Enhancing Learning Of Fraction Through The Use Of Virtual Manipulatives

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1. Introduction

Educational theorists have long endorsed a theoretical basis for the use of concrete manipulatives by referring to the works of Montessori [15], Piaget [20], and Bruner [2]. McNeil and Jarvin [10] point out that according to these early theorists, "children do not come into the world with the capacity for abstract thought. Instead, they must construct abstract concepts through their interactions with concrete objects in the environment" (p. 310). More recent studies indicate that students who use physical manipulatives to explore mathematical topics perform better than students who do not use them [21]. In fact, the Singapore Mathematics Curriculum advocated the use of the concrete-pictorial-abstract (C-P-A) development of concepts at the Primary levels [13]. As pointed out by Ng [19], "[W]henever possible, activities should enable pupils to progress from concrete and pictorial levels to abstract representation", and that "great care must be demonstrated to show how the concrete representations are linked to their abstract forms".

Balka [1] believes that using concrete manipulatives allows students to not only make an important linkage between conceptual and procedural knowledge, but it allows them to recognize relationships among different branches of mathematics as well as view mathematics as an integrated whole. This notion of using concrete objects in educational settings is supported also by the research of Sternberg and Grigorenko [27], Moch [14], and Sowell [24].

With the ease and availability of technology becoming more prevalent, teachers now have the opportunity to use manipulative tools that further promote this linkage between conceptual and procedural knowledge via an electronic channel. Lee and Chen [8] and Yuan, Lee, and Wang [29] advocate that one such technological tool, the virtual manipulative, plays an important role in mathematics education. Mindellhall, Swan, Northcote, and Marshall [11], who define a virtual manipulative as "a virtual representation of a physical manipulative which, through various dynamic processes may help develop mathematical conceptual understanding" (p. 9) strongly support this view.

Moyer, Bolyard, and Spikell [17] also make a case for a clear distinction between the static from the dynamic digital representation of the concrete materials used for the teaching and learning of mathematics. Unlike the static representations which are essentially pictures, they argue for dynamic representations that can be manipulated in the same way as concrete manipulatives as true virtual manipulatives. They feel that the ability to manipulate these representations on the computer “connects the user with the real teaching and learning power of virtual manipulatives, that is, the opportunity to make meaning and see relationships as a result of one’s own actions”.

On a similar note, Moyer and Bolyard [16] feel that the students’ own manipulations of these dynamic objects are key to discovering mathematical properties and relationships. They believe that the pictorial image of the virtual manipulative does not solely portray the meaning of the mathematical ideas behind them and that the interactive nature of the virtual manipulative is vital to the students’ understanding of the mathematics. Suh, Moyer, and Heo [28] also observed that by allowing students to manipulate “on-screen objects to test hypotheses and experiment with ideas, the virtual manipulatives may more closely model the dynamic nature of thinking”, which may enhance “students’ thinking and creativity”. At the same time, they also noted that it is important to encourage students to make connections between different modes of representation of mathematical concepts so as to develop students’ representational fluency.

Mathematics educators believe that students in every grade level should be given ample opportunities to develop their mathematical literacy and competency by acquiring an understanding of mathematics as a dynamic, interesting, and ever-evolving subject ([9], [3], [7]). Ng [19] observed that the range of information and communication technologies (ICT), including virtual manipulatives, is available to help motivate and engage children in the learning process. Silk, Higashi, Shoop, and Schunn [23] share this belief and feel that mathematics taught within dynamic and well-designed technology lessons provide students additional opportunities to understand mathematics in differing contexts. That is, technology engages and empowers students with its powerful ability to make interdisciplinary connections through realistic applications, analysis of information, and interactive simulations.

