SPECIALISING AND CONJECTURING IN MATHEMATICAL INVESTIGATION

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This paper introduces a new framework to model the interactions of the processes of specialising and conjecturing when students engage in mathematical investigation. The framework posits that there is usually a cyclic pathway alternating between examining specific examples (specialising) and searching for pattern (conjecturing), instead of a linear pathway as in many other theoretical models. The framework also distinguishes between observing a pattern and formulating it as a conjecture, unlike most models that treat an observed pattern as a conjecture to be proven or refuted. I will then use the framework to analyse and explicate a secondary school student's specialising and conjecturing processes while he attempted an open investigative task.

INTRODUCTION

There is quite a number of theoretical models developed by educators on the processes in mathematical investigation, e.g. Height (1989) and Bastow, Hughes, Kissane and Mortlock (1991). But many of these models remain theoretical in the sense that there are very few empirical studies on these processes in mathematical investigation despite a thorough search of past and current literature. Moreover, most of these models show a linear pathway from one process to another when in reality, based on empirical data such as those from Yeo (2013), many of the pathways are cyclic. Many theoretical models also oversimplify some of the processes, such as equating an observed pattern to a conjecture to be proven orrefuted, when empirical data suggest that some students will go back to try more examples after observing a pattern in order to be more certain that there is indeed a pattern before formulating it as a conjecture. Therefore, there is a need for a more comprehensive framework to more accurately describe the interactions of the processes in mathematical investigation.

This paper will describe a new framework called the Model for Cognitive Processes in Mathematical Investigation (or the Investigation Model in short), which was developed as part of my doctoral study (Yeo, 2013) to analyse and explicate the cognitive processes when secondary school students attempted open investigative tasks. In particular, this paper will focus on two of the processes called specialising and conjecturing. Specialising is the process of examining special cases or trying specific examples to search for patterns in order to generalise, and conjecturing is the process of searching for patterns and formulating conjectures based on the patterns observed. Specialising and conjecturing, together with justifying (conjectures) and generalising,

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are the four main mathematical thinking processes identified by Mason, Burton and Stacey (2010).

Some researchers (e.g. Clement, 2000) believe that one of the most important needs in basic research on thinking processes is the need for insightful explanatory models of these processes. This type of explanatory models is often iconic in nature and the purpose of the model is to give satisfying explanations for patterns in observations (Lesh, Lovitts, & Kelly, 2000). Schoenfeld (2002) explained that the descriptive power of a model will be high if the model can capture the essence of the phenomenon. Therefore, this paper will illustrate how the Investigation Model can be used to analyse, describe and explicate the processes of specialising and conjecturing in mathematical investigation.

THEORETICAL FRAMEWORK

Based on existing theoretical models of mathematical investigation in literature, I had modified and designed an explanatory framework to model the types and interactions of cognitive processes in mathematical investigation (Yeo, 2013), which is reproduced in Figure 1. The left side of the Investigation Model shows the three phases and eight stages of mathematical investigation. The right side of the model shows the types of processes (indicated by unshaded boxes) and outcomes (indicated by shaded boxes), and their interactions. It is necessary to include outcomes in the model because the processes do not just interact among themselves but they also interact with the outcomes. Most of the stages are named after the main process(es) in that stage.

The process of specialising occurs in the stage of 'Specialising and Using Other Heuristics', but it is beyond the scope of this paper to examine 'other heuristics' such as deductive reasoning. The stage of 'Conjecturing' consists of the process of searching for patterns and two outcomes: 'Observed Pattern' and 'Formulated Conjecture'. Many theoretical models usually show a single pathway from specialising (or trying examples) to pattern searching. But the Investigation Model on Figure 1 allows for a cyclic pathway alternating between specialising and searching for patterns. Unlike other models, the Investigation Model also separates the formulation of a conjecture from the observation of a pattern because some students will go back to specialising some more after observing a so-called pattern because they are not sure whether there is really a pattern. Only after trying more examples and finding the same pattern will the students finally treat it as a conjecture to be proven or refuted.

