Student Teachers' Understanding of Fundamentals in Mathematics

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In the context of teacher education's emphasis on general pedagogical skills and, in particular, mathematical pedagogical skills, there has been much discussion by mathematics teacher educators on the type and extent of mathematics content knowledge needed by teachers of mathematics for teaching at various school levels. Galbraith, in his 1982 *Educational Studies in Mathematics* paper, conceptualized and studied the *Mathematical Vitality* of prospective teachers from different teacher preparation programmes in Australia. For teacher education programs which include disciplinary mathematics, teacher educators at universities need to question and determine the nature of mathematics content knowledge they are developing in their student teachers. The B.Sc (Ed) program in Singapore is a four-year program which prepares primary as well as secondary school teachers. This paper presents the findings of a study on a small sample of student teachers who have taken disciplinary mathematics as a content subject at university level for five semesters. The instrument used**,** which is a modification of Galbraith's *Mathematical Vitality* instrument, tests the student teachers' understanding of basic mathematical terminology, of fundamental logic of mathematics and of what constitutes a valid mathematical proof. Of the 20 student teachers, three answered more than half of the 12 items wrongly while eight gave three or fewer wrong answers. The paper will also discuss the responses to individual items**,** and some implications for the curriculum and teaching of Mathematics to future student teachers.

Keywords: mathematics teachers' subject knowledge, mathematics teacher education, mathematics vitality

Introduction

At university level, mathematics is being taught to an ever-widening spectrum of students who have a slate of diverse reasons and objectives for reading the subject. While few will go on to become research mathematicians, the majority take mathematics courses specially tailored as a service subject in order to complete their degrees in Engineering, Business, Science, Computer Science, Economics and so on. Many of these university undergraduates are thus only interested in Mathematics for its practical applications to their future profession. There is, however, one group of undergraduates for whom mathematics in general will remain relevant and these are student teachers who will be teaching mathematics almost daily in their profession as teachers. In fact, for Singapore mathematics undergraduates of the 1970s and 1980s, teaching was the most natural and popular career upon graduation. While mathematics teachers may not need to understand or use deep level mathematics as is the case of mathematicians, research scientists and engineers, their knowledge of mathematics cannot be at a superficial level of merely carrying out algorithms or applying formulae at a level slightly above what they have to teach in the curriculum. As mathematics educators, they have the all-important role of developing in their students an early conception of what mathematics is all about. The mathematical preparation of potential mathematics teachers at the university level is thus a significant strand in any discussion of the teaching of university mathematics. As an example, the *Preparation of Primary and Secondary Mathematics Teachers* was one of the working groups in the 1998 ICMI study conference on *the Teaching and Learning of Mathematics at University Level.*

Over the last two to three decades, mathematics teacher educators have wrestled with the issue of content knowledge for teaching with the concept of pedagogical content knowledge (PCK) as specialized knowledge just for teaching which is subject specific and different from subject matter knowledge. There has also been much research and discussion on teacher knowledge and its impact on their teaching and their students' learning (Ball, Lubienski & Mewborn, 2001 and Mewborn, 2003) and reports/reviews on the mathematical preparation of teachers such as Tucker et al (2001). However, there is little conclusive evidence to show links between teachers' mathematics subject knowledge, particularly as measured by the number of mathematics courses taken at university level or highest degree, and their students' achievements.

Ball, Thames & Phelps (2008) have taken the discussion further to distinguish and identify subject matter knowledge (SMK) for teachers, a concept which is not to be confused with pedagogical content knowledge. They were "struck by the relatively uncharted arena of mathematical knowledge necessary for teaching the subject that is not intertwined with knowledge of pedagogy, students, curriculum or other non-content domains." They looked into specific tasks which teachers are expected to engage in, tasks such as responding to students' "why" questions or recognizing what is involved in representing a particular mathematical idea, and noted that "these tasks require knowing how (mathematical) knowledge is generated and structured in the discipline and how such considerations matter in teaching", at the same time lamenting that the knowledge and skills for such tasks are "not typically taught to teachers in the course of their formal mathematics preparation".