The National Council of Teachers of Mathematics’ *Principles and Standards for School Mathematics* [18] and other national documents (Singapore’s Mathematics Syllabus documents for primary and secondary levels by the Ministry of Education [12], [13]) provide a rich vision for mathematics teachers of all levels on the use of educational technology to enhance the teaching and learning of mathematics for their technology-rich classrooms of the twenty-first century. In fact, the Singapore Ministry of Education has been encouraging the use of ICT in the classroom and constantly updating its ICT Masterplan (mp) over the years to cater to the ever-changing landscape of ICT and Education:

- mp1 – The first Masterplan for ICT in Education (1997 — 2002) laid a strong foundation for schools to harness ICT, particularly in the provision of basic ICT infrastructure and in equipping teachers with a basic level of ICT integration competency, which achieved a widespread acceptance for its use in education. (http://www.moe.gov.sg/media/press/2008/08/moe-launches-third-masterplan.php)
- mp2 – The second Masterplan for ICT in Education (2003 — 2008) built on this foundation to strive for an effective and pervasive use of ICT in education by, for example, strengthening the integration of ICT into the curriculum, establishing baseline ICT standards for students, and seeding innovative use of ICT among schools. (http://www.moe.gov.sg/media/press/2008/08/moe-launches-third-masterplan.php)
- mp3 – The third masterplan for ICT in Education (2009-2014)represents a continuum of the vision of the first and second Masterplans i.e. to enrich and transform the learning environments of our students and equip them with the critical competencies and dispositions to succeed in a knowledge economy. (http://www.moe.gov.sg/media/press/2008/08/moe-launches-third-masterplan.php)

Although research studies by Crawford and Brown [6] and Stellingwerf and Van Lieshout [26] have shown that virtual manipulatives can play an important role in supporting a student's conceptual understanding of general mathematical topics, few studies have been conducted that investigate the interplay between virtual manipulatives and equivalent fractions ([25], [22]). This paper recognizes both the opportunities and the challenges provided by new technologies and demonstrates the connective power of virtual manipulatives and the teaching of equivalent fractions. It showcases a model of how a teacher used a virtual manipulative investigation for exploring concepts with equivalent fractions during the primary school years.

2. Method

The study adopted a quasi-experimental mixed research approach towards investigating the impact on the use of virtual manipulative to teach the concept of equivalent fractions to Primary Three students in the Singapore School system was investigated. The sample, which was a convenient sample, consisted of two Primary Three classes in a neighbourhood school, i.e. an average achieving public school.

2.1 The Virtual Manipulative – *Fractions*

In the study, the teacher concerned sought to "enrich and transform the learning environment" of his Primary Three students in the learning of equivalent fractions using the virtual manipulative *Fractions*[1] [5].

[1] Readers may click on "*Fractions*" to try out this virtual manipulative (with the compliments from Marshall Cavendish International (Singapore) Pte Ltd).

Fractions is a virtual manipulative, which is included as part of the Primary School textbook series, Shaping Mathematics – a commonly adopted textbook package used in Singapore Primary Schools. It allows the user to compare fractions and determine if two fractions are equivalent. Two circles are presented on the screen as pictorial representations of the two fractions to be compared. By clicking on the up-down arrows next to the respective numerators and denominators, once could change the value of the proper fractions and view the respective pictorial representations of these fractions, similar to the case of using fraction discs. The pictorial representation of one fraction is being shaded yellow while the other is shaded blue. However, one constraint of *Fractions* is that there is a limitation on the size of denominator if a fraction for investigation; the largest denominator that can be represented is 24. A screen capture of *Fractions* is shown in Figure 1.

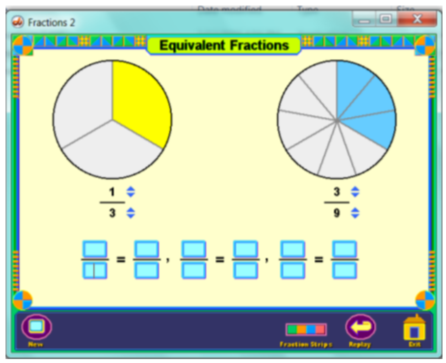


Figure 1. Screen Capture of Virtual Manipulative *Fractions*

After representing the two fractions pictorially, *Fractions* allows the user to drag one circle over another for comparison. The new representation provides a good simulation of an overlay of the two "fraction discs" – the overlapping part would be coloured green to distinguish it from the non-overlapping part so as to better allow the user to compare the two fractions. Figure 2 illustrated how the two fractions 1/3 and 3/9, for example, could be compared to determine if they are equivalent.