Although the focus of this paper is on specialising and conjecturing, it is necessary to consider what follows after a conjecture is formulated because one of the processes in the Justifying Stage, called naïve testing, looks rather similar to the specialising process when students go back to try more examples to be more certain of a pattern. However, naïve testing of a conjecture is different in the sense that there must be a conjecture first, and the students are supposed to try to justify or refute the conjecture by using either a formal proof (Tall, 1991) or a non-proof argument based on the underlying structure (Mason et al., 2010). But very often the students are not able to think of a proof or argument, so some of them will test the conjecture by trying to find if there are counter examples to refute it: this is called naïve testing by Lakatos (1976).

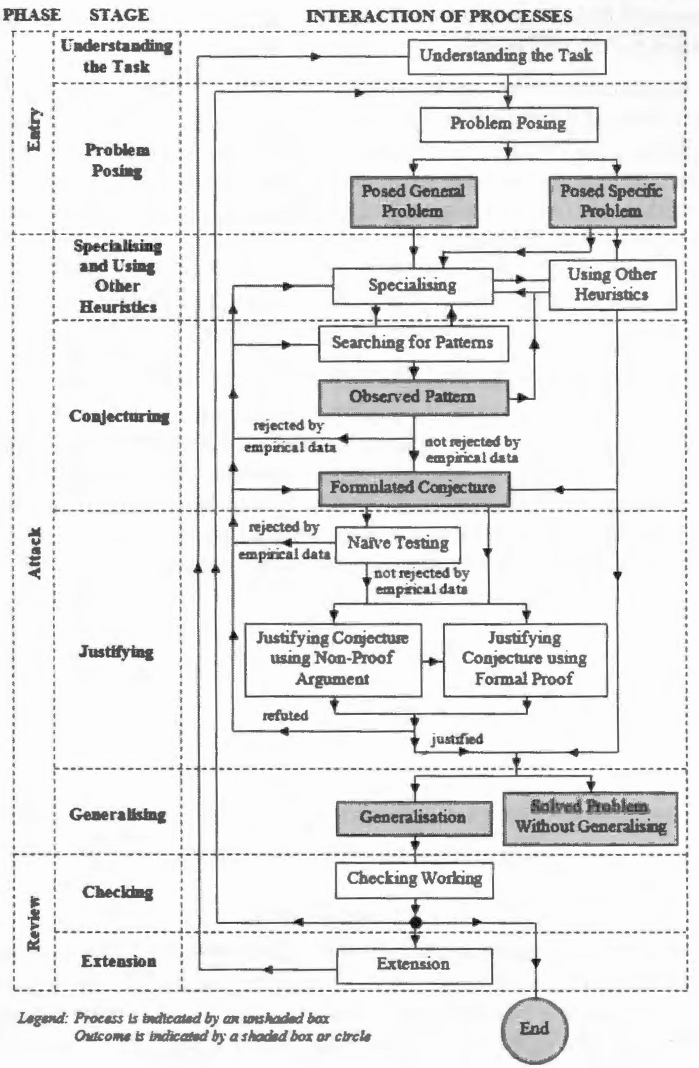

Figure 1: Model for Cognitive Processes in Mathematical Investigation

METHOD AND ANALYSIS

The Investigation Model will be used to analyse the cognitive processes of a secondary school student Ben (pseudonym) when he attempted the following posttest task:

> Choose any number. Add the sum of its digits to the number itself to obtain a new number. Repeat this process for the new number and so forth. Investigate.

Ben was one of the students in my doctoral study (Yeo, 2013) who had undergone a teaching experiment on mathematical investigation consisting of six two-hour lessons: he had been taught how and what to investigate when given open investigative tasks, including cognitive processes such as p roblem posing, specialising, conjecturing, justifying and generalising. Ben was videotaped thinking aloud while he attempted two pretest and two posttest tasks. The verbal protocols were then transcribed and coded to identify the types of processes and their interactions. The following episodes were chosen to illustrate how the Investigation Model can be used to describe and explain Ben's processes of specialising, conjecturing and naïve testing.