The concept of *Mathematical Vitality* as an attribute of a mathematically aware student of mathematics was defined by Galbraith (1982) and this attribute included having a deeper understanding of mathematical processes, the ability to carry out a mathematical analysis and the ability to construct a logical defence of mathematical statements. These are expected abilities of university graduates who major in mathematics. While students can graduate with a university mathematics degree with the knowledge of much mathematics content, what lasts much longer is this *Mathematical Vitality* because it is an attribute which becomes part of the learner's mathematical disposition.

We thus conceptualized *Mathematical Vitality* as the attribute of having basic understanding of the discipline of mathematics in terms of its language and structure, and in particular, the ways and processes of determining and accepting mathematical truths. As there has been research evidence to show that proofs are difficult, not only for potential mathematics teachers but also for university mathematics undergraduates in general (Jones, 2000, Knuth, 2002 and Weber, 2003), we also include an understanding of mathematical proofs as a component of *Mathematical Vitality*. However, it should be pointed out that at this stage, our study of *Mathematical Vitality* is not focused on student teachers' conceptions of proof or their particular difficulties with proofs as discussed in Knuth (2002) or Weber (2003), but on their basic understanding of when a proof is considered valid mathematically.

Although *Mathematical Vitality* may not be content in terms of mathematical topics, theorems and definitions, it is certainly a constituent part of subject matter knowledge for teachers as described by Ball, Thames and Phelps (2008). This is because *Mathematical Vitality* does not deal with knowledge of pedagogy or student understanding, but is important for teachers to have in order that they can in turn help their students build up some understanding of the structures of the discipline.

As mathematics teacher educators at Singapore's National Institute of Education (NIE), the sole teacher education institute in Singapore responsible for pre-service teacher preparation, we are particularly concerned with the mathematical preparation of future mathematics teachers. In particular, for student teachers who read disciplinary mathematics in their program, there was concern as to whether they would graduate with the understanding of these fundamentals and whether their teacher preparation program had adequately developed *Mathematical Vitality* in these potential mathematics teachers. In this context, we undertook a study *Mathematical Vitality of Mathematics Student Teachers* to determine the state of our student teachers' *Mathematical Vitality* at various points of their tertiary mathematics journey. This paper reports on the initial findings on the senior student teachers' Mathematical Vitality. Subsequent phases of the study will focus on intervention in terms of curriculum and teaching.

Context of the Study

The NIE has three groups of programs that prepare teachers for teaching various subjects in the school system. This study is concerned with the four-year Bachelor of Science (Education) ((B.Sc (Ed)) program where there is adequate curriculum space and time for the development of student teachers' mathematics content knowledge. The program has two tracks, for primary teaching and secondary teaching respectively.

All programs for teacher preparation at NIE include two or three Curriculum Studies (CS) subjects, each of which comprise methods courses for developing student teachers' pedagogical content knowledge and pedagogical skills necessary for teaching that specific subject. The number of CS subjects taken by the B.Sc (Ed) student teachers is two for potential secondary teachers and three for potential primary teachers. In addition, the undergraduates in the B.Sc (Ed) program have to take Academic Subjects (AS) which are normal university level disciplinary subjects. Those in the secondary track take two such subjects, one at major and the other at minor level, and these two subjects correspond to their CS subjects, for example, Mathematics and Biology. Those in the primary track take one AS which is usually but not necessarily linked to one of the three teaching subjects. They are also required to take three subject knowledge (SK) areas which correspond to the three teaching subjects. The courses in the SK areas are different from normal university level discipline courses. They cover content which is related to primary school curriculum but with a deeper treatment and from a broader contextual perspective. Subject knowledge courses are not deemed necessary for the secondary track since the academic subjects take care of SMK for the student teachers. Table 1 shows program structure of the four-year B.Sc (Ed) program. A subject comprises a few courses and courses are normally of two or three academic units (AUs) where 1 AU is equivalent to 12 hours of contact teaching time.

