NOTICING AFFORDANCES OF A TYPICAL PROBLEM

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Typical mathematics problems, such as examination-type questions, are often used in classrooms to develop students' procedural fluency. In this article, we describe and analyse what a secondary school mathematics teacher noticed about the affordances of such a problem, as well as how she orchestrated a mathematically productive discussion using the adapted problem in class. The findings suggest that a teacher's productive noticing of the affordances offered by typical problems can enhance the learning experiences of mathematics students.

INTRODUCTION

Using high cognitive-demand tasks is critical for orchestrating productive discussions (Smith & Stein, 2011) during lessons. However, besides the development of concepts, teachers are also mindful about the concomitant development of procedural skills to prepare students for tests and examinations. Hence, the practice of using *examination-type questions* with a more teacher-centred teaching approach is prevalent in Singapore classrooms (Foong, 2009; Ho & Hedberg, 2005). This preference for using typical problems—standard examination or textbook problems—may reflect teachers' belief that it is "important to prepare students to do well in tests than to implement problem-solving lessons" (Foong, 2009, p. 279), a classroom reality that cannot be ignored. Given teachers' strong preference for using typical problems in the classroom, it would be interesting to explore the use of such questions to orchestrate rich discussions. This paper is therefore framed by the following question: Whether, and if so, how typical problems, such as examination-type questions, can be used to orchestrate productive mathematical discussions? Drawing on our preliminary findings from a larger study, we present a case study of Ms. Alice, a proficient secondary school mathematics teacher, to highlight how her productive noticing of the affordances offered by typical problems could provide a mathematical learning experience, aimed at developing students' understanding of matrices.

THEORETICAL CONSIDERATIONS

Orchestrating Learning Experiences

With the aim of supporting teachers to plan for more skilful improvisation, Smith and Stein (2011) propose five productive practices—anticipating, monitoring, selecting, sequencing and connecting—as a way to make "student-centered instruction more manageable" (p. 7). Anticipating, which occurs during lesson planning, refers to predicting students' likely responses to the tasks. The other practices pertain directly to

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the actual work of orchestrating discussions after teachers set students to work on the task: monitoring students' responses while circulating in the classroom, selecting particular students' answers, and purposefully sequencing these selected answers for presentation. Last but not least, teachers support students in making sense of the mathematical ideas by connecting these responses to make a mathematical point. To enact these practices, teachers would need to draw on appropriate mathematical knowledge to interpret students' responses during lessons, and make the necessary pedagogical moves for advancing students' thinking (Smith & Stein, 2011).

Mathematics Teacher Noticing

According to Kilpatrick, Swafford, and Findell (2001), mathematics teachers who are proficient at orchestrating discussions should be able to examine the mathematical possibilities of instructional materials, adapt them for different student profiles, analyse students' reasoning, and respond to the different methods students use in their work. Doing this work of ambitious teaching requires developing a keen awareness of the mathematical connections and having a different act in mind (Mason, 2002). Therefore, developing teachers' eyes to see and the mind to make sense of these mathematical connections is critical for orchestrating learning experiences. Mathematics teacher noticing is an emerging construct that lies at the heart of these components of teaching expertise. It refers to what teachers attend to and how they interpret their observations to make instructional decisions (Mason, 2002; Sherin, Jacobs, & Philipp, 2011). Many studies (e.g., van Es (2011)) on teacher noticing used video studies and investigated what teachers noticed without giving explicit instructional aspects for teachers to direct their attention; other studies (e.g., Goldsmith and Seago (2011)) used teaching or learning artifacts to focus teachers' attention to specific features of instruction. More recently, Choy (2015) brings task design into the realm of teacher noticing and his findings suggest that an explicit focus for noticing, and an emphasis on pedagogical reasoning can increase the likelihood of teachers making instructional decisions which promote students' reasoning. Building on the Three-Point Framework by Yang and Ricks (2012), Choy (2015) highlights mathematics concepts, students' confusion, and teachers' courses of action as critical foci to facilitate productive noticing.

