) = 1.998830614 & Q_{1500}^1(f) = 1.998359368
\end{array}$$

With the help of Matlab (see Matlab Example 7 error1d.m), we may use Formula 13 to compute

$$E_n^1(f) = \int_{u_{n1}}^{b} f - \sum_{k=2}^{n}\left[\frac{a_{nk}}{2}\left(f(u_{n,k-1}) + f(u_{nk})\right)\right].$$

For example, when $n = 1000$ *is used, we get* $E_n^1(f) = -3.663451630525485e - 004$ *in* $[a_{n,1}, 1]$. *When* $n = 1500$ *is used, we get* $E_n^1(f) = -2.443817648409486e - 004$ *in* $[a_{n,1}, 1]$. *As expected, we see that* $\left|E_{1500}^1(f)\right| < \left|E_{1000}^1(f)\right|$.

2.2 Highly Oscillatory in 1-D

In this section, we describe how we integrate a function that is highly oscillatory near the singularity. An HK-integrable function is called non-absolute if f is integrable but not $|f|$. In other words, we describe ways of computing a type of non-absolute HK-integrable function in this sub-section.

Theorem 8 *Let* $\{x_r\} \to a^+$, *and* $A_r = \lim_{n\to\infty} Q_n^1(f)$ *in* $[x_{r+1}, x_r]$, *where* $x_0 = b$. *If* $\sum_{r=0}^{\infty} A_r$ *converges, then* f *is HK-integrable over* $[a, b]$ *and*

$$\int_a^b f = \sum_{r=0}^{\infty} A_r. \tag{24}$$

Proof. $A_r = \lim_{n\to\infty} Q_n^1(f)$ in $[x_{r+1}, x_r]$, so f is HK-integrable on $[x_{r+1}, x_r]$ and $I_r = \int_{x_{r+1}}^{x_r} f$.

For each non-negative integer r, choose a positive function δ_r on $[x_{r+1}, x_r]$ such that $|f(P) - \int_{x_{r+1}}^{x_r} f| < \frac{\epsilon}{2^{r+1}}$ whenever P is a δ_r-fine partition on $[x_{r+1}, x_r]$. Let $I_r = (x_{r+1}, x_r)$ and define a positive function δ on $(a, b]$ by

$$\delta(x) = \begin{cases} min\{\delta_0(b), b - x_1\} & x = b \\ min\{\delta_r(x), \rho(x, \mathcal{C}I_r) & x \in I_r \\ min\{\delta_r(x_{r+1}), \delta_{r+1}(x_{r+1}), \ell(I_r), \ell(I_{r+1})\} & x = x_{r+1} \end{cases},$$

where $\mathcal{C}I_r = [x_{r+1}, x_r]$.

$$|f(P) - \int_a^b f| \leq \sum_{r=0}^{\infty} |f(P_r) - \int_{x_{r+1}}^{x_r} f| < \sum_{r=0}^{\infty} \frac{\epsilon}{2^{r+1}} = \epsilon$$

So f is HK-integrable and $\int_a^b f = \sum_{r=0}^{\infty} \int_{x_{r+1}}^{x_r} f = \sum_{r=0}^{\infty} A_r$. ■

Similarly, we have the following Theorem.

Theorem 9 *Let* $\{x_r\} \to a^+$, *and* $A_r = \lim_{n\to\infty} Q_n^r(f)$ *in* $[x_{r+1}, x_r]$, $r = 0, 1, 2, ...,$ *with* $x_0 = b$. *If for each* r, *there is an* $M > 0$, *and a positive integer* N_r *such that if* $n \geq N_r$, *we have*

$$|Q_n^r(f)| < \frac{M}{2^r}. \tag{25}$$

Then f *is HK-integrable over* $[a, b]$ *and*

$$\int_a^b f = \sum_{r=0}^{\infty} A_r. \tag{26}$$

Proof. Let $\{x_r\} \to a^+$. For each r, since $A_r = \lim_{n\to\infty} Q_n^r(f)$ in $[x_{r+1}, x_r]$, we can find positive integer $N_r^* \geq N_r$ such that if $n \geq N_r^*$, we have

$$|Q_n^r(f)| < \frac{M}{2^r} \tag{27}$$

$$|A_r - Q_n^r(f)| < 2^{-r} \text{ or} \tag{28}$$

$$\left| \int_{x_{r+1}}^{x_r} f - Q_n^r(f) \right| < 2^{-r}. \tag{29}$$

We shall show Theorem 3 holds. For all $\epsilon > 0$, we need to find r so that the following is true:

$$\left|\int_a^{x_r} f\right| \le \sum_{m=r}^{\infty}\left|\int_{x_{m+1}}^{x_m} f\right| \tag{30}$$

