Directed complete poset models of T_1 spaces

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Abstract. A poset model of a topological space X is a poset P such that the subspace $\mathrm{Max}(P)$ of the Scott space ΣP is homeomorphic to X, where $\mathrm{Max}(P)$ is the set of all maximal points of P. Every T_1 space has a (bounded complete algebraic) poset model. It was, however, not known whether every T_1 space has a directed complete poset model and whether every sober T_1 space has a directed complete poset model whose Scott topology is sober. In this paper we give a positive answer to each of these two problems. For each T_1 space X, we shall construct a directed complete poset E that is a model of X, and prove that X is sober if and only if the Scott space $\Sigma\, E$ is sober. One useful by-product is a method for constructing more directed complete posets whose Scott topology is not sober.

Given a poset P, one can define various different intrinsic topologies on P, such as the Alexandrov topology, the interval topology and the order convergence topology. Another intrinsic topology on posets, which was first defined on complete lattices in the 1970's by Scott and later generalized to arbitrary posets, is the Scott topology [**5**]. One of the significant results on the Scott topology is that a T_0 space is an injective object in the category of T_0 spaces if and only if it is homeomorphic to some $(L, \sigma(L))$ where L is a continuous lattice and $\sigma(L)$ is the Scott topology on L [**15**]. The Scott topology on a poset is usually only T_0, thus most of the traditional topological spaces are not of the form $(P, \sigma(P))$. Thus, in order to obtain the general topological spaces from posets with their Scott topology, one has to consider certain canonical subspaces of Scott spaces. One such type of subspaces are the maximal point spaces. Given a poset, let $\mathrm{Max}(P)$ denote the set of all maximal points of P ($e \in \mathrm{Max}(P)$ if $b \in P$ and $b \geq e$ imply $e = b$). It is a little surprise that almost every traditional topological space is homeomorphic to the maximal point space of some poset. For example, the real line with the Euclidean topology is homeomorphic to the maximal point space $\mathrm{Max}(\mathrm{I}\mathbb{R})$, where $\mathrm{I}\mathbb{R} = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ is the poset of all closed intervals with the inverse inclusion order.

Following [**14**], a poset model of a topological space X is a poset P together with a homeomorphism $\phi : X \longrightarrow \mathrm{Max}(P)$. In [**16**] and [**4**], it was proved that every T_1 space has a bounded complete algebraic poset model. It was, however, still unclear whether each T_1 space has a directed complete poset model (or, dcpo model, for short).

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In this paper we give a positive answer to this problem. In Section 1, we show that every T_1 space has a dcpo model (in fact a locally quasialgebraic dcpo model). In Section 2, we prove that a T_1 space is sober if and only if the Scott topology of its dcpo model constructed in Section 1 is sober. These results provide us with a method for constructing more directed complete posets with non-sober Scott topology (note that previously, very few such dcpos have been constructed). In Section 3, for some special spaces we construct a simpler dcpo model.

Some other special types of models of topological spaces have been studied by many authors (for example, see [**1**][**9**][**11**][**13**][**5**]). In [**3**], the similar type of problems for T_0 spaces were considered.

1. Dcpo models of T_1 spaces

In this section, using the result in [**16**], we show that every T_1 space has a dcpo model. A poset P is called bounded complete if for any $D \subseteq P$, sup$D = \bigvee D$ exists in P whenever D has an upper bound in P. This is equivalent to that every nonempty subset of P has an infimum.

A nonempty subset D of a poset P is a directed set if every two elements in D have an upper bound in D. A poset P is called a *directed complete poset*, or dcpo for short, if for any directed subset of $D \subseteq P$, sup$D = \bigvee D$ exists in P.

If X is a subset of a poset P, then $\downarrow X = \{y \in P : y \leq x \text{ for some } x \in X\}$ and $\uparrow X = \{y \in P : y \geq x \text{ for some } x \in X\}$. For any element $a \in P$, one simply writes $\downarrow a$ for $\downarrow\{a\}$ and $\uparrow a$ for $\uparrow\{a\}$.