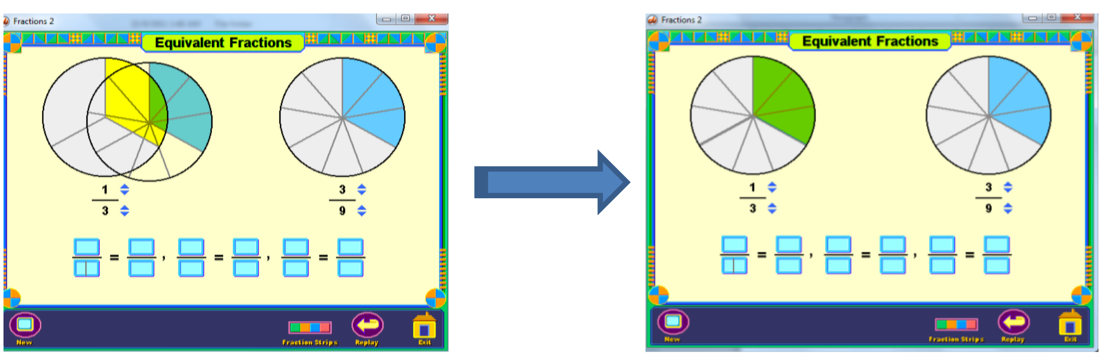


Figure 2. Manipulation of Fractions

The dynamism of the representations in *Fractions* which allows the user the level of manipulation much the same way as that in the case of concrete manipulatives certainly allow us to classify it as a true virtual manipulatives (Moyer, Bolyard, and Spikell [17]).

2.2 The Lesson

Although the C-P-A approach is advocated, the teacher's past teaching strategy adopted for the introduction of equivalent fractions has been rather deductive and rule-based in nature. The inconvenience of having to transport about 40 sets of the concrete manipulatives, such as fraction discs, to the classroom has resulted in teachers often making use of just a demonstration set. Through the demonstration set, the teacher would then illustrate how two or three fractions can be shown to be equivalent, e.g. using fraction discs to illustrate that $1/2$, $2/4$, and $3/6$ are equal. The rule in recognising how two given fractions are equivalent is then explicitly given to students before students make use of the rule to list the first 8 equivalent fractions of a given fraction with denominator less than 12 – the lesson objective.

In this lesson, the teacher concerned will deviate from his past practice. Instead he introduced equivalent fraction to the students through a more inductive approach whereby students could engage in the virtual manipulative *Fractions* to investigate and list equivalent fractions before hypothesising about the rule that governs the relationships between equivalent fractions. This was particularly made possible by the establishment of baseline standards for the students under mp2 which ensure that all students are well equipped with the skills needed to handle the basic operation of laptop computer as the lesson will be conducted in a normal classroom context using laptop computers.

For the one-hour lesson, the teacher started by using laminated circle, which were similar to fraction discs but allowing pupils to colour the parts instead, to help students to recap the concept of fraction as parts of a whole which they learnt in Primary Two (Figure 3).

Figure 3. Using Laminated Circle To Recap Fractions As Part Of Whole

The teacher then got the students to work in pairs to represent the fractions $1/2$ and $2/4$ on their respective laminated circle. He then led the students to observe that they both represent the same part of a whole and introduced the term equivalent fractions. The process was repeated using the laminated circles until the students have listed the first eight equivalent fractions of $1/2$. However, instead of the usual practice of next telling the students the rule that governs the numerators and denominators of equivalent fractions, the teacher introduced the virtual manipulative *Fractions* to the students so that they could see the equivalence of the two representations – the concrete representation and the virtual manipulative representation. Students were then given time to work on *Fractions* or the laminated circle to list the first eight equivalent fractions of $1/3$ (refer to

Worksheet shown in Figure 4). The lesson then ended with the teachers getting the students to share the equivalent fractions of 1/3 in a class discussion.

Activity 1: Create your equivalent fractions

Part A: Given the fraction on the left, create at 8 equivalent fractions using either the fraction strips provided or the computer.