$$\le \sum_{m=r}^{\infty}\left|\int_{x_{m+1}}^{x_m} f - Q_n^m(f)\right| + \sum_{m=r}^{\infty}|Q_n^m(f)| \tag{31}$$

$$\le \sum_{m=r}^{\infty} 2^{-m} + \sum_{m=r}^{\infty}\frac{M}{2^m} \tag{32}$$

$$\le \frac{1}{2^{r-1}} + \frac{M}{2^{r-1}} = \frac{1}{2^{r-1}}(M+1) < \epsilon, \tag{33}$$

which implies that if $r > \left[\dfrac{\ln(\dfrac{M+1}{\epsilon})}{\ln 2} + 1\right] + 1$, then Theorem 3 holds, where $[x]$ denotes the smallest integer greater than or equal to x. ■

Follow the ideas mentioned above, we can generalize the previous result as follows:

Theorem 10 *Let $\{x_r\} \to a^+$, and $A_r = \lim_{n\to\infty} Q_n^r(f)$ in $[x_{r+1}, x_r]$, $r = 0, 1, 2, ...,$ with $x_0 = b$. If for each r, there is an $M > 0$, and a positive integer N_r such that if $n \ge N_r$, we have*

$$|Q_n^r(f)| < Mg(r). \text{ and} \tag{34}$$

$$\sum_{r=1}^{\infty} g(r) < \infty. \tag{35}$$

Then f is HK-integrable over $[a, b]$ and

$$\int_a^b f = \sum_{r=0}^{\infty} A_r. \tag{36}$$

Proof. Let $\{x_r\} \to a^+$. For each r, since $A_r = \lim_{n\to\infty} Q_n^r(f)$ in $[x_{r+1}, x_r]$, we can find positive integer $N_r^* \ge N_r$ such that if $n \ge N_r^*$, we have

$$|Q_n^r(f)| < Mg(r) \tag{37}$$

$$|A_r - Q_n^r(f)| < g(r) \text{ or} \tag{38}$$

$$\left|\int_{x_{r+1}}^{x_r} f - Q_n^r(f)\right| < g(r). \tag{39}$$

Given $\epsilon > 0$, we shall find necessary r from the following observation:

$$\left|\int_a^{x_r} f\right| \le \sum_{m=r}^{\infty}\left|\int_{x_{m+1}}^{x_m} f\right| \tag{40}$$

$$\le \sum_{m=r}^{\infty}\left|\int_{x_{m+1}}^{x_m} f - Q_n^m(f)\right| + \sum_{m=r}^{\infty}|Q_n^m(f)| \tag{41}$$

$$\le \sum_{m=r}^{\infty} g(m) + \sum_{m=r}^{\infty} Mg(m) \tag{42}$$

$$\le \left(\sum_{m=r}^{\infty} g(m)\right)(1+M) < \epsilon. \tag{43}$$

Thus we solve for r such that

$$\left(\sum_{m=r}^{\infty} g(m)\right) < \frac{\epsilon}{1+M}. \tag{44}$$

■

Example 11 *We define* $f(x) = \frac{1}{x}\sin(\frac{1}{x})$ *if* $x \neq 0$, *and* $f(0) = 0$. *We shall show that* f *is HK-integrable though not Lebesgue integrable over* $[0,1]$ *when we prove its two dimensional extension in the next section. We demonstrate how we approximate the integral* $\int_0^1 f$. *We first note the followings:*
(1) We can't apply the uneven partition and quadrature over the interval $[0,1]$ *in one step. Instead, we construct a sequence* $\{x_n\}$ *converges to* 0.
(2) In other words, we select

$$x_i = 5^{-(i-1)} \tag{45}$$

for $i = 1, 2, ...$ *We approximate the integral of* f *in each* $I_i = [x_{i+1}, x_i]$ *and denote the integral of* f *over* I_i *by* A_i *when applying the closed type quadrature* $Q_n^2(f) = \sum_{k=1}^{n}\frac{a_{nk}}{2}\left(f(u_{n,k-1}) + f(u_{nk})\right)$, *where* $u_{n0} = x_{i+1}$,*and* $u_{nk} = x_{i+1} + \sum_{i=1}^{k} a_{ni}$.
(3) Therefore,

$$\int_0^1 f = \sum_{i=1}^{r} A_i, \tag{46}$$

for some r. *If we use the matrix* $a_{nk} = \dfrac{2k}{n(n+1)}$ *and the closed type quadrature in each* $I_i = [x_{i+1}, x_i]$ *for* $i = 1, 2, ...6$,*and with the help of Matlab and the closed type* $Q_n^2(f)$ *,we obtain the following (see Matlab Example 11 closed1d.m and example11.m):*

$$\begin{array}{ll} A_1 = 6.038477253680541e-001, n = 2000 & A_4 = 7.832784472836141e-003, n = 2500 \\ A_2 = -1.844869863026124e-002, n = 2000 & A_5 = -1.350711522280514e-003, n = 2500 \\ A_3 = 3.305131047013245e-002, n = 2500 & A_6 = -1.996700125703929e-004, n = 4500. \end{array} \tag{47}$$

Thus

$$\int_0^1 \frac{1}{x}\sin\frac{1}{x}dx \tag{48}$$

is approximately equal to $\sum_{i=1}^{6} A_i = 6.247327401459105e-001$.