The structure shows that the academic subject forms a large portion of the program, 30% when taken at major level. So, a primary track potential mathematics teacher who also takes AS mathematics would spend 55 AUs or 43% of his/her program on mathematics courses: 39 AUs for AS mathematics, 10 AUs for CS mathematics and 6

AUs for SK mathematics[8]. In the secondary track, a potential mathematics teacher has to take both AS and CS mathematics although the AS mathematics could be at either major or minor level.

Table 1. Structure of the B.Sc(Ed) Program

Track	Subject Specific			Compulsory Courses			TOTAL
	CS	AS*	SK	Education Studies	Practi-cum	Others	
Primary	3 subjects (10 AUs each)	1 subject (39 AUs)	3 subjects (6 AUs each)	5 courses (12 AUs)	4 stints (21 AUs)	3 courses (8 AUs)	128 AUs
Secon-dary	2 subjects (12 AUs each)	2 subjects Major: 39 AUs Minor: 24 AUs	Nil	5 courses (12 AUs)	4 stints (21 AUs)	4 courses (11 AUs)	131 AUs

Note: Each AS subject comprises 13 courses of 3 AUs each. A subset of 8 courses is taken by those doing it at minor level.

Within the B.Sc (Ed) program, all mathematics courses are offered by the Mathematics and Mathematics Education (MME) academic group, the department which houses both mathematicians and mathematics educators and is responsible for the curriculum and delivery of these courses. MME has the main responsibility for the development of mathematical SMK and PCK of the B.Sc (Ed) student teachers and hence it is important for the faculty to understand the mathematical standards of our student teachers beyond assessment results and to investigate if our graduates do indeed achieve the stated objectives of our courses. Moreover, since our undergraduates are being prepared for just the teaching career, MME can design and deliver focused AS mathematics courses which are better suited for teachers, provided it is clear on what is required to develop good teachers of mathematics.

Method, Instrument and Sample

In this context, the current study is designed as the first stage in finding out the *Mathematical Vitality* of our student teachers where we have been responsible for their mathematics learning at tertiary level. The authors sought and received permission to modify and use items from the *Mathematical Vitality Test* as designed by Galbraith (1982). The instrument comprised 12 items in multiple-choice format and space was provided for explanations if any. The items are given in the Annex together with an explanation of what mathematical understanding was being examined for each item. The instrument also sought background data of the participants in terms of the mathematics courses which they had completed in their program so far.

The population of this study is the whole cohort of student teachers of the B.Sc(Ed) program who take AS mathematics as a major. They were from both the primary and secondary tracks and the study was conducted when they had completed at least ten

[8] It must be noted that while any AS mathematics student teacher in the primary track would always choose CS mathematics and correspondingly SK mathematics, the converse is not true since a primary track student teacher is trained to teach 3 subjects i.e. has 3 CS and 3 SK subjects, but has only 1 academic subject which could be any one from the slate of Mathematics, Physics, Chemistry, Biology, English, English Literature, History, Geography, Art, Music or Drama.

mathematics courses out of the thirteen required for AS mathematics. They were invited to participate through taking the test on a voluntary basis and were assured that the results would have no bearing on their course assessment and moreover produce findings to help MME improve the mathematics curriculum. The test could be taken at any time during a two-day period at a fixed venue. Since the student teachers knew that it was for research purposes, there was no concern that earlier participants would tell others about the items and this flexibility in test administration time was intended to encourage higher rates of participation. Nevertheless, of the more than eighty student teachers in the cohort, only twenty turned up to take part in the study.