A subset X is called a lower set (upper set) if $X = \downarrow X$ ($X = \uparrow X$, respectively).

A subset U of a poset P is Scott open if i) $U = \uparrow U$ and ii) for any directed subset D, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$, whenever $\bigvee D$ exists. All Scott open sets of a poset P form a topology on P, denoted by $\sigma(P)$ and called the Scott topology on P. The space $(P, \sigma(P))$ is denoted by ΣP, called the Scott space of P. A subset F of P is closed with respect to the Scott topology if it is a lower set and closed under suprema of directed subsets (i.e. for any directed set $D \subseteq F$, $\bigvee D \in F$ whenever $\bigvee D$ exists). It is then clear that for any point $a \in P$, the closure $cl(\{a\})$ of $\{a\}$ in ΣP equals $\downarrow a$.

For two elements a and b in a poset P, a is way-below b, denoted by $a \ll b$, if for any directed subset D of P, if $\bigvee D$ exists and $b \leq \bigvee D$ then there exists $d \in D$ such that $a \leq d$. An element x is called compact if $x \ll x$. The set of all compact elements of P will be denoted by $\mathrm{K}(P)$.

A poset P is called a continuous poset if for every element $a \in P$, the set $\{x \in P : x \ll a\}$ is a directed set and

$$a = \bigvee \{x \in P : x \ll a\}.$$

A poset P is called algebraic, if for any $a \in P$, the set $\{x \in K(P) : x \leq a\}$ is a directed set and $a = \bigvee \{x \in K(P) : x \leq a\}$. For any algebraic poset P, the family $\{\uparrow u : u \in \mathrm{K}(P)\}$ is a base of the Scott topology on P (see Corollary II-1.15 of [**5**]).

In the following we shall always assume that the topology on the set $\mathrm{Max}(P)$ of maximal points of poset P is the subspace topology inherited from ΣP.

THEOREM 1. [**16**][**4**] *Every T_1 space has a bounded complete algebraic poset model.*

The following is a sketch of the proof of Theorem 1 in [**16**]. Let X be a T_1 space. Take $\mathcal{A}$ to be the set of all filters of nonempty open sets of X that have a nonempty intersection. Then, with respect to the inclusion order, $\mathcal{A}$ is a bounded complete algebraic poset. The compact elements of $\mathcal{A}$ are of the form $L(U) = \{V \in \mathcal{O}(X) : U \subseteq V\}(U \in \mathcal{O}(X))$. Also, $\mathrm{Max}(\mathcal{A}) = \{N(x) : x \in X\}$, where $N(x) = \{U \in \mathcal{O}(X) : x \in U\}$ is the open neighbourhood filter of $x \in X$. Define $\phi : X \longrightarrow \mathrm{Max}(\mathcal{A})$ by

$$\phi(x) = N(x), x \in X.$$

Then, as X is T_1, ϕ is a bijection. Note that as $\mathcal{A}$ is algebraic, the subsets of $\mathcal{A}$ of the form $\uparrow L(U)$ $(U \in \mathcal{O}(X))$ form a base of the Scott topology on $\mathcal{A}$. Now for any $U \in \mathcal{O}(X)$, $\phi^{-1}(\uparrow L(U) \bigcap \mathrm{Max}(\mathcal{A})) = \{x : L(U) \subseteq \phi(x)\} = \{x : U \in \phi(x)\} = U$. So ϕ is continuous. For any open set U of X, $\phi(U) \cap \mathrm{Max}(P) = \{\phi(x) : x \in U\} = \{N(x) : x \in U\} = \uparrow L(U) \cap \mathrm{Max}(\mathcal{A})$, which is open in $\mathrm{Max}(\mathcal{A})$. Hence ϕ is also an open mapping, therefore it is a homeomorphism between X and $\mathrm{Max}(\mathcal{A})$.

A poset P is called a local dcpo if every upper bounded directed subset has a supremum [**17**] (such dcpos are called cups in [**3**][**4**]). Clearly, every bounded complete poset is a local dcpo.