3 Two Dimensional Henstock Integrals

The Henstock integral and our quadratures can be stated in higher dimensions. First we shall define Henstock integral in higher dimension. Let A be a fixed interval in the $n-$dimensional Euclidean space. A division $D = \{(I, x)\}$ of A is a collection of finite number of interval-point pairs (I, x) such that the intervals are pair-wise non-overlapping and their union is A. If the union is a subset of A,we call such D a partial division. For example, in one dimension a division D of an interval $[a, b]$ is given by $a = x_0 < x_1 < ... < x_n = b$. If $I = [x_{i-1}, x_i]$ and $x \in I$,we use $(D) \sum f(x) |I|$ to denote $\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$. Note that in higher dimension $|I|$ represents the volume of the I. A division $D = \{(I, x)\}$ is said to be $\delta - fine$ if for each interval-point (I, x), we have $x \in I \subset S(x, \delta(x))$, where $x$is a vertex of I and $S(x, \delta(x))$ is an open sphere with center x and radius $\delta(x)$. For simplicity, we give the definition of Henstock integration on only two dimensions below.

Definition 12 *A real-valued function f is said to be Henstock-Kurzweil integrable (or simply HK-integrable) with value I on an interval $A = [a, b] \times [c, d]$ if for every $\epsilon > 0$ there is a positive function δ on $[a, b] \times [c, d]$ such that*

$$\left| \sum_{i=1}^{n} f(x_i) |A_i| - I \right| < \epsilon \tag{49}$$

for each δ-fine partition P of A, where $|A_i|$ denotes the area of $A_i, i = 1, 2, ..., n$. In such case, we write $\int_A f\, dA$ or simply $\int f$.

First, we introduce uneven partition and integration quadratures.

For a real-valued function $f : [a, b] \times [c, d] \to R$, we define the following **open type 2D quadrature**:

$$\begin{aligned} Q_{nm}^{3}(f) = \sum_{l=2}^{m} \sum_{k=2}^{n} \frac{a_{nk} b_{ml}}{4} (f(u_{n,k-1}, v_{m,l-1}) \\ + f(u_{nk}, v_{m,l-1}) + f(u_{n,k-1}, v_{ml}) + f(u_{nk}, v_{ml})) \end{aligned} \tag{50}$$

Similar to the 1-D case, we open type 2-D quadrature $Q_{nm}^{3}(f)$ to handle singularities lying on the edges. We also consider the following **closed type 2-D quadrature**:

$$\begin{aligned} Q_{nm}^{4}(f) = \sum_{l=2}^{m} \sum_{k=2}^{n} \frac{a_{nk} b_{ml}}{4} (f(u_{n,k-1}, v_{m,l-1}) + f(u_{nk}, v_{m,l-1}) + \\ f(u_{n,k-1}, v_{ml}) + f(u_{nk}, v_{ml})) + \frac{a_{n1} b_{m1}}{4} f(u_{n1}, v_{m1}) \\ \sum_{k=2}^{n} \frac{a_{nk} b_{ml}}{4} (f(u_{n,k-1}, v_{m1}) + f(u_{nk}, v_{m1})) + \\ \sum_{l=2}^{m} \frac{a_{nk} b_{ml}}{4} (f(u_{n,1}, v_{m,l-1}) + f(u_{n1}, v_{ml})) \end{aligned} \tag{51}$$

in which $u_{n0} = a, u_{nk} = a + \sum_{p=1}^{k} a_{np}$ for $k = 1, 2, ..., n$, $v_{m0} = c$, and $v_{ml} = c + \sum_{q=1}^{l} b_{mq}$ for $l = 1, 2, ..., m$; note that $u_{nn} = b$ and $v_{mm} = d$. The following theorem is analogy to Theorem 6 in 1D, which we omit its proof.

