Title	Maximal point spaces of dcpos satisfying the Lawson condition
Author(s)	Xiaoyong Xi and Dongsheng Zhao
Source	*Topology and Its Applications*, *230*, 417-424
Published by	Elsevier

Maximal point spaces of dcpos satisfying the Lawson condition

Xi Xiaoyong and Zhao Dongsheng*

Abstract. A directed complete poset (dcpo, for short) P is said to satisfy the Lawson condition if the restriction of the Scott topology and the Lawson topology on the set of maximal points coincide. In this paper we investigate such dcpos, in particular their maximal point spaces. The main result is that for any dcpo satisfying the Lawson condition, the maximal point space is well-filtered and coherent (the intersection of any two saturated compact sets is compact). The relationship of such dcpos with other classes of dcpos are also considered.

1. Introduction

A poset model of a T_1 space X is a poset P such that X is homeomorphic to the $\mathrm{Max}(P)$ of all maximal points of P equipped with the relative Scott topology. Although in general, the Scott topology on a poset is at most T_0, the maximal point spaces of posets could be very general. In [**4**][**25**], it was proved that every T_1 space has a bounded complete algebraic poset model. In [**27**], it was proved that every T_1 space has a directed complete poset model, equivalently, a topological space is T_1 if and only if it has a dcpo model. This result established a two way bridge between classical topology and relatively new domain theory.

In general, given a class $\mathcal{P}$ of dcpos and a class $\mathcal{T}$ of topological spaces, we can ask the following problems: (i) which spaces have a model in $\mathcal{P}$? (ii) which posets have their maximal point spaces lying in $\mathcal{T}$?

2000 *Mathematics Subject Classification.* 06B35, 06B30, 54A05.

Key words and phrases. directed complete poset; Scott topology; Lawson topology; maximal point space; well-filtered space; coherent space; Lawson condition ; Lawson compact dcpo.

* Corresponding author.

Kamimura and Tang [**10**] characterized spaces that have a bounded complete algebraic dcpo model with a countable base. Edalat and Heckmann[**5**] proved that every complete metric space has a domain model.

Several other authors have discovered more special properties of maximal point spaces of domains (see [**2**][**11**][**13**][**15**][**16**][**18**][**19**][**21**] [**22**]). In particular, Martin proved that the maximal point spaces of domains are Baire and Choquet[**17**].

In [**13**], Lawson proved that the maximal point spaces of ω-continuous dcpos satisfying the Lawson condition are exactly the Polish spaces. In this result, the Lawson condition plays a crucial role. Given any complete metric space X, the domain $\mathbf{B}X$ of closed formal balls is a model of X (Example V-6.8 of [**7**]) and satisfies the Lawson condition [**5**].

The main objective of this paper is to study the maximal point spaces of general dcpos that satisfy the Lawson condition. Our main results are: (i) The maximal point space of every dcpo satisfying the Lawson condition is well-filtered and coherent; (ii) if every Scott closed set of a dcpo satisfies the Lawson condition, the (Scott spaces of the) dcpo is well-filtered.

For other recent work concerning well-filteredness and coherence of spaces see Jia, Jung and Li [**8**], Xi and Lawson [**23**] and Xi and Zhao [**27**].

2. Posets satisfying the Lawson condition

For any subset A of a poset P, let $\downarrow A = \{x \in P : x \leq y \text{ for some } y \in A\}$ and $\uparrow A = \{x \in P : x \geq y \text{ for some } y \in A\}$. A nonempty subset D of a poset P is a directed set if every two elements in D have an upper bound in D. A poset P is called a *directed complete poset* (or *dcpo* for short) if for any directed subset of $D \subseteq P$, $\sup D = \bigvee D$ exists in P.

A subset U of a poset P is Scott open if i) $U = \uparrow U$ (called an upper set) and ii) for any directed subset D, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$, whenever $\bigvee D$ exists. All Scott open sets of a poset P form a topology on P, denoted by $\sigma(P)$ and called the Scott topology on P. The space $(P, \sigma(P))$ is denoted by ΣP, called the Scott space of P. It follows that a subset F of P is Scott closed if (i) $F = \downarrow F$ (called a lower set), and (ii) for any directed subset D of P, $D \subseteq F$ implies $\bigvee D \in F$ if $\bigvee D$ exists. For more about the Scott topology and related structures see [**6**][**7**].