Lemma 1. *For any algebraic local dcpo P, there is a dcpo $\hat{P}$ such that Max(P) and Max$(\hat{P})$ are homeomorphic.*

Proof. Let P be an algebraic local dcpo. We use $\leq_P$ to denote the partial order on P.

Let $\hat{P} = \{(x, e) : x \in P, e \in \mathrm{Max}(P) \text{ and } x \leq_P e\}$. Define the binary relation $\leq$ on $\hat{P}$ as follows:

$$(x, e) \leq (y, d) \text{ if and only if either } e = d \text{ and } x \leq_P y, \text{ or } y = d \text{ and } x \leq_P d.$$

(a) The relation $\leq$ is a partial order on $\hat{P}$.

Here we just check the transitivity of $\leq$. Assume that $(x, d) \leq (y, e) \leq (z, f)$. Firstly, we have $x \leq_P y \leq_P z$.

We consider the following cases:

(a1) $d = e$. If $z = f$, then by $x \leq_P z$, we have $(x, d) \leq (f, f) = (z, f)$. If $z \neq f$, then by $(y, e) \leq (z, f)$ we must have $f = e$, and hence $f = e = d$. Thus again, $(x, d) \leq (z, d) = (z, f)$.

(a2) $d \neq e$. Then from $(x, d) \leq (y, e)$, it follows that $y = e$. Then as $(e, e) = (y, e) \leq (z, f)$, we must have have $(z, f) = (e, e) = (y, e)$ (note that $e, f \in \mathrm{Max}(P)$). Then again we get $(x, d) \leq (z, f)$.

(b) By the definition of $\leq$, $\mathrm{Max}(\hat{P}) = \{(e, e) : e \in \mathrm{Max}(P)\}$.

(c) Now we verify that $\hat{P}$ is a dcpo.

Given any directed subset $\mathcal{D}$ of $\hat{P}$, we prove the existence of $\bigvee\mathcal{D}$ by considering two cases:

(c1) There are $(x_1, e_1), (x_2, e_2) \in \mathcal{D}$ with $e_1 \neq e_2$. Then we have $x_2 = e_2$, so $(e_2, e_2) = (x_2, e_2) \in \mathcal{D}$ will be the largest element of $\mathcal{D}$. Hence $\bigvee \mathcal{D} = (e_2, e_2)$.

(c2) For all $(x, d), (x', d') \in \mathcal{D}$, $d = d'$. Then $\mathcal{D} = \{(x_i, d) : i \in I\}$, where $d \in \mathrm{Max}(P)$ and $\{x_i : i \in I\}$ is a directed subset of P with d as an upper bound. So $\bigvee_P\{x_i : i \in I\}$ exists as P is a local dcpo, and $\bigvee\mathcal{D} = (\bigvee_P\{x_i : i \in I\}, d)$.

It thus follows that $(\hat{P}, \leq)$ is a dcpo.

(d) Define $f : \mathrm{Max}(P) \longrightarrow \mathrm{Max}(\hat{P})$ by $f(e) = (e, e)$ for each $e \in \mathrm{Max}(P)$. Then f is a bijection. For any Scott open set $U \in \sigma(P)$, the set $\overline{U} = \{(x, e) \in \hat{P} : x \in U\}$ is an upper set in $\hat{P}$. Let $\mathcal{D}$ be a directed subset of $\hat{P}$ such that $\bigvee \mathcal{D} \in \overline{U}$. If $\mathcal{D}$ has a largest element, then trivially $\bigvee \mathcal{D} \in \mathcal{D} \cap \overline{U}$. Otherwise, $\mathcal{D} = \{(x_i, e) : i \in I\}$, where e is a fixed element in $\mathrm{Max}(P)$ and $\{x_i : i \in I\}$ is a directed subset of P. Then $\bigvee \mathcal{D} = (\bigvee_P \{x_i : i \in I\}, e) \in \overline{U}$. Thus $\bigvee_P \{x_i : i \in I\} \in U \in \sigma(P)$, so $x_{i_0} \in U$ for some $i_0 \in I$, which implies $(x_{i_0}, e) \in \mathcal{D} \cap \overline{U}$. All these show that $\overline{U}$ is a Scott open set of $\hat{P}$. Now $f(U \cap \mathrm{Max}(P)) = \{(d, d) : d \in U \cap \mathrm{Max}(P)\} = \overline{U} \cap \mathrm{Max}(\hat{P})$, implying that f is an open mapping.