Theorem 13 *If f is Riemann integrable over $[c, b] \times [d, f]$ for each $c \in (a, b]$ and $d \in (e, f]$, and is improper Riemann integrable over the interval $[a, b] \times [e, f]$. Then*

$$\int\int_{[a,b]\times[e,f]} f = \lim_{m\to\infty} \lim_{n\to\infty} Q^3_{nm}(f), \tag{52}$$

or

$$\int\int_{[a,b]\times[e,f]} f = \lim_{m\to\infty} \lim_{n\to\infty} Q^4_{nm}(f). \tag{53}$$

3.1 2D Error

We consider a rectangle in the xy plane enclosed by four points $(a, b), (a + h, b), (a, b + k)$ and $(a + h, b + k)$. Next we define a function on this rectangle and we label $A = (a, b, f(a, b)), B = (a+h, b, f(a+h, b)), C = (a, b+k, f(a, b))$, and $D = (a+h, b+k, f(a+h, b+k))$. We first note that the volume for the trapezium bounded by $ABCD$ is

$$\left[\frac{f(a, b) + f(a + h, b)}{2}\right] hk. \tag{54}$$

We note that this trapezium is taking the weights of two points, A and B into consideration. Thus, if we replace h and k by uneven partitions a_{nk} and by b_{ml} respectively, we have a **two point quadrature**, say

$$\left[\frac{f(a, b) + f(a + a_{nk}, b)}{2}\right] a_{nk}b_{ml}, \tag{55}$$

and we write

$$\int\int_{[a,a+a_{nk}]\times[b,b+b_{ml}]} f = \left[\frac{f(a, b) + f(a + a_{nk}, b)}{2}\right] a_{nk}b_{ml} + E^2_{mn}(f). \tag{56}$$

In view of 2D closed quadrature 49, we may write

$$\begin{aligned}\int\int_{[a,a+a_{nk}]\times[b,b+b_{ml}]} f = \frac{a_{nk}b_{ml}}{4}&(f(u_{n,k-1}, v_{m,l-1}) + f(u_{nk}, v_{m,l-1}) \\ &+ f(u_{n,k-1}, v_{ml}) + f(u_{nk}, v_{ml})) + E^4_{mn}(f),\end{aligned} \tag{57}$$

which we call it a **four point quadrature** in 2D. Intuitively, the four point quadrature gives a better approximation since we are taking the average of weights by using four points; in other words, we have

$$E^4_{mn}(f) < E^2_{mn}(f). \tag{58}$$

Next, we should describe an error when three points are used. We now replace C by $C^* = (a, b + k, f(a, b + k))$ and we consider the plane P determined by three points, A, B, and C^*; we set the normal vector for P to be $\overrightarrow{n} = \overrightarrow{AB} \times \overrightarrow{AC^*}$. Assume $\overrightarrow{n} = (n_1, n_2, n_3)$, then the plane equation for P can be written as

$$n_1(x - a) + n_2(y - b) + n_3(z - f(a, b)) = 0, \tag{59}$$

or

$$n_1x + n_2y + n_3z = n_1a + n_2b + n_3f(a, b). \tag{60}$$

If we use the plane P described above and apply the theorem below modified from [6], we see that volume below shall give good approximation of the error when 2D closed type quadrature 49 is used.

Theorem 14 *Let S be a surface determined by the differentiable function $f(x, y)$ in $(a, b) \times (c, d)$ and continuous in $[a, b] \times [c, d]$. Let P by the plane determined by three points $(a, c, f(a, c)), (b, c, f(b, c))$ and $(a, d, f(a, d))$,and R be the region bounded by S, by the plane P, and by the perpendicular projection from the surface to the plane P. If the plane equation for P is $Ax + By + Cz = D$. Then the volume of R is given by*

$$\int\int r(x, y)dpdq \tag{61}$$

$$= \int_c^d \int_a^b r(x, y) \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{vmatrix} dxdy,$$

where $\begin{bmatrix} p(x, y) \\ q(x, y) \\ r(x, y) \end{bmatrix} = [p_1, p_2, p_1 \times p_2]^{-1} \begin{bmatrix} x \\ y \\ f(x, y) - \frac{D}{C} \end{bmatrix}$ *, and p_1 and p_2 are two orthonormal basis in the plane P.*

Remark: If we apply the uneven partition matrices, a_{nk} and b_{ml} in each rectangle $[a, a + a_{nk}] \times [b, b + b_{ml}]$, the volume R represented by the double integral in Theorem 14 can be interpreted as the estimate of the error for $\int\int_{[a,a+a_{nk}]\times[b,b+b_{ml}]} f$, when three points $(a, b, f(a, b)), (a + a_{nk}, b, f(a + a_{nk}, b))$ and $(a, b + b_{ml}, f(a, b + b_{ml}))$ are used; we denote such error by $E^3_{mn}(f)$. Clearly, we have

$$E^4_{mn}(f) \le E^3_{mn}(f) \le E^2_{mn}(f). \tag{62}$$

3.2 Special Type of Functions With Singularities

As we have discussed on 1-D monotone function, there is a reason we pick an uneven partition using a *uniformly regular* matrix to make our convergence faster. In the following example, we describe how we can make use of the uneven partitions to approximate an improper Riemann integral $\int\int_{[a,b]\times[c,d]} f$, where $f : [a, b] \times [c, d] \to R$ and posses singularities near $x = a$ and $y = c$.