In the following, we shall always assume that the topology on the set $\mathrm{Max}(P)$ of maximal points of a poset P is the relative Scott topology, and call the space $\mathrm{Max}(P)$ the maximal point space of P.

A poset model of a topological space X is a poset P such that $\mathrm{Max}(P)$ is homeomorphic to X [**12**]. Every space having a poset model is T_1.

The Lawson topology on poset P, denoted by $\lambda(P)$ is the common refinement of the Scott topology and the lower topology (Definition III-1.5 of [**7**]).

Definition 1. *A poset P is said to satisfy the Lawson condition if the restriction of the Scott topology and the Lawson topology on the set of maximal points coincide.*

Remark 1. It is easily observed that a poset P satisfies the Lawson condition if and only if for any $x \in P$, there there is a Scott closed set $F \subseteq P$ such that

$$\uparrow x \cap \mathrm{Max}(P) = F \cap \mathrm{Max}(P).$$

Example 1. (1) A poset is called bounded complete if every upper bounded subset has a supremum. Every bounded complete dcpo satisfies the Lawson condition [**27**].

(2) A poset P is called Lawson compact if it is compact with respect to the Lawson topology. By [**23**], every Lawson compact dcpo satisfies the Lawson condition. Note that every bounded complete dcpo is Lawson compact, thus this reduces the fact that every bounded complete dcpo satisfies the Lawson condition.

(3) For any metric space X, the set $\mathbf{B}X$ of all closed formal balls is a poset satisfying the Lawson condition (Theorem 13 of [**5**]). The poset $\mathbf{B}X$ is a dcpo provided X is complete.

(4) The dcpo X constructed in [**9**] by Johnstone does not satisfy the Lawson condition. For the element $x = (2, 2)$, $\uparrow x \cap \mathrm{Max}(X) = \{\mathbb{N} - \{1\}\} \times \{\infty\}$. For any Scott closed set C of X, $\uparrow x \cap \mathrm{Max}(X) = C \cap \mathrm{Max}(X)$ implies $C = X$.

Another property related to the Lawson condition is the following.

Definition 2. *A poset P is said to satisfy the strong Lawson condition if for any $x \in P$,*

$$\downarrow(\uparrow x)$$

is Scott closed.

Remark 2. (1) If every element of P is under some maximal point (for example, if P is a dcpo), then P satisfies the strong Lawson condition iff for any $x \in P$, $\downarrow(\uparrow x) \cap \mathrm{Max}(P)$ is Scott closed.

Thus if a dcpo satisfies the strong Lawson condition, it satisfies the Lawson condition.

(2) In [**27**] and [**23**], it was actually proved that every bounded complete dcpo and every Lawson compact dcpo satisfies the strong Lawson condition, respectively.

Example 2. The following is a dcpo that satisfies the Lawson condition but not the strong Lawson condition.

Let $L = \{c\} \cup \{a_n :\in \mathbb{N}\} \cup \{b_n :\in \mathbb{N}\} \cup \{e, \hat{e}\}$. The order on L is given by

$$c \leq a_i \ (i \in \mathbb{N}),$$
$$b_i < a_i \ (i \in \mathbb{N}),$$
$$b_1 < b_2 < ... < e < \hat{e}.$$

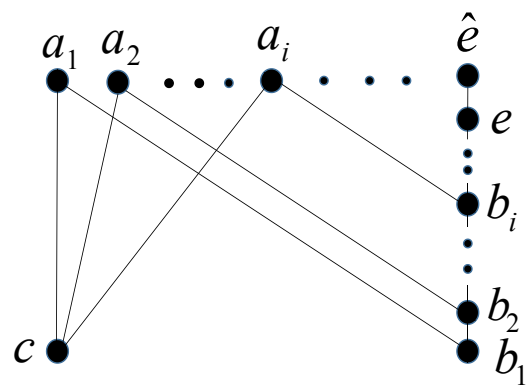

Figure 1. dcpo L

Then L is a dcpo satisfying the Lawson condition. But $\downarrow(\uparrow c \cap \mathrm{Max}(L)) = L - \{e, \hat{e}\}$ is not Scott closed.