$\frac{1}{3}$								

Part B: Answer the following questions

I am ______________________________

I chose the fraction strips / computer * to help me in my activity

*delete one accordingly

I finish my activity in _______ minutes

Figure 4. Worksheet to list the equivalent fractions of 1/3

In order to investigate for a possible difference in impact on different ability students of such an approach, the same lesson was conducted to two different classes, one of higher mathematics ability (Primary 3A) and another of lower mathematics (Primary 3F), as reflected by their mathematics performance in Primary 2. To minimise the teacher factor, the same teacher taught both the classes.

3. Instruments

As the lesson is only of 60-minute duration, both quantitative and qualitative data were collected to better triangulate any observable outcomes.

3.1 Fraction Pre-Test

The general mathematics achievement of the Primary 3A and 3F have been determined by the respective classes' performance at the end of Primary Two in mathematics, as reflected by their aggregated scores on mathematics tests and examinations over a period of a year. However, as the lesson concerned was on fractions and fractions was first introduced at Primary 2 (refer to Table 1 for the coverage of fractions at Primary 2), it would help to also determine if the differences in the two classes' learning of fractions was as reflected by their general performance in mathematics in Primary 2.

Fraction of a whole including:

- interpreting of fraction as part of a whole,
- reading and writing fractions,
- comparing and ordering
  - unit fractions
  - like fractions

(Denominator of given fractions should not exceed 12)
Exclude fraction of a set of objects.

Table 1: Coverage of fractions at Primary 2 [12]

A pre-test on fractions taught in Primary 2 was to help better determine if the two classes are indeed different in their entering behaviour for the given lesson. Two sample questions from the pre-test are show in Figure 5.

3.2 Worksheet On Equivalent Fractions

The Worksheet on equivalent fractions (Figure 4) was given to all the students during the lesson to determine if students, when given a choice, would use the concrete or the virtual manipulative *Fractions* to find the equivalent fractions of $1/3$. It will also provide the number of fractions equivalent to the fraction $1/3$ that the students managed to list, and this would be referred to as the Equivalent Fractions Listing scores.

3.3 Engagement Survey

To determine the level of engagement of the students, the Engagement Survey was conducted at the end of the lesson for both the classes. The survey is a modified version of the PETALS^{TM} Engagement Indicator Questionnaire developed by the Singapore Ministry of Education. It provided scores for the overall level of engagement, as well as that for the three sub-scales on behavioural, cognitive, and affective engagement. The Questionnaire was modified to cater to the needs of the Primary 3 students by simplifying the language and reducing the Likert Scale from an 11-point one to a 3-point one. Six sample questions from the Engagement Survey are shown in Figure 6.

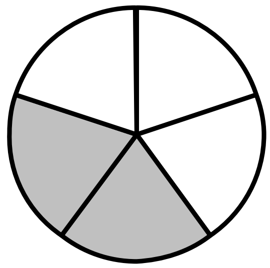
3.4 My Math Learning Journal

To supplement and triangulate the information collected through the Engagement Survey, the students in both classes completed a Journal – My Math Learning Journal, a day after the lesson. The Journal required students to "Write a letter to your friend to tell him or her about the math lesson you had in school".

3.5 Interview Of Teacher

To provide further insight to the students' response to the lesson as well as the effectiveness of the lesson, a short interview with the teacher concerned was carried out.

7. Look at the figure below and answer the following questions.



(a) ☐ of the circle is shaded.

(b) $\frac{2}{5}$ and ☐ make 1 whole.

8. Arrange the fractions in order, beginning with the greatest.

$\frac{1}{3}$ $\frac{1}{12}$ $\frac{1}{2}$ $\frac{1}{9}$

Greatest			

Figure 5. Sample Questions From The Pre-Test

What happened during the lessons?	Yes	Not sure	No
I am excited to learn.			
I follow classroom instructions.			
I am willing to do work that is hard.			
I enjoy doing activities that my teachers give me.			
I obey classroom rules			
I come up with different ides when I do my work.			