Example 15 *We define $f(x, y) = \dfrac{1}{\sqrt{xy}}$ if $(x, y) \in (0, 1] \times (0, 1]$ and $f(x, y) = 0$ when $xy = 0$. It is easy to see that f is HK-integrable over $[0, 1] \times [0, 1]$.*
If we use $a_{nk} = \dfrac{6k^2}{n(n + 1)(2n + 1)}, b_{ml} = \dfrac{6l^2}{m(m + 1)(2m + 1)}$, we can use Maple to compute the following (see Maple Examplr15.mws):

$m = n$	$Q^3_{nm}(f)$	$Q^4_{nm}(f)$
100	3.996446426	3.998131303
150	3.998194031	3.999119400
200	3.998877289	3.999481040

It is easy to see the closed type quadrature $Q^4_{nm}(f)$ gives a better approximation than the open type $Q^3_{nm}(f)$ in this case.

3.3 Highly Oscillatory Functions With Singularities

In this sub-section, we describe how a highly oscillatory, non-absolute function can be proved to be HK-integrable from theoretical point of view and we also describe how we can prove a non-absolute function to be HK-integrable from computational point of view. The next theorem describes conditions for a non-absolute integral from theoretical point of view.

Theorem 16 *Let $X \subset A = [a,b] \times [c,d]$ be closed and of measure 0 and f be Lebesgue integrable on each interval $I_i, i = 1, 2, ...,$pair-wise non-overlapping and $\cup_{i=1}^{\infty} I_i = A - X$. Assume the following function is well defined; $H(I) = \sum_{i=1}^{\infty} \int_{I_i \cap I} f$ for every interval $I \subset A$. Then f is Henstock integrable on A with the integral value $H(A)$ if and only if the following (SL) condition on X holds: for every $\epsilon > 0$ there exists $\delta(x) > 0$ such that for any $\delta - fine$ partial division $D = \{(I,x)\}$ of A with $x \in X$ we have $(D)\sum_{x \in X} |H(I)| < \epsilon$.*

Proof. The necessity follows easily from Henstock lemma (see [4] or [2]). To proves sufficiency, we assume $f(x) = 0$ when $x \in X$. Then for every $\epsilon > 0$,there is $\delta(x) > 0$ such that for any $\delta - fine$ partition $D = \{(I,x)\}$ of I_i we have

$$(D)\sum |f(x)\,|I| - F(I)| < \epsilon 2^{-i}, \tag{63}$$

for $i = 1, 2, ...,$ and for any $\delta - fine$ partial partition $D = \{(I,x)\}$ of A with $x \in X$ we have $(D)\sum |H(I)| < \epsilon$. Hence for any $\delta - fine$ partition $D = \{(I,x)\}$ of A with $D = D_1 \cup D_2$, where D_1 has x in some I_i and D_2 otherwise, we have

$$\left|(D)\sum f(x)\,|I| - H(A)\right| \leq (D_1)\sum |f(x)\,|I| - H(I)| + (D_2)\sum |H(I)| < 2\epsilon. \tag{64}$$

Therefore, f is HK integrable over A and the proof is complete. ■

We remark that [4] has proved a similar result in which he gave sufficient conditions so that f is integrable. More precisely, Pfeffer requires $H(I)$ to exist for all such $\{I_i\}$. A version of this Theorem has appeared in [5]. We shall see how this can be used for our numerical computations below. For simplicity we assume the function has singularities along the $x = a$ and $y = c$, the domain of our integration is $[a,b] \times [c,d]$, and $b - a = d - c$. We write $D_{ij} = [x_{i+1}, x_i] \times [y_{j+1}, y_j]$, where $\{x_i\} \to a^+, \{y_j\} \to c^+$ as $i, j \to \infty$. Then we compute the integral A_{ij} of f on each D_{ij} by applying the closed type quadrature $Q^4_{nm}(f)$. For convenience we write $D_i = D_{ii} \cup D_{1i} \cup .... \cup D_{i-1,i} \cup D_{i1} \cup .... \cup D_{i,i-1}$, $A_1 = A_{11}$ and $A_i = A_{ii} + A_{1i} + ... + A_{i-1,i} + A_{i1} + ... + A_{i,i-1}$ for $i = 2, 3, ...$ When there is no confusion, we use $Q^i_n(f)$ to denote the quadrature $Q^4_{nm}(f)$ on D_i. Hence the estimate of the integral of f on $[a,b] \times [c,d]$ is given by the sum $\sum_{i=1}^{r} A_i$ for some r. To prevent a subinterval to be too thin due to the partition, we may use the same uneven partitions $a_{nk} = b_{ml}$ on $[a,b]$ and $[c,d]$.