Conversely, assume that W is a Scott open set of $\hat{P}$. Let $\underline{W} = \uparrow\{x \in K(P) : (x, e) \in W$ for some $e \in \mathrm{Max}(P)\}$, where $K(P)$ is the set of all compact elements of P. By its definition, $\underline{W}$ is an upper set of P. Let $D = \{x_i : i \in I\}$ be a directed subset of P such that $\bigvee_P \{x_i : i \in I\} \in \underline{W}$. Then there exists $e \in \mathrm{Max}(P)$ and $x \in K(P)$ such that $(x, e) \in W$ and $x \leq_P \bigvee_P \{x_i : i \in I\}$. Since $x \in K(P)$, there is x_{i_0} such that $x \leq x_{i_0}$, which implies $x_{i_0} \in D \cap \underline{W}$. These show that $\underline{W}$ is a Scott open set of P.

Denote $W' = \{d \in \mathrm{Max}(P) : (d, d) \in W\}$. Now $f^{-1}(W \cap \mathrm{Max}(\hat{P})) = \{d \in \mathrm{Max}(P) : f(d) = (d, d) \in W\} = W'$. We claim that $W' = \underline{W} \cap \mathrm{Max}(P)$. In fact, for any $d_0 \in W'$, the set $\{(x, d_0) : x \leq_P d_0, x \in K(P)\}$ is a directed subset of $\hat{P}$ whose supremum in $\hat{P}$ equals (d_0, d_0). As $(d_0, d_0) \in W$ and W is a Scott open set of $\hat{P}$, there is $x_0 \leq d_0, x_0 \in K(P)$ such that $(x_0, d_0) \in W$. Thus $d_0 \in \underline{W} \cap \mathrm{Max}(P)$. Conversely, if $d_0 \in \underline{W} \cap \mathrm{Max}(P)$, then $d_0 \in \mathrm{Max}(P)$ and there is $(x, e) \in W$ with $x \in K(P)$ and $x \leq_P d_0$. Then $(x, e) \leq (d_0, d_0)$, implying $(d_0, d_0) \in W$ because W is an upper set in $\hat{P}$. Hence $d_0 \in W'$. All these show that $f^{-1}(W \cap \mathrm{Max}(\hat{P})) = W' = \underline{W} \cap \mathrm{Max}(P)$, which is an open set of $\mathrm{Max}(P)$. Therefore f is continuous.

The combination of the above proofs shows that f is an homeomorphism between $\mathrm{Max}(P)$ and $\mathrm{Max}(\hat{P})$. □

Remark 1. The reader may wonder whether in the proof of Lemma1 one can replace $\hat{P}$ by the subposet $\downarrow\mathrm{Max}(P)$ of P. In general, this is not possible as we cannot confirm that $\downarrow\mathrm{Max}(P)$ is a dcpo.

Remark 2. The following facts on the dcpo $\hat{P}$ constructed from an algebraic local dcpo P in the proof of Lemma 1 will be used in later parts of this paper.

(i) If $\mathcal{D}$ is a directed subset of $\hat{P}$ and it does not have a largest element, then there is $d \in \mathrm{Max}(P)$ and a directed subset $\{x_i : i \in I\}$ of P such that $\mathcal{D} = \{(x_i, d) : i \in I\}$, and in this case $\bigvee \mathcal{D} = (\bigvee_P \{x_i : i \in I\}, d)$.