Figure 6. Six Sample Questions From The Engagement Survey

4. Results and Discussion

Table 2 lists the Pre-test scores of the two classes on Primary Two fraction concepts.

	Primary 3A	Primary 3F	Combined
n	40	38	78
Mean	18.83	16.74	17.81
SD	1.71	2.24	2.23

Table 2: Pre-test Scores

The Standardized Mean Difference (SMD) for the Pre-test scores is 0.94, which reflects a rather large effect size[2] between the two classes. In other words, the two classes may not be considered as equivalent classes, and Primary 3A does appear to have achieved significantly better than Primary 3F in terms of the Primary Two fraction concepts.

Table 3 provides a breakdown of the number of fractions equivalent to $\frac{1}{3}$ that the students from the two classes listed correctly while Table 4 summarises the mode of representation that the students used to list the equivalent fractions.

	Primary 3A	Primary 3F	Combined
n	37	38	75
Mean	5.64	4.29	4.87
SD	3.21	2.85	3.07

Table 3: Equivalent Fractions Listing Scores

	Primary 3A	Primary 3F	Combined
Laminated Circle (Concrete)	4	0	4
Fractions (Virtual Manipulative)	26	19	45
Did not respond	7	19	26

Table 4: Mode Of Representation Used To List Equivalent Fractions

The SMD for the Equivalent Fractions Listing scores is 0.44, though reflecting a large but not as large an effect size between the two groups. So, the performance level between the two classes appears to have narrowed, when compared with that for the Pre-test. Furthermore, it is observed that most of the students used the virtual manipulative (more than 90% of those who responded) to list the equivalent fractions. Due to time constraint, a large number of the students in Primary 3F who did not indicate the choice of representation used to list the equivalent fractions; in fact less than five students in each class was observed to be using the laminated circle to list the equivalent fractions. Thus,

[2] According to Cohen ([5], pp.284-288), SMD=0.10 is a small effect size, SMD=0.25 is a medium effect size, and SMD=0.40 or larger is a large effect size.

the use of the virtual manipulative *Fractions* appears to have narrowed down the achievement gaps between the two groups. The teacher also commented that he observed that both groups appeared to be more engaged and motivated, pointing towards an observed overall improvement in mathematics performance in both groups. Table 5 shows a breakdown of the Engagement Levels of both groups.

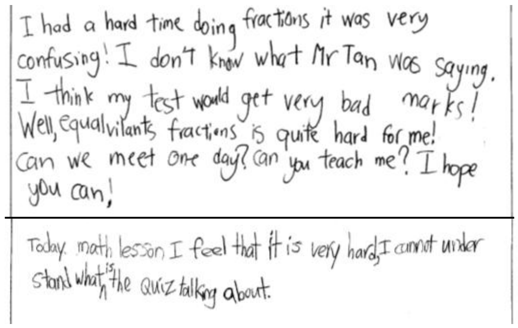
	Primary 3A			Primary 3F			Combined		
Measure	n	Mean	SD	n	Mean	SD	n	Mean	SD
Overall Engagement	38	38.87	0.60	38	40.94	0.50	76	39.91	0.55
Behavioural Engagement	38	12.95	0.60	38	13.28	0.50	76	13.12	0.55
Cognitive Engagement	38	12.08	0.70	38	13.44	0.50	76	12.76	0.61
Affective Engagement	38	13.84	0.50	38	14.21	0.40	76	14.03	0.45

Table 5: Engagement Levels

As there are a total of 5 items for each of the sub-scales, the overall engagement score has a maximum score of 45 while each of the sub-scales has a maximum score of 15. The engagement scores thus do agree with the teacher's observation that the students in both classes are highly engaged. In particular, both classes are not only engaged but highly engaged affectively. The teacher attributed the slightly higher engagement scores of Primary 3F to a possible inflated self-report score due to the use of the virtual manipulative *Fractions*, which has allowed for more opportunities for trial and error – an encouragement for these weaker students who often do not succeed at their first trial with problems. Unlike the laminated circle where checking for equivalence of two fractions require the cumbersome effort of actual drawing and erasing of lines to divide the circles into parts, the dynamic nature of the virtual manipulative *Fractions* facilitates such checking by requiring the user to simply carry out the actions of clicking and dragging on the mouse attached to the laptop computers. In fact, the teachers observed that the few students who used the laminated circles appeared to be frustrated with the constant need to ensure the lines drawn have indeed divided the circles into equal parts and the tedious effort to erase these lines using the tissue paper provided when a trial failed. The teacher also shared that the use of the concrete fraction discs in past years was frustrating for the students as they needed to constantly locate the relevant pieces needed to carry out the comparison from among the mass of pieces of different fractional values that made up the fraction discs set (Figure 7).