Findings and Discussion of Items

General Results

Among the 20 participants, nine were from the primary track and eleven were from the secondary track. Fourteen of them have taken ten mathematics courses while the remaining six had completed eleven or twelve courses. Eight of these courses were compulsory core courses (Calculus 1 and 2, Algebra 1 and 2, Statistics 1, Finite Mathematics, Number Theory and Computational Mathematics) while the other courses were electives with different combinations of choices for the various student teachers.

Although there was no time restriction, most of the student teachers completed the test in 20 to 30 minutes with three taking only 10 to 15 minutes and two taking 45 minutes. The results showed a wide variation among the student teachers. The test instrument had 12 items and the number of correctly answered items ranged from 2 to 11. The items are not given different marks based on difficulty and here the score on the test shall be taken to mean the number of correctly answered items.

Figure 1 shows the number of student teachers attaining each score from 0 to 12. The mean score is 7.7, the median is 7.5 and the mode is 7 where 5 student teachers obtained correct answers to 7 items. Half the student teachers managed to get 8 or more items correct while only 3 of them got less than 6 correct. There was one outlier case who obtained only two correct answers.

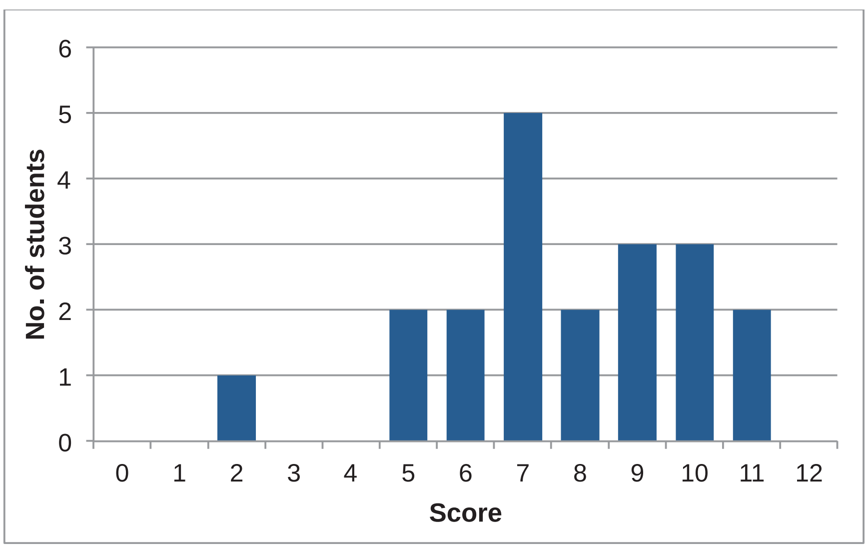

Figure 1. Distribution of student teachers across the scores.

Discussion of individual items

Table 2 shows the performance of student teachers on individual items with respect to the choices made. The correct choice is marked in bold and underlined. From table 2, it can be seen that 6 of the items, items 1, 4, 6, 7, 10 and 11, had at least 70% choosing the correct answer. In fact, items 1 and 11 were the items where almost everyone selected the correct option and the only person who chose a wrong option in item 1 was the student teacher who only obtained two correct answers among the 12 items.

Table 2. Student teachers' choice of options in the items

Item Number	Choice				
	A	B	C	D	E
1	**19**	0	0	1	0
2	13	**7**	0	0	0
3	3	3	3	5	**6**
4	2	0	0	**14**	4
5	**11**	6	0	2	1
6	0	2	1	2	**15**
7	**14**	1	1	3	1
8	0	**12**	5	0	3
9	5	1	**10**	4	
10	**16**	1	2	1	
11	0	0	0	**20**	0
12	2	**10**	1	7	

The following sections will discuss performance in the various items clustered according to the concepts or processes being tested.

Items testing basic knowledge

There were two items designed to test basic knowledge. These were item 2 which tested the knowledge of convention that the square root sign (√) referred to the positive root only and item 4 which checked on their knowledge that π is irrational and the consequences thereof. The students performed fairly well on item 4 although it is of concern that 6 of them either did not realise that π is irrational or could not deduce that the product of a rational number and an irrational number is necessarily irrational.