(ii) The set of maximal points of $\hat{P}$ equals $\{(d, d) : d \in \mathrm{Max}(P)\}$.

A subset F of a dcpo P is way-below a subset G, denoted by $F \ll G$, if for any directed subset $D \subseteq P$, $\bigvee D \in \uparrow G$ implies $D \cap \uparrow F \neq \emptyset$. In particular, if $x \in P$ and $F \ll \{x\}$ then we just write $F \ll x$. A dcpo P is quasicontinuous if for any $x \in P$, the family

$$fin(x) = \{F : F \text{ is finite and } F \ll x\}$$

is a directed family (for any $F_1, F_2 \in fin(x)$ there is $F \in fin(x)$ such that $F \subseteq \uparrow F_1 \cap \uparrow F_2$) and for any $x \not\leq y$ there is $F \in fin(x)$ satisfying $y \notin \uparrow F$ (see Definition III-3.2 of [**5**]).

A dcpo P is called quasialgebraic, if for any $x \in P$, the family

$$compfin(x) = \{F \subseteq P : F \text{ is finte}, F \ll F \text{ and } x \in \uparrow F\}$$

is directed and whenever $x \not\leq y$, there is $F \in compfin(x)$ such that $y \notin \uparrow F$.

Remark 3. (1) Using a similar argument as Exercise III-3.19 (i) of [**5**], one can show that a dcpo P is quasialgebraic if for any x there is a directed family $\mathcal{F} \subseteq comfin(x)$ such that whenever $x \not\leq y$, there is $F \in \mathcal{F}$ such that $y \notin \uparrow F$.

(2) Every algebraic dcpo is quasialgebraic and every quasialgebraic dcpo is quasicontinuous (see Exercise III-3.19 (i) of [**5**]).

A poset P is called locally quasialgebraic if for each $a \in P$, the subposet $\downarrow a$ is quasialgebraic.

To make the proof of the next major lemma clear, we first prove the following.

Lemma 2. *Let P be a bounded complete algebraic poset and $d, e \in Max(P)$. If $u \in K(P)$ and $u \leq d$ and $u \leq e$, then $\{(u,d),(u,e)\} \ll \{(u,d),(u,e)\}$ holds in $\Downarrow(d,d)$.*

In particular, if $u \in K(P)$ and $u \leq d$ then $(u,d) \ll (u,d)$ holds in $\downarrow(d,d)$.

Proof. We only need to prove the first part. Let d, e and u satisfy the above conditions. Put $F(u,e) = \{(u,d),(u,e)\}$. Then $F(u,e) \subseteq \downarrow(d,d)$. We now prove that $F(u,e) \ll F(u,e)$ holds in $\Downarrow(d,d)$.

Let $\mathcal{D} = \{(y_i, e_i) : i \in I\} \subseteq \downarrow(d,d)$ be a directed subset such that $\bigvee\mathcal{D} \in \uparrow F(u,e)$.

If $\mathcal{D}$ has a largest element, say (y_{i_0}, e_{i_0}), then trivially $(y_{i_0}, e_{i_0}) = \bigvee\mathcal{D}$, so $(y_{i_0}, e_{i_0}) \in \mathcal{D} \cap \uparrow F(u,e)$.

Now assume that $\mathcal{D}$ does not have a largest element. By Remark 2 (i), there exists $c \in \mathrm{Max}(P)$ such that $\mathcal{D} = \{(y_i, c) : i \in I\}$ and $\{y_i : i \in I\}$ is a directed subset of $\downarrow d$. In this case $\bigvee\mathcal{D} = (\bigvee_P\{y_i : i \in I\}, c)$. Then, by $\bigvee\mathcal{D} \in \uparrow F(u,e)$, it follows that $u \leq \bigvee_P\{y_i : i \in I\}$, therefore there is y_{i_1} such that $u \leq y_{i_1}$ because $u \in K(P)$.

If $c = d$, then $(y_{i_1}, c) = (y_{i_1}, d) \geq (u,d)$, implying $(y_{i_1}, c) \in \mathcal{D} \cap \uparrow F(u,e)$.