Note that this dcpo L is even algebraic.

Proposition 1. *If (X, d) is a metric space such that every closed ball $\overline{B}(x, r) = \{y : d(x, y) \leq r\}$ is compact, then the poset $\mathbf{B}X$ of closed formal balls of X satisfies the strong Lawson condition.*

Proof. Note that for every closed formal ball (x, r),

$$\uparrow(x, r) \cap \mathrm{Max}(\mathbf{B}X) = \{(y, 0) : d(x, y) \leq r\} \neq \emptyset.$$

Thus
$\downarrow(\uparrow(x,r)) = \{(y,s) : \text{there exists } z \in X, d(x,z) \leq r \text{ and } d(y,z) \leq s\}$
$= \{(y,s) : \overline{B}(y,s) \cap \overline{B}(x,r) \neq \emptyset\}$.

Let $\{(x_i,r_i) : i \in I\} \subseteq \downarrow(\uparrow(x,r))$ be a directed subset such that $(w,c) = sup\{(x_i,r_i) : i \in I\}$. Then $\{\overline{B}(x_i,r_i) : i \in I\}$ is a filter base of nonempty closed compact sets of (X,d), and $\overline{B}(x_i,r_i) \cap \overline{B}(x,r) \neq \emptyset$ for each $i \in I$. Therefore $\bigcap\{\overline{B}(x_i,r_i) : i \in I\} \cap \overline{B}(x,r) \neq \emptyset$. Let $u \in \bigcap\{\overline{B}(x_i,r_i) : i \in I\} \cap \overline{B}(x,r)$. Then $(w,c) \leq (u,0) \in \uparrow(x,r) \cap \mathrm{Max}(\mathbf{B}X) \subseteq \downarrow\uparrow(x,r)$. This deduces that $\downarrow(\uparrow(x,r))$ is Scott closed. □

Remark 3. If a metric space is complete and every closed ball is compact, then the space is separable and every closed ball is totally bounded (Remark 2 of [**14**]), thus by Proposition 1 of [**14**], the domain $\mathbf{B}X_\perp$ of closed formal balls with a bottom element adjoint, is an FS-domain. As every FS-domain is Lawson compact and every Lawson compact dcpo satisfies the strong Lawson condition [**23**], $\mathbf{B}X_\perp$ satisfies the strong Lawson condition. It then follows easily that $\mathbf{B}X$ also satisfies the strong Lawson condition. Thus if in Proposition 1 the space (X,d) is complete, we have an alternative proof.

Remark 4. Consider the metric space $X = (\mathbb{Q}, d)$ of all rational numbers with the ordinary metric. For this metric space, $\mathbf{B}X$ is isomorphic to the poset $CB(X)$ of all closed balls with the reverse inclusion order. Consider $B(0,\sqrt{2}) = \mathbb{Q} \cap [-\sqrt{2},\sqrt{2}]$ in $CB(X)$. The set $\{B(2, 2-\sqrt{2}+\frac{1}{n}) : n \in \mathbb{N}\}$ is a directed subset of $\downarrow(\uparrow B(0,\sqrt{2}))$ and $Sup\{B(2, 2-\sqrt{2}+\frac{1}{n}) : n \in \mathbb{N}\} = B(2, 2-\sqrt{2})$, and $B(2, 2-\sqrt{2}) \notin \downarrow(\uparrow B(0,\sqrt{2}))$. Thus $CB(X)$ does not satisfy the strong Lawson condition.

The strong Lawson condition will be used in next section.

3. Maximal point spaces of dcpos satisfying the Lawson condition are well-filtered

In this section we prove the main results of this paper. The following result due to Xi and Lawson [**23**] will be needed.

Lemma 1. *([**23**]) Let L be a dcpo.*

(1) If for every $x \in L$, $\uparrow x$ is Lawson compact, then L satisfies the strong Lawson condition.

(2) For any Lawson compact set $F \subseteq L$, $\downarrow F$ is Scott closed.