Affordances of a Mathematics Task

A typical problem, as described earlier, can certainly be used very procedurally by a teacher but can it be used in a more productive manner? What kinds of affordances do such problems offer to the teacher? In this context, using the perceptual psychologist, Gibson's (1986) ideas, we can emphasise that: (1) an affordance for using a typical problem exists relative to the action and capabilities of the teacher, (2) the existence of the affordance is independent of the teacher's ability to perceive it, and (3) the affordance does not change as the needs and goals of the teacher change. Gibson also highlighted that affordances in relation to an observer could be positive or negative which in our context may lead to productive or less productive use of the problems in class by the teacher. Hence, to perceive the affordances of a typical problem means to be able to *notice* the characteristics of the task in relation to the particular understandings of the related concept in order to adapt the task for use in classrooms. But what should a teacher notice about a task so as to recognise its affordances? In this paper, we adopt Choy's (2015) notion of productive noticing to investigate what a teacher noticed about the affordances of a typical mathematics problem.

METHODOLOGICAL CONSIDERATIONS

The data reported in this paper came from a larger study on orchestrating learning experiences in a secondary school mathematics classroom in Singapore. The study followed a design-based research approach to develop a toolkit for teachers as a means of supporting their orchestration of learning experiences, as well as to develop a theory about teachers' noticing in the context of orchestrating learning experiences. In this paper, we examine the practices of Alice (pseudonym), one of the three teachers who took part in the study. Alice is a Senior Teacher at Coventry Secondary School (pseudonym), which is a government-funded school performing slightly above average in the national examinations. As a Senior Teacher, Alice has a strong mathematical background and has been actively involved in mentoring novice teachers in her school.

Data were collected and generated through voice and video recordings of the lesson, as well as voice recordings of a pre-lesson discussion and a post-lesson discussion with Alice. The findings were developed through identifying themes related to what Alice noticed about the content, her students' confusion, and her own courses of action, with reference to the framework developed by Choy (2015). In addition, the lesson was also analysed by identifying segments which corresponded to Smith and Stein's (2011) five practices. In this paper, we present our preliminary findings through snapshots of how Alice noticed the affordances of a typical problem, and how she deployed the problem in class to orchestrate a mathematically productive discussion.

FINDINGS AND DISCUSSION

In this section, we present our analysis of Alice's lesson on Matrices for Secondary Three (Grade 9) students. Her students had learnt how to multiply two matrices prior to this lesson. The learning experience stipulated in the curriculum document was for students to apply matrix multiplications to solve contextual problems, and for them to justify if two matrices can be multiplied by checking the order of the matrices. During the introductory phase of the lesson, Alice used a modified version of a typical problem (See Figure 1 for the typical problem) and orchestrated a mathematically productive learning experience using students' responses to the modified problem (See Figure 2).

Perceiving the Affordances of a Typical Mathematics Problem

Alice selected a typical problem (See Figure 1) from a past examination paper as the source of the introductory problem to be used during the lesson. This is a typical examination-type contextual problem involving matrices. There are two parts to the problem: the first part requires students to perform a routine matrix multiplication that involves pre-multiplying a 3×1 matrix by a 2×3 matrix to obtain a 2×1 matrix $\begin{pmatrix} 185 \\ 184 \end{pmatrix}$ as the solution; the second part requires them to explain the meaning of the product which in this context represents the total points gained by Theresa and Robert for the awards.

Example 1

[Nov 2013] Teresa and Robert attend the same school. They keep a record of the awards they have earned and the points gained. The matrices show the numbers of awards and the points gained for each award.

$$\begin{array}{c} \\ \text{Teresa} \\ \text{Robert} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Gold} & \text{Silver} & \text{Bronze} \end{array} \\ \begin{pmatrix} 29 & 10 & 5 \\ 30 & 6 & 8 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ \text{Gold} \\ \text{Silver} \\ \text{Bronze} \end{array} \begin{array}{c} \text{Points} \\ \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \end{array}$$

(a) Find $\begin{pmatrix} 29 & 10 & 5 \\ 30 & 6 & 8 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$

(b) Explain what your answer to (a) represents.

Figure 1: The Typical Problem used during the Lesson

Instead of presenting the problem as it was given, Alice made two modifications to the problem (See Figure 2). First, she provided information about the awards obtained and the points for each award within the stem of the problem instead of representing them in matrices as in the original problem. Second, she asked for the total number of points obtained by each person instead of finding the matrix product directly. By doing so, Alice modified the problem in a way that required students to formulate their solution in terms of a matrix multiplication. Moreover, because the order of the matrices were not given, students had to decide on the order of the matrices before using the appropriate matrix multiplication to find the answer. In addition, the modified problem did not require students to use a matrix method, which then reduced the problem to a straight-forward arithmetic one. This provided opportunities for Alice to emphasise the connections between matrix multiplication and arithmetic which could potentially provide some meaning to matrix operations.