Episode 1: Interaction between specialising and pattern searching

Figure 2 shows the first portion of Ben's working for the task. He started with the number 12345 and added the sum of its digits to itself to obtain 12360. Then he made a serious mistake in misinterpreting the task: instead of repeating the process for the new number 12360, he repeated the process for a completely new random number 242 and obtained 250. Despite him not recovering from his error throughout the investigation, the Investigation Model is still able to capture his thinking processes.

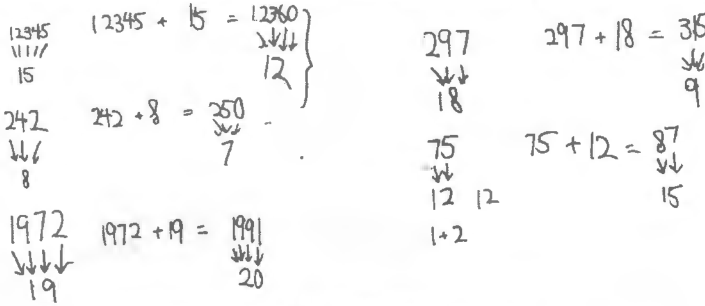

Figure 2: First part of Ben's working

The following protocols continued from when Ben started trying the fifth number 75 (this will be called Example 5) at the right end of Figure 2. Square brackets were used in the transcript to enclose the transcriber's comments such as what the student was writing. An ellipsis was used to indicate a short pause of three seconds or less.

20 03:43 If I try to use a small number like ... 75 [write: 75] ... 75 [draw an arrow from each digit of 75 downward] I add it together I have [write: 12] 12 ... [continue writing in another line] 75 + 12 = 87 [stop writing] ... 87 [draw an arrow from each digit of 87 downward] equals to 15 [write: 15] ... 15 which is 1 ...[start writing] 1 + 2 [stop writing] ... added to 12 [write: 12] ...

21 04:20 So 19 [point pen at 19 in Example 3] … is ...

22 04:25 If I try with another, another two-digit number like [turn to new p. 2] … 27 [write: 27] [draw an arrow from each digit of 27 downward] 27 equals to, add together is 9 [write: 9] 9 … [continue writing in another line] 27 + 9 [stop writing] … will give me 36 [write: 36] 36 [draw an arrow from each digit of 36 downward]. If I add them together, I will still get 9 [write: 9].

23 04:50 There is no difference …

24 04:54 If I tr y 50 [write: 50] … 50 [draw an arrow from each digit of 50 downward] if I add it together, I will get [write: 5] 5 … [continue writing in another line] 50 + 5 = 55 [stop writing] ... 55 [draw an arrow from each digit of 55 downward] you add it up, you get a maximum of 10 [write: 10].

25 05:15 [Pause 4 seconds]

In Line 20, Ben was trying Example 5 (specialising), and in Line 21, he was searching for patterns when he pointed his pen at the number 19 in his previous Example 3. After failing to find any patterns, Ben tried a new Example 6 (Line 22) and continued to search for patterns but to no avail (Line 23). Then Ben tried Example 7 (Line 24) and paused for four seconds, presumably to search for patterns. Thus Ben was alternating between specialising and pattern searching as illustrated by the Investigation Model in Figure 3. The numbers in the figure represent the line numbers in Ben's transcript but with a di fference. For example, Line 20 is coded as ' Specialising' and L ine 21 as 'Searching for Patterns, but in Figure 3, it is more helpful to use the line numbers to indicate the pathways so that we can see that Ben moved from 'Specialising' to 'Searching for Pa tterns' (indicated by 20), then back to 'Specialising' (indicated by 21), and then to 'Searching for Patterns' again (indicated by 22), and so forth.