Theorem 17 *Let $\{x_i\} \to a^+$, $\{y_j\} \to c^+$and $A_i = \lim_{n\to\infty} Q^i_n(f)$. If for each i, there is an $M > 0$, and a positive integer N_i such that if $n \geq N_i$, we have*

$$\left|Q^i_n(f)\right| < \frac{M}{2^i}. \tag{65}$$

Then f is HK-integrable over $[a,b]\times[c,d]$ and

$$\int_{[a,b]\times[c,d]} f = \sum_{i=0}^{\infty} A_i. \tag{66}$$

Proof. Let $\{x_i\}\to a^+$, $\{y_j\}\to c^+$.For each i, there is an $M>0$ and a positive integer i such that if $n\geq N_i$, since $A_i=\lim_{n\to\infty} Q_n^i(f)$, we have

$$\left|Q_n^i(f)\right| < \frac{M}{2^i} \text{ and} \tag{67}$$

$$\left|A_i - Q_n^i(f)\right| < \frac{1}{2^i}. \tag{68}$$

Similar to the 1-D case, we observe the following: Given $\epsilon>0$, we shall find necessary i from the following observation:

$$\left|\int_{[a,x_i]\times[c,y_i]} f\right| \leq \sum_{m=i}^{\infty}\left|\int_{D_m} f\right| \tag{69}$$

$$\leq \sum_{m=i}^{\infty}\left|\int_{D_m} f - Q_n^m(f)\right| + \sum_{m=i}^{\infty}\left|Q_n^m(f)\right| \tag{70}$$

$$\leq \sum_{m=i}^{\infty} 2^{-m} + \sum_{m=i}^{\infty}\frac{M}{2^m} \tag{71}$$

$$\leq \frac{1}{2^{i-1}} + \frac{M}{2^{i-1}} = \frac{1}{2^{i-1}}(M+1) < \epsilon, \tag{72}$$

■

Example 18 *We define $f(x,y)=\dfrac{1}{xy}\sin(\dfrac{1}{xy})$ if $(x,y)\in(0,1]\times(0,1]$ and $f(x,y)=0$ when $xy=0$.We shall show first the integral exists in the sense of HK-integral. We next describe how we compute $\int\int_{[0,1]\times[0,1]} f$.*

We shall describe a numerical method of computing the integral of f on $[0,1]\times[0,1]$ and provide the theoretical justification later. We write $D_{ij}=[x_{i+1},x_i]\times[y_{j+1},y_j]$, where $x_i=\dfrac{1}{2^{i-1}}, y_j=\dfrac{1}{2^{j-1}}$ for $i,j=1,2,....$ Then we compute the integral A_{ij} of f on each D_{ij} applying the quadrature $Q^4_{nm}(f)$. For convenience we write $A_1=A_{11}$ and $A_i=A_{ii}+A_{1i}+....+A_{i-1,i}+A_{i1}+....+A_{i,i-1}$ for $i=2,3,....$ Hence the estimate of the integral of f on $[0,1]\times[0,1]$ is given by the sum $\sum_{i=1}^{r} A_i$ for some r. With the choice

$$a_{nk}=\frac{6k^2}{n(n+1)(2n+1)}, \text{ and } b_{ml}=\frac{6l^2}{m(m+1)(2m+1)} \tag{73}$$

with $m=n=600$ and $r=5$; with the exception that we use $m=n=900$ on A_{55} for accuracy. All *computation are done by **Matlab** below (see Matlab Example 18, closed2d.m and example18.m):*

$$\begin{array}{ll} A_{11} = 3.556911460387095e-001 & A_{24} = A_{42} = 1.114718337841618e-003 \\ A_{22} = 4.235363358640941e-002 & A_{34} = A_{43} = -6.399340315928442e-005 \\ A_{12} = A_{21} = -1.748938295935638e-001 & A_{55} = 1.554142287227561e-005 \\ A_{33} = 1.114718337841618e-003 & A_{15} = A_{51} = 1.114718337841618e-003 \\ A_{13} = A_{31} = 4.235363358640941e-002 & A_{25} = A_{52} = -6.399340315928442e-005 \\ A_{23} = A_{32} = -1.544234937543672e-002 & A_{35} = A_{53} = -1.142140513702673e-004 \\ A_{44} = -1.142140513702673e-004 & A_{45} = A_{54} = -7.395592409418208e-005 \\ A_{14} = A_{41} = -1.544234937543672e-002 & \end{array} \tag{74}$$

By summing the estimate above we obtain

$$\int\int_{[0,1]\times[0,1]} \frac{1}{xy}\sin(\frac{1}{xy}) \approx \sum_{i=1}^{5} A_i = 7.603759560620742e-002. \tag{75}$$