Lemma 2. *For any two Lawson compact subsets K_1, K_2 of a dcpo L, $K_1 \cap K_2$ is Lawson compact if $(\downarrow K_1) \cap K_2 = K_1 \cap K_2$.*

Proof. By Lemma 1(2), $\downarrow K_1$ is Scott closed (hence also Lawson closed), thus $(\downarrow K_1) \cap K_2$ is Lawson compact, implying $K_1 \cap K_2$ is Lawson compact if the condition is satisfied. $\square$

Corollary 1. *If L is a dcpo and $K_1, K_2 \subseteq Max(L)$ are Lawson compact, then $K_1 \cap K_2$ is Lawson compact.*

Lemma 3. *Let L be a dcpo. For any Scott open set U and filtered family $\mathcal{K}$ of Lawson compact subsets of L, if $\bigcap \mathcal{K} \subseteq U$ then $K \subseteq U$ for some $K \in \mathcal{K}$.*

Proof. The proof of this lemma has much similarity to that for Proposition 2.4 of [**23**].

Let $\mathcal{K}$ be a filtered family of non-empty Lawson compact sets of L and U a Scott open set such that

$$\bigcap \mathcal{K} \subseteq U.$$

Assume that $K - U \neq \emptyset$ for every $K \in \mathcal{K}$. Let

$$\mathcal{C} = \{F \subseteq U^c : F \text{ is Scott closed and } F \cap K \neq \emptyset \text{ for all } K \in \mathcal{K}\}.$$

Then $\mathcal{C} \neq \emptyset$ because $U^c \in \mathcal{C}$. Using Zorn's Lemma, one deduces that $\mathcal{C}$ has maximal chain, say $\mathcal{B}$. For each $K \in \mathcal{K}$ and $B \in \mathcal{B}$, $B \cap K \neq \emptyset$. It then follows that $\bigcap \mathcal{B} \cap K \neq \emptyset$ due to the compactness of K (every Lawson compact set is Scott compact). Hence $C_0 = \bigcap \mathcal{B}$ is a minimal member of $\mathcal{C}$.

The nonempty Scott closed set C_0 must have a maximal element (as it is a dcpo), say y_0. Since $y_0 \notin U$ and $\bigcap \mathcal{K} \subseteq U$, there is $K_0 \in \mathcal{K}$ such that $y_0 \notin K_0$. Note that $K_0 \cap C_0$ is a closed subset of the Lawson compact set K_0, it is also Lawson compact. Thus $\downarrow(K_0 \cap C_0)$ is Scott closed by Lemma 1(2).

Then $y_0 \in C_0 - \downarrow(K_0 \cap C_0)$. Hence $\downarrow(K_0 \cap C_0)$ is a proper nonempty Scott closed subset of C_0.

Now for any $K \in \mathcal{K}$, there is a $K' \in \mathcal{K}$ such that $K' \subseteq K_0 \cap K$. Then $\emptyset \neq K' \cap C_0 \subseteq (K_0 \cap K) \cap C_0 \subseteq \downarrow(K_0 \cap C_0) \cap K$, implying $\downarrow(K_0 \cap C_0) \in \mathcal{C}$. This contradicts the minimality of C_0.

Hence there must be a $K \in \mathcal{K}$ satisfying $K \subseteq U$. $\square$

A subset of a space is saturated if it equals the intersection of all open sets containing it. A space is called coherent if the intersection of any two compact saturated sets is compact (Definition VI-6.2. of [**7**]). A space is called well-filtered if for any filtered family $\mathcal{C}$ of compact saturated sets and each open set U with $\bigcap \mathcal{C} \subseteq U$, there is $K \in \mathcal{C}$ with $K \subseteq U$ (Definition I-1.24.1.).

Theorem 1. *If L is a dcpo satisfying the Lawson condition, then*

(1) Max(L) is coherent, and
(2) Max(L) is well-filtered.

PROOF. (1) Note that $\mathrm{Max}(L)$ is a T_1 space and every subset in a T_1 space is saturated. Thus (1) follows from Corollary 1 and the fact that if $K_1, K_2 \subseteq \mathrm{Max}(L)$ are compact, then they are Lawson compact because L satisfies the Lawson condition.