Item 2 was the single item where the wrong option (A) was chosen by a sizeable majority. In their earlier learning of mathematics, they had been so conditioned to write down both positive and negative square roots when solving $x^2 = a$ (where $a > 0$) that it was quite natural for 13 of them to think that $\sqrt{9}$ was ± 3. This result is not surprising as similar results were reported for the same item in Galbraith (1982) while testing prospective teachers in Australia and in Wong (1990) for testing PGDE (Sec) student teachers in Singapore. However, this is not too worrying since the misconception can be corrected through explicitly making the notation clear and emphasising the distinction between $y^2 = x$, $y = \pm \sqrt{x}$, $y = \sqrt{x}$ and $y = -\sqrt{x}$.

Items testing the use of examples as proof

Items 3, 11 and 12 tested whether the participants would accept a range of examples (whether a finite or infinite number of them) as proof of a result. For item 3, it is clear from the spread of choices and that only 6 chose the correct option (E) that this was a difficult item for the student teachers. It is also an item where every distractor was chosen by a few student teachers. Distractors (A), (B) and (C) were designed to test the misconception that an infinite number of objects satisfying a property points towards all such objects satisfying the property and it is a matter of concern that a total of 9 participants chose these options. Very few wrote additional explanations but it was worrying that one wrote “infinitely many means all”. There were 5 student teachers who chose option D and a closer study of the four explanations given showed that they did not have the same misconception. In fact two wrote “infinitely many does not imply all” while two others tried to give counter-examples such as right-angled triangle satisfying Pythagoras’ Theorem. However, a lack of understanding of the phrase “necessarily correct” and not taking into consideration the phrase “more likely” resulted in their choosing Option (D) as the closest to their conclusion.

In item 12, however, a classroom situation and actual well-known geometrical result were presented and the participants were to determine if the teaching activity was an acceptable “proof” of the result. Half of the 20 participants stated correctly that although many examples were used to verify the result, the proof was invalid as one must prove it for ALL triangles. Nevertheless, 7 of the participants did accept the activity as valid because it covered different types of triangles but it was reassuring that 3 of the 7 explained that they felt the activity was valid in the context of teaching but more formal proofs were necessary for mathematics.

While student teachers had problems with item 3 and, to a lesser extent item 12, they had no problems with item 11. Perhaps this is due to the fact that the “proofs” used specific numerical examples while items 3 and 12 did not and hence all the participants were able to determine that the “proofs” were invalid.

Items testing mathematical logic

Items 5 to 8 checked on the participants’ ability to apply mathematical logic to particular examples. In general, items 6 and 7 were well done. For item 6, 15 of the participants understanding that a proven result does not imply the truth or otherwise if its converse and for item 7, 14 of the participants understood how to select a counterexample to disprove an if-then statement.

For items 5 and 8, 11 and 12 participants respectively obtained correct answers, with 8 participants getting both correct and 5 getting both wrong. What was particularly interesting was that among the 5 participants who selected Option (C) in item 8, believing that the converse of a true statement was also true, 4 of them also gave the wrong option in item 5 with three of the four accepting that the contrapositive of a theorem was true but felt it still needed to be proved.

Items dealing with specific proofs

Items 9, 10 and 11 dealt with specific proofs of results and student teachers were to determine whether the proofs were correct or valid and where there were errors. Item 11 has already been mentioned and Item 10 was also not difficult, with 16 student teachers being able to ascertain that the proof by contradiction was valid. In fact, 5 of the 16 wrote in explanation that the given example was a *proof by contradiction*.

However, only 10 participants managed to recognise that the proof in item 9 was incorrect because it was proving the converse instead of the result, choosing option (C).

A closer inspection of the answer scripts showed that 3 of these 10 also wrote explanations which contained some mathematical error.