Claim: $(HK)\int\int_{[0,1]\times[0,1]} f$ *exists.*
We put

$$I_1 = D_{11}, I_2 = D_{22}, I_3 = D_{12}, I_4 = D_{21} \text{ and so on in Theorem 15.} \tag{76}$$

We show these D_{ij} *as follows in Figure 1:*

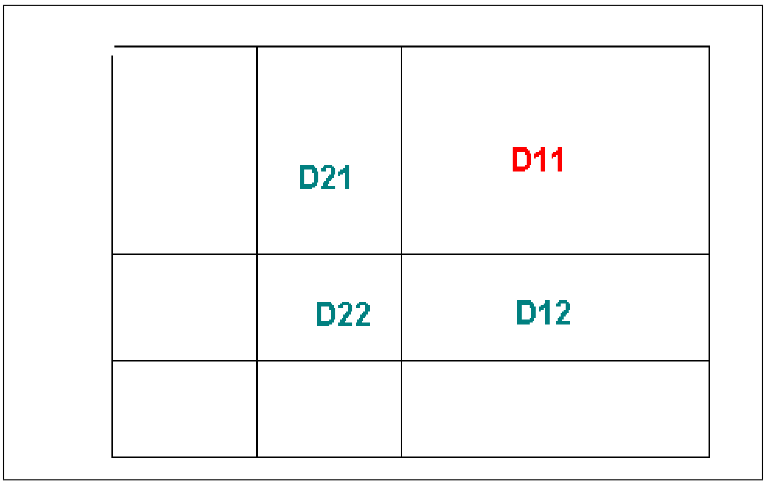

Figure 1.

We shall show that the conditions in Theorem 16 are satisfied for the function f *and therefore the above process gives an estimate of the integral value. First, we show that* $H(I)$ *exists when* $I \subset [0,1]\times[0,1]$. *It is easy to see that* $H(I)$ *exists when* $I \subset (0,1]\times(0,1]$. *Note that keeping* y *constant we have* $\frac{d(\cos\frac{1}{xy})}{dx} = \frac{1}{x^2y}\sin\frac{1}{xy}$. *Using integration by parts, we obtain for* $\epsilon > 0$

$$\int_{\epsilon}^{1}\left(\frac{1}{xy}\right)\sin\left(\frac{1}{xy}\right)dx = x\cos\left(\frac{1}{xy}\right)\Big]_{x=\epsilon}^{x=1} - \int_{\epsilon}^{1}\cos\left(\frac{1}{xy}\right)dx. \tag{77}$$

Then it follows that

$$H([0,1]\times[0,1)) = \lim_{i,j\to 0}\int_{[2^{-i},1]\times[2^{-j},1]} f = \int_0^1 \cos\left(\frac{1}{y}\right)dy - \int_0^1\int_0^1 \cos\left(\frac{1}{xy}\right)dxdy. \tag{78}$$

Hence $H(I)$ is defined for any $I \subset [0,1]\times[0,1]$. In fact, we have also shown above that the repeated integral of f on $[0,1]\times[0,1]$ exists and equal to $H([0,1]\times[0,1])$. Following the same argument above, we can prove that $|H(I)| \leq 2\gamma(\beta-\alpha)$ when $I = [0,\gamma]\times[\alpha,\beta] \subset [0,1]\times[0,1]$. Since the function is symmetric, the same inequality holds for $I = [\alpha,\beta]\times[0,\gamma] \subset [0,1]\times[0,1]$. Now for $\epsilon > 0$ we choose $4\gamma < \epsilon$ and we see that the (SL) condition holds. Consequently, the condition in Theorem.. are satisfied. The proof of this example is complete.

3.4 Avoiding or Ignoring the singularities

It is well-known from [1] that in dealing with numerical integration of functions with singularities, we apply very often the method of avoiding the singularity (our open type quadrature) or that of ignoring the singularity (our closed type quadrature). It is intuitive to see if a function is monotone with singularity near end point in 1-D or along the edges in 2-D, the closed type quadrature or ignoring the singularities work better as saw in Example 7 and 15.