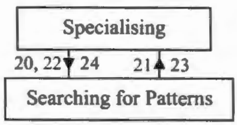

Figure 3: Alternating between specialising and searching for patterns

Episode 2: Difference between observed pattern and formulated conjecture

The following protocols picked up from when Ben started trying Example 10 using the number 5000.

40 09:05 If I try to add ... if I try example of an even number [write: 5000] 5000 ... 5000 [draw an arrow from each digit of 5000 dow nward] if I a dd it together, I will have 5 [write: 5] ... If I try ... so [continue writing in another line] 5000 + 5 = 5005 [stop writing] —

41 09:32 — which is an odd number [write: (odd no.)] ... So from an even number [point pen at 5000], I obtain an odd number ...

42 09:44 Let me try an odd number now. [Write: 5001] 5001. [Draw an arrow from each digit of 5001 downward] If I add it all together, I will have 6 [write: 6] ... [continue writing in another line] 5001 + 6 [write: 5001 + 6 = 5007].

43 09:56 It will still give me an odd number: 5005, 5007.

44	10:02	[Pause 5 seconds]
45	10:07	I try five thousand [write: 5] and ... 5222 [write after the digit 5: 222]. [Draw an arrow from each digit of 5222 downward] It will give me [write: 11] 11 [continue writing in another line] 5222 + 11 = 5233 [stop writing]
46	10:28	— which is still an odd number ...
47	10:32	Every time I add them together [draw a big brace from 5005 to 5233], I get an odd number ... Is it the same for every single pattern? ...

In Line 40, Ben was trying Example 10, and in Line 41, he observed a pattern that the next number was an odd number. Sometimes a student may observe a pattern immediately after trying an example, so it is not easy to distinguish the exact juncture between searching for patterns and observing a pattern. However, Ben was not sure whether there was really a pattern because he continued to try two more examples (Example 11 in Line 42 and Example 12 in Line 45) and he observed that it was still the same pattern (Lines 43 and 46). So he said, "Is it the same for every single pattern?" (Line 47) What he probably means is whether the pattern is the same for every single example. At this moment, Ben was more certain that there was a pattern and this was coded as when he formulated his conjecture. In other w ords, there is a difference between 'observed pattern' and 'formulated conjecture': Ben did not treat the pattern as a conjecture when he first observed it, but he went back to try more examples to see if the pattern could withstand the test of a few more examples before formulating it as a conjecture. Figure 4 shows the pathways of Ben's processes and outcomes in Episode 2 as modelled by the Investigation Model.

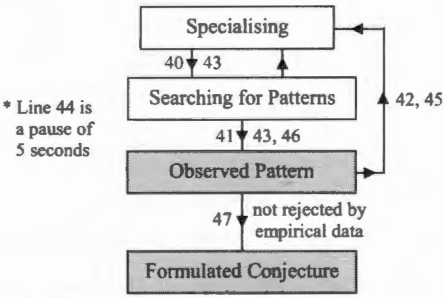

Figure 4: Observed pattern vs. formulated conjecture

Episode 3: Difference between specialising and naïve testing

The following protocols showed what happened immediately after Ben formulated his conjecture in the previous episode. Instead of trying to think of a non-proof argument or a formal proof to justify his conjecture, Ben decided to test his conjecture by trying to find if there are counter examples to refute it.

48	10:40	Write: 2987] 2897. [Draw an arrow from each digit of 2987 downward] If I add it all together, I get ... 26 [write: 26] ... [continue writing] 2897 + 26 = [stop writing] ... 2 ... 2903 ...
49	11:12	2923 [write: 2923].