Of the remaining 10 who did not choose option (C), 5 deemed the proof valid (option (A)) while 4 chose (D) but either gave no reason or mathematically incorrect reasons.

Conclusion and Implications

Overall, the results showed that student teachers were not very strong in their understanding of fundamental mathematical logic, mathematical language or proof techniques. Considering that the participants had already completed at least ten of their thirteen courses for the academic subject mathematics, only half of them managed to get 8 or more items correct out of the 12 items. The difficulty could be due to two features of the instrument: (i) many of the questions were not contextualised with explicit numerical examples or specific geometrical properties and (ii) the rather unfamiliar language used. This is also a cause for concern because it meant that student teachers lacked fluency in mathematical language and were unable to draw general conclusions.

In terms of the concepts tested, the better performance of item 6 relative to item 9 indicated that while they understood that one could not make conclusions about the converse of a proven statement, a substantial number could not determine that the given proof was proving the converse of a statement instead of the statement itself and was thus a wrong proof. In the experience of faculty teaching university mathematics, this result is probably not surprising as students are quite likely to construct a proof from the conclusion to the premise instead of the other way round. This could be due to the fact that most of the theorems learned in secondary or high school were the if-and-only-if type of results.

It also appears that mathematical logic is not well understood by the student teachers as seen by their inability to argue out the correct responses, especially where the examples were not specific enough. The relationship between a statement and its contrapositive, for example, as tested in items 5 and 8, was only understood by slightly more than half the participants. It has been a debated issue in the department whether various aspects of mathematical logic coupled with methods of proofs should be a covered in a specific separate course or whether they should be just taught within the various courses covering different topics. While it would be difficult to teach the logic without content topics, the current practice of having them implicitly done within various topics has not been successful as shown by these results. Faculty could make a concerted effort to continually emphasise the various methods of proof and the understanding of in mathematical logic in their teaching of topics, to the extent of explicitly naming the methods or terms. From the fact that some students actually used terms like "contrapositive" and "proof by contradiction", we deduce that some of their mathematics lecturers had taught the terms but as indicated by those who could not select the right options, the understanding of fundamental mathematical ideas and processes of proof has not been internalised by all those who have taken a substantial number of mathematics courses.

The findings could not be directly compared to the studies by Galbraith (1982) and Wong (1990) but for some similar items, a greater proportion of the students in the current study were able to select the correct answer as compared to those in Wong (1990). This could be due to the currency of their tertiary mathematics courses, because the sample in Wong's study comprised students who had graduated one or more years previously, while the students in the current sample are still enrolled in undergraduate mathematics courses. Galbraith (1982) has argued that pre-service teachers seemed to approach mathematics with an instrumental view and has also conjectured that simply taking more mathematics courses would not enhance *Mathematical Vitality*. He had suggested more research into establishing a link between student performance and *Mathematical Vitality* as a measure of relational mathematics, both of the teachers themselves and their approach to teaching mathematics. Further study would need to be taken after the students in this study graduate to (a) know if they have retained their *Mathematical Vitality* and (b) establish links, if any, between their *Mathematical Vitality* and their teaching approaches.

One specific limitation of this set of data is the small sample size and the researchers will work towards persuading more student teachers to participate in the next data collection which will be among the new first-year students. Nevertheless, although the findings cannot be generalised, they are useful for informing faculty of their students' mathematical understanding.

As is the case with most tertiary institutions, mathematics courses are taught by experts in each area and it is quite the norm for each faculty member to cover a systematic collection of definitions, theorems and applications with the assumption that their students will somehow, through doing mathematics, learn to be fluent in mathematical language and develop a general understanding of the logical processes of arriving at mathematical truths. While this may be the case, there are students who fail to acquire such *Mathematical Vitality*, as is shown through the findings of this study, although they had performed better than those in an earlier study. The authors conjecture that enhancing the *Mathematical Vitality* of undergraduates is possible but requires a concerted, conscious and consistent endeavour on the part of faculty members in at least a large majority of mathematics courses to do so through agreed means and approaches. The results of the current study form the starting point for just such an endeavour, and the fact that mathematicians and mathematics educators work together in the same department holds promise of successful collaboration.