(2) If L satisfies the Lawson condition, then every (saturated) compact set of $\mathrm{Max}(L)$ is a Lawson compact set of L. Now applying Lemma 3, we deduce that $\mathrm{Max}(L)$ is well-filtered. □

A poset model of a T_1 space X is a poset P such that X is homeomorphic to the maximal point space of P.

EXAMPLE 3. Let $X = (\mathbb{N}, \tau_{cof})$ be the space of all natural numbers equipped with the co-finite topology. Then every subset of X is compact, so it is coherent but not well-filtered ($\bigcap\{\mathbb{N} - \{1, 2, ..., k\} : k \in \mathbb{N}\} = \emptyset$ but each $\mathbb{N} - \{1, 2, ..., k\} \neq \emptyset$). By Theorem 1, it does not have a dcpo model satisfying the Lawson condition. In [**27**], it was proved that $(\mathbb{N}, \tau_{cof})$ does not have a bounded complete dcpo model. Here we deduce that it even does not have a dcpo model satisfying the Lawson condition.

REMARK 1. *(1) For a subset A of a dcpo L, consider the following properties:*

(i) A is Lawson compact;

(ii) A is Scott compact and $\downarrow(A \cap C)$ is Scott closed for every Scott closed C;

(iii) A is Scott compact and $\downarrow A$ is Scott closed.

Then clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

(2) If L has a bottom element, then L is Scott compact and for any Scott closed set C, $\downarrow(L \cap C) = C$ is Scott closed. But L need not be Lawson compact.

(3) From the proof of Lemma 3, it is seen that the sets in $\mathcal{K}$ satisfying condition (ii) are enough.

LEMMA 4. *Let L be a well-filtered dcpo (i.e. ΣL is well filtered). Then the poset $(k(L), \supseteq)$ of all nonempty saturated compact subsets of L is a dcpo model of Max(L).*

PROOF. Firstly, by Remark 3 of [**27**], the intersection of any filtered family of nonempty compact saturated sets of L is nonempty compact saturated. Thus $(k(L), \supseteq)$ is a dcpo where the supremum of a directed subset $\mathcal{D} \subseteq k(L)$ is the intersection $\bigcap \mathcal{D}$. Also $A \in \mathrm{Max}(k(L))$ iff $A = \{y\}$ for some $y \in \mathrm{Max}(L)$, hence $f : \mathrm{Max}(k(L)) \longrightarrow \mathrm{Max}(L)$ is a bijection, where $f(\{y\}) = y$ for each $\{y\} \in \mathrm{Max}(k(L))$.

For any Scott open set $U \subseteq L$, let $U^k = \{F \in k(L) : F \subseteq U\}$. Then U^k is a Scott open set of $k(L)$ because L is well-filtered. Clearly $f^{-1}(U \cap \mathrm{Max}(L)) = U^k \cap \mathrm{Max}(k(L))$. Hence f is continuous. Conversely, let E be a Scott open set of $k(L)$. Then the set $E^s = \{x \in P : \uparrow x \in E\}$ is a Scott open set of L. In addition, $f(E \cap \mathrm{Max}(k(L))) = E^s \cap \mathrm{Max}(L)$, showing that f is an open mapping. Therefore f is a homeomorphism. □

Different classes of posets may model the same class of spaces. For example, the class of all dcpos and the class of all posets both model the class of all T_1 spaces. The class of all FS-domains, the class of all Lawson compact domains and the class of all bounded complete domains all model the same class of spaces (Corollary 3.6 [**17**]).

The theorem below is another such type of result.

Theorem 2. *The class of bounded complete dcpos and the class of Lawson compact dcpos model the same spaces.*

Proof. If L is bounded complete, then it is Lawson compact (Theorem III-1.9 of [**7**]).

Conversely, let L be Lawson compact. Then ΣL is compact, well-filtered and coherent (Theorem 4.2 of [**23**]). Since the intersection of any two members of $k(L)$ is in $k(L)$ if the intersection is nonempty, one easily verifies that the poset $(k(L), \supseteq)$ is a bounded complete dcpo, where the supremum of a bounded subset $\mathcal{F}$ equals the intersection $\bigcap \mathcal{F}$. By Lemma 4, it follows that $(k(L), \supseteq)$ is a bounded complete dcpo model of $\mathrm{Max}(L)$. □

Recall that every bounded complete domain is an FS-domain and every FS-domain is Lawson compact. Thus we re-deduce the Corollary 3.6 of [**17**], where the bounded complete domains are called Scott domains and Lawson compact domains are called coherent domains.