> Teresa and Robert attend the same school. They keep a record of the awards they have earned and the points gained. Teresa obtained 29 Gold, 10 Silver, and 5 Bronze awards. Robert obtained 30 Gold, 6 Silver, and 8 Bronze awards. They gained 5 points from each Gold award, 3 points for each Silver award, and 2 points for each Bronze award.
> Find the total number of points that Teresa gained.
> Find the total number of points that Robert gained.

Figure 2: The Modified Problem used during the Lesson

More importantly, Alice anticipated students might not use matrix multiplication, as intended in the original problem (Figure 1), because they had just recently been introduced to matrix multiplication. Instead, students could the answer by performing two separate matrix multiplications or they could use an arithmetic method. During the post-lesson interview, Alice revealed that she had considered students' confusion and anticipated their answers based on her knowledge of the students:

> Why I choose this question is because most of the exam style questions are based on solving problems involving matrices. And this question will extend their thinking and help them to transfer their mathematical thinking into other representations. This is what I find challenging amongst some students… I will know that certain students will give this [answer], exactly which students I don't know…

Hence, we see how Alice's reasoning for the modifications were made. These modifications afforded opportunities for her to build on students' less-than-optimal solutions to reveal students' reasoning, explain the procedure of multiplying two matrices, and connect students' solutions to the intended one.

Orchestrating Discussions with the Adapted Problem

Alice demonstrated her recognition of the affordances offered by the modified task when she orchestrated discussions during the lesson. After setting her students to work on the problem, Alice moved around in the class checking students' responses and helping out students who had queries:

1 Alice: (Walks around the class and comes to Student S1.) Can you write this for me on the board?

2 S1: Ok. (Walks to the whiteboard and writes the following.)

$T = 5 \times 29 + 3 \times 10 + 2 \times 5 = 185$

$R = 5 \times 30 + 3 \times 6 + 2 \times 8 = 184$ *[arithmetic solution]*

3 Alice: (Walks around while waiting for Student S1 to finish writing.) Ok. Most of you have written what [Student S1] has written. 5 points for 29 gold, 3 points for 10 silver and 2 points for 5 bronze. Most of you have written in this manner. The last few days, we have been talking about matrices, right? Would you like to convert this to a matrix problem?
(Student S2 raises his hands.) Have you written it in matrix form? (Student S2 nods and Alice goes over to take a look.) Okay. Can you write your answer on the board?

4 S2: (Walks to the board and writes the following.)

$$T = \begin{pmatrix} 29 & 10 & 5 \end{pmatrix} \times \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = 185 \text{ and } R = \begin{pmatrix} 30 & 6 & 8 \end{pmatrix} \times \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = 184$$

[two separate matrix products]

5 Alice: Any other answers from [Student S2's] answer? (Walks around the class and selects Student S3's answer) Can you write this on the board?

6 S3: (Walks to the board and writes the following.)

$$\begin{pmatrix} 29 & 10 & 5 \\ 30 & 6 & 8 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 29 \times 5 + 10 \times 3 + 5 \times 2 \\ 30 \times 5 + 6 \times 3 + 8 \times 2 \end{pmatrix} = \begin{pmatrix} 185 \\ 184 \end{pmatrix}$$ *[a single matrix produced]*

7 Alice: Thank you all three of you. [Student S1] has written using an arithmetic method. Most of you have written in this manner. This one comes very naturally to you, ok? [Student S2] has written Robert and Theresa's award separately. He has tried to use the matrix method, (points to Student S1's solution.) Something like this, ok? Let's check whether the order of matrix is correct or not.

(Alice goes through the method of matrix multiplication and gets the class to check the order of Student S2's matrices.)

... Ok. Student S3 has written Robert's and Theresa's together so that you only write this matrix once (points to the column matrix [5 3 2]). Don't need to write two times, correct or not? See. Over here. You have to write two times but here, [Student S3] only has to write it once. Let's check the order again...

Alice then asked the class why Student S3's solution was better compared to the other two students. She led the class to see that Students S3's method is a more economical process as the "points matrix" is written only once and that would be useful if there were, say, 100 students. She also highlighted the use of matrices to represent large amount of data. Following this, Alice initiated another short discussion:

8 Alice: I would like to bring this problem a little bit further. Notice that Student S3 presented the information this way. Is there another way to represent the same information?