Continual curriculum review and investigation into our own teaching and our student teachers' learning are necessary for mathematics teacher education to remain relevant and effective. In addition to considering international issues, for example those documented by working groups 1 and 2 of the ICMI Centenary Symposium (Menghini et al, 2008), local findings on mathematical understanding of our student teachers will contribute towards targeted and specifically designed enhancements which will not be based on fashion or political correctness but on professional evidence-based and informed judgement. Further intervention study on modified curriculum or teaching methods will be the next phase of this research study.

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Annex A *Mathematical Vitality Test Items*

This test comprises 12 multiple choice questions. For each question, select only **one answer** (A, B, C, D or E). You are encouraged to explain your answer in the space provided.

Item No	Question	What is being tested
1	Which of the following statements is correct? (A) If $x = 4$, then $x^2 = 16$. (B) If $x^2 = 16$, then $x = 4$. (C) $x = 4$ if and only if $x^2 = 16$. (D) If $x \neq 4$, then $x^2 \neq 16$. (E) None of the above.	Understanding of the if-then logic
2	Which of the following statements is correct? (A) $\sqrt{9} = \pm 3$ (B) $\sqrt{9} = +3$ (C) $\sqrt{9} = -3$ (D) $\sqrt{9} = \|9\|$ (E) None of the above	Knowledge of basic symbol/terminology of square root sign
3	It has been proven that infinitely many triangles possess property H. *Statement S*: All triangles possess property H. Which of the following statements is *necessarily* correct? (A) *Statement S* is true but further proof is needed. (B) *Statement S* is true and no further proof is needed. (C) *Statement S* is more likely to be true than false. (D) *Statement S* is more likely to be false than true. (E) None of the above.	Having an infinite number of the object satisfying the statement has no bearing on the truth of the statement applied to all such objects.
4	The radius of a circle is k cm where k is a positive integer. Select whichever of the following you believe to be correct. (A) The circumference could be exactly 88 cm. (B) The circumference could be exactly 9 cm. (C) Either (A) or (B) could be correct. (D) Neither (A) nor (B) could be correct. (E) It is not possible to decide in favour of any of the above alternatives.	Knowledge that π is irrational and that the product of a rational with an irrational must be irrational.
5	*Theorem X*: If three positive integers a, b and c satisfy condition P, then they satisfy condition Q. *Statement Y*: If three positive integers a, b and c do not satisfy condition Q, then they do not satisfy condition P. *Theorem X* has been proved. Which of the following statements is *necessarily* correct concerning *Statement Y*? (A) *Statement Y* is true and does not need further proof. (B) *Statement Y* is true but needs to be proved. (C) *Statement Y* is false and does not need disproof. (D) *Statement Y* is false but need disproof. (E) None of the above.	Understanding the relation of a statement's contrapositive to the original statement.