Figure 7. Concrete Fraction Dics Pieces

In fact, of all the Journal entries submitted by the students, only two contains negative remarks with regards to the lesson, one from each class (Figure 8).


I had a hard time doing fractions it was very confusing! I don't know what Mr Tan was saying. I think my test would get very bad marks! Well, equalvilants fractions is quite hard for me! Can we meet one day? Can you teach me? I hope you can!

Today math lesson I feel that it is very hard, I cannot under stand what is the quiz talking about.

Figure 8. Negative Comments From Journal

The concern of these two students, though, appeared to be more of their performance in the Pre-test than on the actual lesson itself. In fact, there are four others, two from each class, who though expressed that the content covered is difficult/hard, also shared that they found the experience to be fun/interesting/rewarding (refer to sample comment shown in Figure 9), reflecting the point that the learning of difficult/complex content could also be fun/interesting/rewarding.

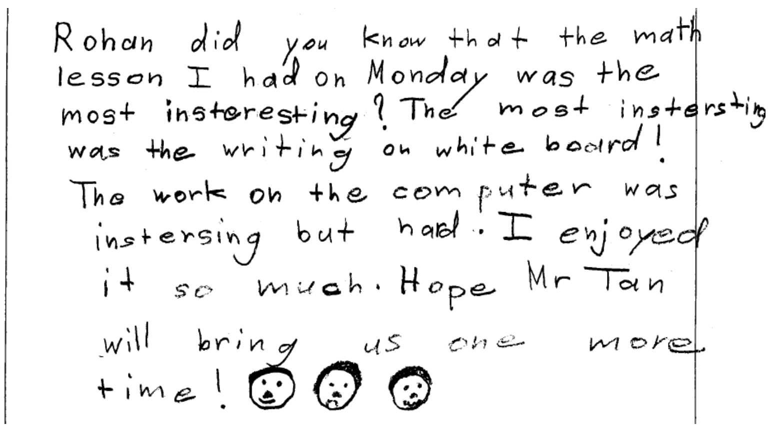
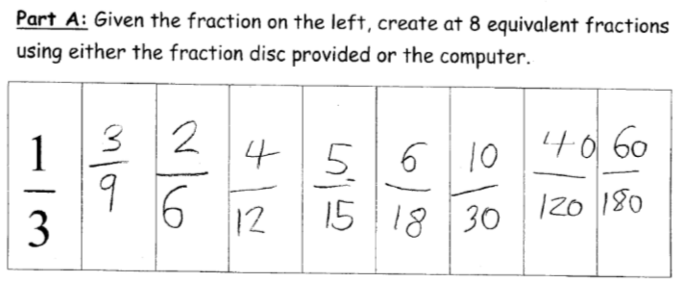

Rohan did you know that the math lesson I had on Monday was the most insteresting? The most insterstin was the writing on white board! The work on the computer was instersing but hard. I enjoyed it so much. Hope Mr Tan will bring us one more time!

Figure 9. Sample Comment That Reflect 'Hard' But Interesting Experience

Interestingly, more than half of the respondents for the Journal entries (43 out of 76 or 57%) expressly and accurately described the content that was covered for the lesson even though the Journal was completed a day later.