(After some time, Student S4 highlights a possible way.)

9 S4: Change column and row. (Student S4 goes up and writes a 3 × 2 matrix.)

This response got students thinking about the order of the corresponding "points matrix". Another Student S5 went up and wrote the correct matrix product as a 2 × 1 matrix but pre-multiplied the 3 × 2 matrix to the 1 × 3 matrix. Alice then orchestrated a short discussion for Student S5 to realise his mistake, who then correctly wrote:

$$(5 \quad 3 \quad 2)\begin{pmatrix} 29 & 30 \\ 10 & 6 \\ 5 & 8 \end{pmatrix} = (185 \quad 184)$$

In this short vignette, we see how Alice orchestrated a mathematically productive discussion using Smith and Stein's (2011) five practices. Alice monitored students' answers to the questions carefully when she was circulating the classroom. Even though she asked for volunteers to answer the questions, it was clear that she was deliberate in her selection and sequencing of students' responses (See Lines 1 to 6). By beginning with an arithmetic solution, she was able to connect Student S1's arithmetic operations to how matrix multiplications are performed through the sequencing of Student S2's and Student S3's matrix solutions. Alice also highlighted the different ways to express the given information as matrices (Lines 8 and 9), which was an important idea for the lesson, and gave the motivation for using a matrix approach. The reason for writing the problem as a product of two matrices (Student S3's solution) was made explicit when Alice moved from Student S2's solution to the solution offered by Student S3.

ALICE'S NOTICING AND AFFORDANCES OF THE TASK

The two vignettes highlight how Alice went beyond solving the original problem procedurally and identified students' experiences that could be enhanced. She modified a typical examination question to emphasise at least three key ideas in matrix multiplication during the classroom discussion. First, Alice used the arithmetic solution to write out explicitly how matrix multiplication is performed (See Line 2). Next, she emphasised the order of matrices and when matrices could be multiplied (Line 7). Lastly, she highlighted how contextual information represented in matrices could be captured in different ways (e.g., using 2×3 and 3×1 matrices or using 1×3 and 3×2 matrices). Drawing from how Alice orchestrated the learning experience in class, we argue that she noticed the affordances of such a typical problem. More specifically, Alice attended to students' possible confusion that there was only one way to represent information using matrices, and used her understanding of the relationships between arithmetic and matrix operations to modify the problem. Moreover, Alice's orchestration of the discussion in class suggests she was more attuned to students' particular solutions and thus she was able to sequence the presentation to enhance students' learning experience.

In many ways, Alice's noticing can be classified as extended in that she modified the problem based on her interpretation of content requirements, and attended to particular strategies offered by various students. Her strong content mastery could be attributed to the professional development activities, such as mathematics-related and general pedagogical courses she had taken in recent years. Alice's use of the typical problem suggests that she was familiar with the syllabus requirements. Even though her post-lesson interview reveals a strong need to fulfil the requirements of the examinations, Alice tried to interpret and adapt the written curriculum to suit the needs of her students. She was reflective and was always ready to learn from her experiences with students. By "recognising possibilities" from her "three worlds of experiences" (Mason, 2002, p. 94), Alice was able to see beyond the given typical problem, had a different act in mind, and proposed another way to ask the question. Furthermore, given her focus on the concept, students' thinking, and how she orchestrated the learning experience of her students, we also classify Alice's noticing as productive (Choy, 2015) because she had recognised the affordances of a typical task that could potentially led students to gain new insights into the use of matrices.

CONCLUDING REMARKS

Despite a high-stakes examination-driven system, Alice's noticing had enabled her to use a typical problem beyond drill-and-practice for examination to emphasise conceptual understanding. Being able to perceive affordances or "notice possibilities" (Mason, 2002, p. 94), as Alice had done, could therefore provide a way to negotiate the murky territory between procedural and conceptual fluency. Adopting a pragmatic approach, she moved in between the two, and capitalised on the affordances of a typical examination problem to (i) reinforce procedural skills, (ii) emphasise examination requirements, and (iii) at the same time, develop conceptual understanding. Notwithstanding the limitations of our preliminary findings, Alice's modification and use of a typical problem highlight the potential of noticing affordances of such questions, and how even typical questions can be used for orchestrating discussions. Given the prevalence of using such typical problems in classrooms, supporting teachers to unlock these problems' affordances may hold promising implications for teaching and learning. It remains to be seen how we can support teachers to notice the affordances of tasks in our future work with them.

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