Corollary 2. *For a T_1 space X, the following are equivalent:*
(1) X has a bounded complete domain model.
(2) X has an FS-domain model.
(3) X has a Lawson compact domain model.

For the implication of (3) and (1), just note that if L is a domain, then $((k(L), \supseteq)$ is a domain.

Lemma 5. *If every Scott closed set of a dcpo L satisfies the Lawson condition, then L satisfies the strong Lawson condition.*

Proof. Assume that every Scott closed set of L (as a dcpo) satisfies the Lawson condition and $x \in L$. There is a Scott closed set C of L such that $C \cap \mathrm{Max}(L) = \uparrow x \cap \mathrm{Max}(L)$. We assume C is the smallest such Scott closed set. For the Scott closed set C, note that $x \in C$. By the assumption,

there is a Scott closed subset $\hat{C}$ of C such that $\hat{C} \cap \mathrm{Max}(C) = \uparrow x \cap \mathrm{Max}(C)$. Clearly $\hat{C} \cap \mathrm{Max}(L) \subseteq C \cap \mathrm{Max}(L)$. Let $e \in C \cap \mathrm{Max}(L)$, then $e \in \mathrm{Max}(C)$ and $e \in \uparrow x$, thus $e \in \uparrow x \cap \mathrm{Max}(C)$, implying $e \in \hat{C}$. Hence $e \in \hat{C} \cap \mathrm{Max}(L)$. It follows that $\hat{C} \cap \mathrm{Max}(L) = C \cap \mathrm{Max}(L)$. Note that $\hat{C}$ is also a Scott closed set of L. By the assumption on the minimality of C, we have $C = \hat{C}$. Now $\mathrm{Max}(C) = C \cap \mathrm{Max}(C) = \hat{C} \cap \mathrm{Max}(C) = \uparrow x \cap \mathrm{Max}(C)$, implying $\mathrm{Max}(C) \subseteq \uparrow x$. Then $C = \downarrow(\mathrm{Max}(C)) \subseteq \downarrow(\uparrow x) = \downarrow(\uparrow x \cap \mathrm{Max}(L)) \subseteq C$. Thus $\downarrow(\uparrow x \cap \mathrm{Max}(L)) = C$ is Scott closed. □

Corollary 3. *If every Scott closed set of a dcpo L satisfies the Lawson condition, then every Scott closed set (as a dcpo) satisfies the strong Lawson condition.*

Lemma 6. *If every Scott closed set of a dcpo L satisfies the strong Lawson condition, then L is well-filtered.*

Proof. Let A be a Scott closed set and K a saturated Scott compact subset of L. By Proposition 2.4 of [**23**], we only need to show that $\downarrow(A \cap K)$ is Scott closed. Assume that $\{x_i : i \in I\} \subseteq \downarrow(A \cap K)$ is a directed subset.

For each $i \in I$, $C_i = \downarrow(\uparrow x_i \cap \mathrm{Max}(A))$ is a Scott closed subset of A. Also $A \cap K$ is a Scott compact subset of A. For each $i \in I$, there is $b \in A \cap K$ such that $x_i \leq b$. Then $b \leq m$ for some $m \in \mathrm{Max}(A)$, which implies $m \in K$ because $K = \uparrow K$. It follows that $m \in A \cap K$ and $m \in \uparrow x_i \cap \mathrm{Max}(A) \subseteq C_i$. Therefore $C_i \cap A \cap K \neq \emptyset$.

Also $\{C_i : i \in I\}$ is a filtered family, we have that $\bigcap\{C_i : i \in I\} \cap A \cap K \neq \emptyset$. Let $y \in \bigcap\{C_i : i \in I\} \cap A \cap K$. For each $x_i, y \in C_i$, hence we have $x_i \in \downarrow(\uparrow y \cap \mathrm{Max}(A))$. Thus $\bigvee\{x_i : i \in I\} \in \downarrow(\uparrow y \cap \mathrm{Max}(A))$, because A satisfies the strong Lawson condition. Hence there is $u \in \mathrm{Max}(A)$ such that $y \leq u$ and $\bigvee\{x_i : i \in I\} \leq u$. Again, as $K = \uparrow K$, we have $u \in K$, thus $u \in A \cap K$. Finally we have $\bigvee\{x_i : i \in I\} \in \downarrow(A \cap K)$.