If $c \neq d$, then as $\bigvee_P\{y_i : i \in I\} \leq d$ and $c \in \mathrm{Max}(P)$, we have $\bigvee_P\{y_i : i \in I\} \neq c$. It thus follows that $\bigvee\mathcal{D} = (\bigvee_P\{y_i : i \in I\}, c) \not\geq (u,d)$, so we must have $\bigvee\mathcal{D} \geq (u,e)$, thus $c = e$ because $\bigvee_P\{y_i : i \in I\} \neq c$. Therefore $(y_{i_1}, c) = (y_{i_1}, e) \geq (u,e)$, implying $(y_{i_1}, c) \in \mathcal{D} \cap \uparrow F(u,e)$.

All these together show that $F(u,e) \ll F(u,e)$ holds in $\downarrow(d,d)$. $\square$

Lemma 3. *If P is a bounded complete algebraic poset, then the dcpo $\hat{P}$ constructed in Lemma 1 from P is locally quasialgebraic.*

Proof. Note first that since P is a bounded complete algebraic poset, for each $a \in P$ the subposet $\downarrow a$ is an algebraic dcpo.

For any $(y,d) \in \hat{P}$, we show that $\downarrow(y,d)$ is quasialgebraic by considering two cases.

(1) $(y,d) \in \hat{P}$ with $y \notin \mathrm{Max}(P)$.

Then the lower part $\downarrow(y,d)$ of (y,d) in $\hat{P}$ is isomorphic to the lower part $\downarrow y$ of y in P. By the remark at the start of the proof, $\downarrow y$ is an algebraic dcpo, so $\downarrow(y,d)$ is an algebraic dcpo.

(2) $(y,d) = (d,d) \in \hat{P}$.

Then $d \in \mathrm{Max}(P)$ and $\downarrow(d,d) = \{(z,e) : e \in \mathrm{Max}(P), z \leq_P d\}$. To prove $\downarrow(d,d)$ is quasialgebraic, we consider two types of elements in $\downarrow(d,d)$.

(i) Let $(x,d) \in \downarrow(d,d)$. Then $(x,d) = (\bigvee_P\{u \in K(P) : u \leq_P x\}, d) = \bigvee\{(u,d) : u \leq_P x, u \in K(P)\}$. By Lemma 2, if $u \leq x$ and $u \in K(P)$, then (u,d) is a compact element of $\downarrow(d,d)$, thus $\{(u,d)\} \ll \{(u,d)\}$ holds in $\downarrow(d,d)$. Consider the family $\mathcal{F} = \{\{(u,d)\} : u \leq_P x, u \in K(P)\}$ of finite sets (in fact singletons). If $\{(u_1,d)\}, \{(u_2,d)\}$ are from $\mathcal{F}$, then there is $u \in K(P) \cap \downarrow x$ such that $u_1 \leq u$ and $u_2 \leq u$, thus $\{(u,d)\} \in \mathcal{F}$ and $\{(u,d)\} \subseteq \uparrow\{(u_1,d)\} \cap \uparrow\{(u_2,d)\}$. Hence $\mathcal{F}$ is a directed family whose every member is way-below itself. In addition, assume $(v,c) \in \downarrow(d,d)$ and $(v,c) \not\geq (x,d)$. If $c \neq d$, then $(v,c) \leq (d,d)$ implies $v \neq c$, thus $(v,c) \notin \uparrow\{(u,d)\}$ for every $\{(u,d)\} \in \mathcal{F}$. If $c = d$, then $(v,c) = (v,d) \not\geq (x,d)$ implies $v \not\geq u_0$ for some $u_0 \in K(P)$ and $u_0 \leq x$, thus $\{(u_0,d)\} \in \mathcal{F}$ and $(v,c) \notin \uparrow\{(u_0,d)\}$.