On the other hand, the Example 18 above is an instant of the principle of avoiding the singularity when handling highly oscillatory functions. Suppose we add an additional term B_r to $\sum_{i=1}^{r} A_i$, where

$$B_r = B_{rr} + B_{1r} + ... + B_{r-1,r} + B_{r1} + ... + B_{r-1,r}, \tag{79}$$

and

$$B_{rr} = \int_{[0,1/2^r]\times[0,1/2^r]} f, \tag{80}$$

$$B_{1r} = \int_{[1/2,1]\times[0,1/2^r]} f, ..., \tag{81}$$

and so on, and using an adaptive trapezoidal rule (by modifying $Q^3_{nm}(f)$ so the singularities along the x or y axes are included in the calculations) on each rectangle with $f(x,y) = 0$ when $xy = 0$. Note that B_{ir} is the integral of f on a rectangles along the $x - axis$ and adjacent to that of A_{ir} for $i = 1,2,...,r-1$. Similarly, B_{ri} is the integral of f on a rectangles along the $y - axis$ and adjacent that of A_{ri} for $i = 1,2,...,r-1$. The resulting quadrature giving the estimate

$$\sum_{i=1}^{r} A_i + B_r \tag{82}$$

can be called the compound adaptive trapezoidal rule in the plane of a closed type (ignoring singularities). Unfortunately, the computation shows the closed type-ignoring singularities gives no pattern of convergence where the open type described in the Example works perfectly. This is predicable because for functions which are highly oscillatory near the singularities, we do not want to calculate the areas (or volumes in higher dimension) for those rectangles near the singularities.

3.5 Some Observations of Numerical Integration in Maple or Mathematica

1. For function such as $F(x) = x^2 \cos\left(\frac{\pi}{x^2}\right)$ if $x \neq 0$ and $F(0) = 0$, we know that $F'(x) = 2x\cos\left(\frac{\pi}{x^2}\right) + \frac{2\pi \sin\left(\frac{\pi}{x^2}\right)}{x}$ if $x \neq 0$ and $F'(0) = 0$. It is known that $F'(x)$ is not Lebesgue but $HK-$integrable in $[0, 1]$,and it follows from the **Fundamental Theorem of Calculus** that $\int_0^1 F'(x)dx = F(1) - F(0) = -1$. The method we discussed in Example 7 will work well in this case. However, both Maple 11 and Mathematica can not give us an answer.

2. They both rely on iterated integrals which are not always the value of the double integral (by Fubini's theorem).

3. For function $f(x, y) = \frac{xy}{(x^2 + y^2)^2}$ if $x^2 + y^2 > 0$ and $f(x, y) = 0$ if $x^2 + y^2 = 0$ in the region $[-1, 1] \times [-1, 1]$,
$$\int\int_{[-1,1]\times[-1,1]} f(x, y)dA \tag{83}$$
does not exist and yet the value of its repeated integrals is 0. Both Maple 11 and Mathematica give the "wrong" answer 0 when repeated integrals $\int_{-1}^1 \int_{-1}^1 f(x, y)dxdy$ are computed.

4. Both Mathematica and Maple can't handle singularities which lie on the diagonal of a region. Singularities lie on a diagonal line. Consider evaluating the following numerical integral
$$\int_0^1 \int_0^1 \cos 2\pi x \cos 2\pi y \left(\ln(x - y)^2 - \ln\left(1 + (x - y)^2\right)\right) dxdy, \tag{84}$$
both Maple and Mathematica could not give an answer due the singularities lie along $x = y$. What we will do is to transform the singularities to the boundary first and apply a quadrature which uses uniformly regular matrices for computations. Note that the function $f(x, y) = \cos 2\pi x \cos 2\pi y \left(\log(x - y)^2 - \log\left(1 + (x - y)^2\right)\right)$ is symmetric with respect to $y = x$, so we consider the integration over the triangle with vertices $O = (0, 0)$, $P = (1, 0)$ and $Q = (1, 1)$. After the transformation with change of variables, $u = x$, and $v = x - y$, the singular points are shifted to x- axis, and the Jacobian is $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \left|\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\right| = 1$. Thus, equation (1) becomes $\int_0^1 \int_0^1 \cos 2\pi u \cos 2\pi(u - v) \left(\ln v^2 - \ln\left(1 + v^2\right)\right) dudv$. By using $a_{nk} = \frac{2(b-a)k}{n(n+1)}$, and $b_{ml} = \frac{2(b-a)l}{m(m+1)}$, and write a corresponding **Matlab** program, we obtain the following information
$$Q_{400,400}(f) = -0.223374393133243$$
$$Q_{600,600}(f) = -0.223411046499008$$
$$Q_{800,800}(f) = -0.223421583469551$$
$$Q_{1000,1000}(f) = -0.223425232050112$$

4 Conclusion

Numerical integration of functions with singularities is always a difficult subject and it becomes even harder when we go from one dimension to higher dimensions. In this paper, we described quadratures involving matrices which allow us to partition an interval unevenly, which is an essence of the Henstock-Kurzweil integration. Authors would like to comment that the computation techniques described here are due to the evolving technological tools which allow users to experiment HK-integral computationally

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Supplementary Files:

[7] Matlab files for Example 7: closed1d.m, open1d.m, error1d.m.

[8] Matlab files for Example 11: closed1d.m, example 11.m.

[9] Maple file for Example 15: Example15.mws.

[10] Matlab files for Example 18: closed2d.m and example 18.m.