50	11:14	[Pause 4 s]
51	11:18	What if I try to... The main reason why I'm getting an odd number is because this is odd [circle 7 in 2897] ... and this is even [circle 6 in 26] ... If I put another odd number and odd number, or even number and even number like ...
52	11:31	[Write: 2572] 25 ... 72. [Draw an arrow from each digit of 2572 downward] If I add them all up, I'll get a total of ... [write: 16] 16 ... [continue writing in another line] 2572 + 16 [stop writing] —
53	11:47	— will, should give me an even number 2588 [write: 2588].
54	11:55	So, therefore, this is an even number [write: (even no.)]. So it doesn't have to be odd number all the time ...

In Line 48, Ben tried Example 13 and found the same pattern (Line 49). Then he paused for four seconds. During this time, he was able to deduce the reason behind his conjecture, which he articulated in Line 51. The deductive argument led him to think of a counter example (Example 14 in Line 52) to ensure that the next term in the sequence was even instead of odd (Line 53), thus refuting his conjecture (Line 54). This kind of naïve testing in the Justifying Stage is different from trying more examples in the Specialising and Conjecturing Stages to be more certain that there is indeed a pattern first because naïve testing happened after the formulation of a conjecture at the end of the Conjecturing Stage. Figure 5 shows the pathways of Ben's processes and outcomes in both Episodes 2 and 3 as modelled by the Investigation Model. It is beyond the scope of this paper to discuss what happens after a conjecture is rejected by naïve testing.

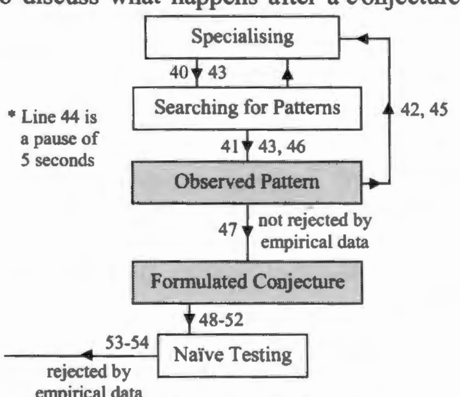

Figure 5: Naïve testing of conjecture

DISCUSSION AND CONCLUSION

The power of an insightful explanatory model on thinking processes lies in its ability to capture the actual processes and explain the interactions accurately (Schoenfeld, 2002). The analysis presented in the previous section has demonstrated how the Investigation Model is capable of faithfully depicting the interactions of the processes of specialising, conjecturing and naïve testing during mathematical investigation. The empirical data from my doctoral study have suggested that the processes and their interactions are much more complex than those modelled by most theoretical frameworks. For example, some students often alternate between specialising and searching for patterns, and a more robust model should capture such phenomenon. Also, such a model should be able to discern between processes or outcomes that look similar, such as an 'observed pattern' and a 'formulated conjecture'. Researchers can then use the framework to analyse other students' cognitive processes at a fine-grained level while teachers can use the model to teach students how to think when engaging in mathematical investigation. Therefore, the Investigation Model can help to make students' thinking processes visible to researchers and teachers.

References

Bastow, B., Hughes, J., Kissane, B., & Mortlock, R. (1991). *40 mathematical investigations* (2nd ed.). Western Australia: Mathematical Association of Western Australia.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-589). Mahwah, NJ: Erlbaum.

Height, T. P. (1989). *Mathematical investigations in the classroom.* Australia: Longman Cheshire Pty.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery.* New York: Cambridge University Press.

Lesh, R. A., Lovitts, B., & Kelly, A. E. (2000). Purposes and assumptions of this book. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 17-33). Mahwah, NJ: Erlbaum.

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (2nd ed.). Harlow, England: Pearson.

Schoenfeld, A. H. (2002). Research methods in (mathematics) education. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 435-487). Mahwah, NJ: Erlbaum

Tall, D. (1991). *Advanced mathematical thinking.* Dordrecht, The Netherlands: Kluwer Academic Press.

Yeo, J. B. W. (2013). *The nature and development of processes in mathematical investigation.* Unpublished doctoral thesis, National Institute of Education, Nanyang Technological University, Singapore. doi:10.13140/RG.2.1.1280.0880