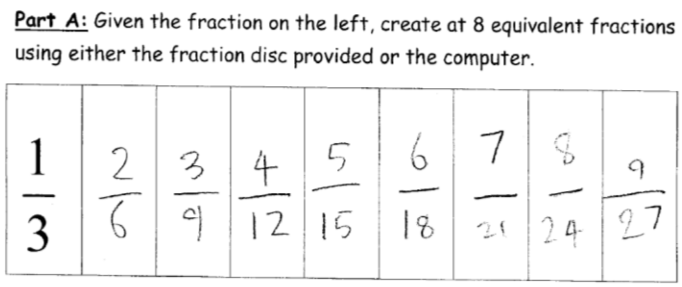
The teacher also observed that the students were engaged in higher order thinking as they went about listing the equivalent fractions using the virtual manipulatives. Students were seen to be hypothesising and observing patterns. The students were only asked to make use of either the concrete or virtual manipulative *Fractions* to list equivalent fractions but not asked to predict how these equivalent fractions may be generated. In fact, the observation of the patterns to predict the rule that governs the numerators and denominators of equivalent fractions has been planned for the following lesson.

However, the students who used the virtual manipulative *Fractions* were observed to be studying the patterns and seeking to list the equivalent fractions using their perceived rule. This has been reflected by some of equivalent fractions that these students listed which has denominators that exceed 24; 24 is the largest number of equal pieces that the virtual manipulative could divide the whole into. A sample response from each of the two classes 3A and 3F are shown on Figures 10 and 11 respectively. During the end-of-lesson class discussion, many of the students were also observed to be articulating possible rules that govern the generation of the equivalent fractions without being prompted by the teacher. The virtual manipulative *Fractions* appear to have become vehicles to enhance students' thinking and creativity, as observed by Suh, Moyer, and Heo [28].


Part A: Given the fraction on the left, create at 8 equivalent fractions using either the fraction disc provided or the computer.

$\frac{1}{3}$	$\frac{3}{9}$	$\frac{2}{6}$	$\frac{4}{12}$	$\frac{5}{15}$	$\frac{6}{18}$	$\frac{10}{30}$	$\frac{40}{120}$	$\frac{60}{180}$

Figure 10. Sample Listing Of Equivalent Fractions From Class 3A


Part A: Given the fraction on the left, create at 8 equivalent fractions using either the fraction disc provided or the computer.

$\frac{1}{3}$	$\frac{2}{6}$	$\frac{3}{9}$	$\frac{4}{12}$	$\frac{5}{15}$	$\frac{6}{18}$	$\frac{7}{21}$	$\frac{8}{24}$	$\frac{9}{27}$

Figure 11. Sample Listing Of Equivalent Fractions From Class 3F

5. Conclusion

The study points towards a positive impact of the use of virtual manipulatives on students' learning of fractions at the primary levels. It appears not only to engage students in their learning, but seem to have a positive effect on narrowing students' achievement. Furthermore, by allowing students to manipulate the "on-screen objects to test hypotheses and experiment with ideas", the experience enhances "students' thinking and creativity" (Moyer, and Heo [28]). In fact, the initial concern of the teacher was the limitation of the virtual manipulative *Fractions* in only being able to generate fractions of denominator up to 24. This has however prompted some of the students instead to go on to observe the patterns of the equivalent fractions with smaller denominators, and hypothesize the rule needed to generate equivalent fractions of denominator beyond 24, as observed above.

Notwithstanding the positive reactions from the participating teacher and students in this study, it must be emphasized that as observed by the teacher in this study, the key success of the use of the virtual manipulative in the mathematics classroom does not lie solely on the power of the manipulative. The provision of the ICT infrastructure under mp1 and the establishment of the baseline ICT standards under mp2 have both contributed to the right tone of learning environment to tap on the power of the virtual manipulative to enrich and transform the learning environments of the students. This in turn enriches the learning experiences of the students.

It remains, though, a question, as raised by the participating teacher, the kind of impact of the use of virtual manipulatives has across different topics. In particular, it would be interesting to look into how the role of virtual manipulatives affects the general pedagogical approach of the mathematics teachers. How would virtual manipulatives affect the C-P-A development of concepts, which is advocated at the Primary levels in the Singapore context [12]? A re-examination of such key pedagogical approach in the light of the availability of such pedagogically rich ICT tool would help teachers to better appreciate and integrate the use of virtual manipulatives into the mathematics classroom.

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