□

By the above theorem and the Theorem 4.2 of [**23**], we deduce the following result.

Theorem 3. *The following are equivalent for any dcpo L:*

(1) L is Lawson compact.

(2) Every Scott closed subset of L satisfies the Lawson condition and L is Scott compact and coherent .

Example 4. (1) The example below shows that the necessity of coherence in the above theorem.

Let $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ with the ordinary order of numbers. Then every Scott closed set satisfies the strong Lawson condition. But X is not Lawson compact because it is not Scott compact.

This is also a dcpo satisfying the strong Lawson condition and is not Lawson compact.

(2) The following example shows the necessity of Scott compactness of L in Theorem 3.

Let $L = \mathbb{N}$ with the discrete order. Then L is coherent and every Scott closed set satisfies the Lawson condition. But L is not Lawson compact.

As a summary, here we list the relationships among the main classes of dcpos discussed in this paper.

(1) Every bounded complete dcpo is Lawson compact.

(2) Every Scott closed subset of a Lawson compact dcpo satisfies the (strong) Lawson condition.

(3) If every Scott closed subset of a dcpo satisfies the Lawson condition, then the dcpo is well-filtered.

(4) The maximal point space of a dcpo satisfying the Lawson condition is well-filtered and coherent.

(5) A T_1 space has a bounded complete dcpo model iff it has a Lawson compact dcpo model.

4. Some related problems for future work

We close the paper with some related problems.

The main result in this paper is that the maximal point spaces of dcpos satisfying the Lawson condition are well-filtered and coherent. A natural question arising is:

(1) Does every well-filtered and coherent T_1 space have a dcpo model satisfying the Lawson condition?

We have no clue for how to prove the answer is positive. But if we wish to construct a counterexample, the space must not be metrizable, or even not Hausdorff and first countable as every such space has a bounded complete dcpo model due to [**27**].

One possible candidate might be the set $\mathbb{R}$ of real numbers with the co-countable topology.

It is even unknown whether every Hausdorff space has a dcpo model satisfying the Lawson condition (note that every Hausdorff space is well-filtered and coherent). To obtain a counterexample, the given space must not be first countable as every Hausdorff k-space has a bounded complete dcpo model due to Zhao and Xi [**27**].

A simple space that is Hausdorff but not first countable is the set $\mathbb{R}$ of real numbers with the topology τ: $U \in \tau$ iff $0 \notin U$ or there exists an open

interval (a, b) and a countable set B such that $0 \in (a, b) - B \subseteq U$. This space is not first countable.

Let $Z(X) = OFil(0) \cup \{\{x\} : x \in \mathbb{R} - \{0\}\}$, where $OFil(0)$ is the set of all filter base $\mathcal{F}$ of open sets in τ with $0 \in \bigcap \mathcal{F}$. Define the order $\leq$ on $Z(X)$ by $A \leq B$ only if either (i) $A, B \in OFil(0)$ and $A \subseteq B$, or (ii) $B = \{x\}$ with $x \neq 0$ and $A \in OFil(0)$ and x is in every member of A. Then, using a similar argument as in [**25**], one can verify that $(X, \leq)$ is a bounded complete dcpo and is a model of $(\mathbb{R}, \tau)$.

The Lawson condition and the strong Lawson condition are quite close. Thus we have the following problem:

(2) Does the class of dcpos satisfying the Lawson condition and that of dcpos satisfying the strong Lawson condition model the same class of spaces?

Acknowledgement

(1) This work was supported by the NSF Project of Jiangsu Province, China (BK20170483), NSF of China (11361028, 61300153, 11671008, 11626207, 11701500) and NIE AcRF Project (RI 3/16 ZDS).

(2) We must thank the reviewer for giving us many valuable comments and suggestions for improving the manuscript.