6	The following are two statements: *Statement X*: If a polygon has property P, then it has property Q. *Statement Y*: If a polygon has property Q, then it has property P. *Statement X* is a theorem in geometry which has been proved. Which of the following statements is *necessarily* correct concerning *Statement Y*? (A) *Statement Y* is true and does not need further proof (B) *Statement Y* is true but needs to be proved. (C) *Statement Y* is false and does not need disproof. (D) *Statement Y* is false but needs disproof. (E) It is not possible to determine whether *Statement Y* is true or false from the information given.	Understanding of the truth of a statement has no bearing on that of its converse.
7.	A statement *S* reads as follows: *S*: A whole number n is divisible by 6 if the sum of its digits is divisible by 6. Select whichever of the following you believe to be true. (A) The number 39 shows that *S* is false. (B) The number 42 shows that *S* is false. (C) The numbers 39 and 42 both show that *S* is false. (D) *S* is false but neither 39 nor 42 is adequate to disprove it. (E) *S* is true.	Ability to use counter examples appropriately to disprove an if-then statement.
8	We know that a given statement *S* is true if $y < 0$. Which of the following statements must be true? (A) If *S* is true, then $y \geq 0$. (B) If *S* is false, then $y \geq 0$. (C) If *S* is true, then $y < 0$. (D) If *S* is false, then $y < 0$. (E) None of the above.	Ability to apply the relationship between a statement and its contrapositive.
9	Result: For $a > 0$ and $b > 0$, $\frac{1}{2}(a+b) \geq \sqrt{ab}$. Proof: Step 1: If $\frac{1}{2}(a+b) \geq \sqrt{ab}$, then $(a + b)^2 \geq 4ab$. Step 2: Hence, $a^2 - 2ab + b^2 \geq 0$. i.e. $(a - b)^2 \geq 0$. Step 3: Since $(a - b)^2 \geq 0$ is always true, $\frac{1}{2}(a+b) \geq \sqrt{ab}$. Do you think the proof is valid? Why? (A) Yes, the proof is valid. (B) No, Step 1 is wrong because we cannot square both sides of an inequality. (C) No, the proof should not begin with what is to be proved and lead to a correct result. (D) No, for other reasons. (Please give/explain your reason(s).)	Ability to recognise the error of proving the converse instead.

10	<u>Result</u>: If $x + 4$ is an odd integer, then x is an odd integer. <u>Proof</u>: Given that $x + 4$ is an odd integer, assume that x is even. Let $x = 2k$ where k is an integer. Then $x + 4 = 2k + 4 = 2(k + 2)$ which is even, since $(k + 2)$ is an integer. Since this contradicts the fact that $x + 4$ is odd, the assumption is wrong and hence x is odd. Which of the following statements is correct? (A) The proof is valid. (B) The proof is wrong because it begins with a wrong assumption. (C) The proof is wrong because we are actually proving that if x is even, then $x + 4$ is even. (D) The proof is wrong but for other reasons. (Please give/explain your reason(s).)	Recognising a proof by contradiction and its validity.
11	<u>Result</u>: If m is a factor of n and n is a factor of k, then m is a factor of k. <u>Proof 1:</u> Consider the numbers: 4, 8 and 24. 4 is a factor of 8 since 4 x 2 = 8. 8 is a factor of 24 since 8 x 3 = 24. Hence 24 = 4 x 2 x 3. So 4 is a factor of 24. <u>Proof 2</u>: 3 is a factor of 6 and 6 is a factor of 24. We see that 3 is a factor of 24. 3 is not a factor of 5 and 5 is not a factor of 7. We see that 3 is not a factor of 7. Which of the following statements is correct? (A) Proof 1 only shows one case. Proof 2 is valid because it shows both factors and non factors. (B) Proof 2 is wrong because the result is not about non-factors. Proof 1 is correct because it deals with numbers which meet the conditions. (C) Both proofs are valid, they just use different numbers. (D) Both proofs are not valid because we need to show that the result is true for all numbers m, n and k. (E) None of the above.	Recognising that numerical examples are not proofs.

12. Result: The sum of the measures of the interior angles of a triangle is 180º.

Proof:

Forty pupils of a class each drew a triangle. Some drew acute-angled triangles, some drew obtuse triangles and others drew right-angled triangles. They cut out their triangles, tore them up and put the three angles together as in the diagram below.

Since the three angles lay on a straight line, they add up to 180º.

Is the above a valid proof for the result?

(A) No, diagrams cannot be used in proofs.

(B) No, because only 40 triangles were used and we must prove for all triangles.

(C) No, because there are no mathematical statements in the proof.

(D) Yes, because different types of triangles were used to verify the result.

Recognising that a good range of examples does not constitute a valid mathematical proof.