References

[1] Ali-Akbaria, M. Honarib, B. and Pourmahdiana, M.: Any T1 space has a continuous poset model, Topology and its Applications **156**(2009), 13:2240 - 2245.

[2] Bennett, H. and Lutzer, D.: Domain representable spaces, Fund. Math. 189(2006), 255-268.

[3] Engelking,R.: *General topology*, Vol. 6, Heldermann Verlag, Berlin, United State of America, 1989.

[4] Erné, M.: Algebraic models for T1-spaces, Topology and its Applications **158**(2011), 7:945-962.

[5] Edalat, A. and Heckmann, R.: A computational model for metric spaces, Theoretical computer science **193**(1998), 53-73.

[6] Goubault-Larrecq, J.: *Non-Hausdorff topology and Domain Theory: Selected Topics in Point-Set Topology*, Vol. 22, Cambridge University Press, 2013.

[7] Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. W. and Scott, D. S.: *Continuous lattices and domains*, Vol. 93, Cambridge University Press, 2003.

[8] Jia, X. Jung, A. and Li, Q.: A note on coherence of dcpos, Topology and its Appl. 209(2016), 235-238.

[9] Johnstone, P.: Scott is not always sober, In: Continuous Lattices, Lecture Notes in Math. 871, Springer-Verlag, (1981), 282-283.

[10] Kamimura, T. and Tang, A.: Total objects of domains, Theoretical Computer Science 34(1984), 3:275-288.

[11] Kopperman, R., Künzi, A. and Waszkiewicz, P.: Bounded complete models of topological spaces, *Topology and its Applications*, 139(2004), 13: 285-297.
[12] Lawson, J. D.: Spaces of maximal points, Math. Structures. Comput. Sci. **7**(1997), 5: 543-555.
[13] Lawson, J. D.: Computation on metric spaces via domain theory, *Topology and its Applications* 85 (1998), 247-263
[14] Lawson, J. D.: Metric space and FS-domains, Theoretical computer science **405**(2008), 73-74.
[15] Liang, L. and Klause, K.: Order environment of topological spaces, Acta Math. Sinica **20**(2004), 5: 943-948.
[16] Martin, K.: Ideal models of spaces, Theoretical computer science **305**(2003), 277-297.
[17] Martin, K.: Domain theoretic models of topological spaces. Electronic Notes in Theor. Comp. Sci. 13 (1998), 173-181.
[18] Martin, K.: Nonclassical techniques for models of computation. Topology Proceedings 24 (1999), 375-405.
[19] Martin, K.: The regular spaces with countably based models. Theor. Comp. Sci. 305 (2003), 299-310.
[20] Scott, D. S.: Continuous Lattices, In: Toposes, Algebraic Geometry and Logic, Lecture Notes in Math. 274, Springer-Verlag, (1972), 97-136.
[21] Waszkiewicz, P.: How Do Domains Model Topologies? Electronic Notes in Theoretical Computer Science 83 (2004), 18-pages.
[22] Weihrauch,K. and Schreiber, U.: Embedding metric spaces into cpo's, *Theoret. Comput. Sci.*, 16 (1981), pp. 524
[23] Xi, X, and Lawson, J.: On well-filtered spaces and ordered sets, Topology and its Applications 228(2017), 1:139-144.
[24] Xi, X. and Zhao, D.: Well-filtered spaces and their dcpo models, Math. Structures. Comput. Sci. 27 (2017), 4:507-515.
[25] Zhao, D.: Poset models of topological spaces, In: Proceeding of International Conference on Quantitative Logic and Quantification of Software, Global - Link Publisher, (2009), 229-238.
[26] Zhao, D. and Xi, X.: Dcpo models of T_1 topological spaces, Math. Proc. Cambridge Phil. Soc. (DOI: https://doi.org/10.1017/S0305004116000888 Published online: 11 October 2016).
[27] Zhao, D. and Xi, X.: On spaces that have a bounded complete dcpo model, Rocky Mountain Journal of Mathematics (to appear).

Department of Mathematics, Jiangsu Normal University, Jiangsu China
E-mail address: `littlebrook@jsnu.edu.cn`

Mathematics and Mathematics Education, National Institute of Education Singapore, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616
E-mail address: `dongsheng.zhao@nie.edu.sg`