(ii) Next, let $(x,e) \in \downarrow(d,d)$ where $e \neq d$. Then $x \leq_P d$. By Lemma 2, for each $u \leq_P x$ with $u \in K(P)$, the finite set $F(u,e) = \{(u,d),(u,e)\}$ satisfies $F(u,e) \ll F(u,e)$ in $\downarrow(d,d)$ and (x,e) in $\uparrow F(u,e)$. Let $\mathcal{G} = \{F(u,e) : u \in K(P) \text{ and } u \leq_P x\}$. Then $\mathcal{G}$ is a directed subfamily of $compfin(x)$ because $K(P) \cap \downarrow x$ is a directed set.

Now let $(v,c) \in \downarrow(d,d)$ and $(v,c) \not\geq (x,e)$. Then $v \leq_P d$ and $v \neq c$ (if $v = c$ then $c = d$, implying $(v,c) = (d,d) \geq (x,e)$). If $c \neq e$ and $c \neq d$, then $(v,c) \notin \uparrow F(u,e)$ for any $F(u,e) \in \mathcal{G}$. If $c = e$ or $c = d$, then it holds that $v \not\geq_P x$. So there is $u_1 \in K(P)$ with $u_1 \leq_P x$ such that $u_1 \not\leq_P v$. Thus $(v,c) \notin \uparrow F(u_1,e) \in \mathcal{G}$.

By Remark 3(1), the dcpo $\downarrow(d,d)$ is quasialgebraic. □

Given a T_1 space, by Theorem 1, there is a bounded complete algebraic poset $\mathcal{A}$ such that $\mathrm{Max}(\mathcal{A})$ is homeomorphic to X. By Lemma 3 and Lemma 1, $\hat{\mathcal{A}}$ is a locally quasialgebraic dcpo and $\mathrm{Max}(\hat{\mathcal{A}})$ is homeomorphic to $\mathrm{Max}(\mathcal{A})$, therefore homeomorphic to X. All these facts together yield the first main result of this paper.

THEOREM 2. *Every T_1 topological space has a dcpo model that is locally quasialgebraic.*

2. Dcpo models of sober spaces

A subset A of a topological space X is irreducible if for any closed sets F_1, F_2 of X, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A topological space X is sober, if for any nonempty irreducible closed set F of X there is a unique point $a \in X$ such that $cl(\{a\}) = F$. Every sober space is T_0 and every T_2 space is sober.

It is well known that the Scott space of any continuous dcpo is sober. A more general result is that the Scott space of any quasicontinuous dcpo is sober (see Proposition III-3.7 of [**5**]).

The following result on irreducible sets is well known and will be used in the proof of Lemma 5.

LEMMA 4. *Let X be a subspace of Y. The following statements are equivalent for a subset $A \subseteq X$:*

(1) A is an irreducible subset of X;

(2) A is an irreducible subset of Y;

(3) $cl_Y(A)$ is an irreducible subset of Y.

The result below may have been obtained by other people already. For readers' convenience, we give a proof here.

Proposition 1. *If P is a poset such that ΣP is sober, then the subspace Max(P) of ΣP is sober.*

Proof. Let P be a poset such that ΣP is sober. Since $\mathrm{Max}(P)$ is a T_1 space, in order to show it is sober, it is enough to prove that every nonempty irreducible closed subset of $\mathrm{Max}(P)$ is a singleton set. For any nonempty irreducible closed set F of $\mathrm{Max}(P)$, by Lemma 4, $cl_P(F)$ is an irreducible closed set of the Scott space ΣP. Thus, by the assumption, there is a unique point $e \in P$ such that $\downarrow e = cl_P(\{e\}) = cl_P(F)$, which implies $F \subseteq cl_P(F) = \downarrow e$. But F is a nonempty set consisting of maximal elements of P, it follows that $e \in F$ and so $F = \{e\}$. Hence $\mathrm{Max}(P)$ is sober. $\square$

Now we prove that every sober T_1 space has a dcpo model whose Scott topology is sober. Note that for any poset Q, in order to show ΣQ is sober, it is enough to prove that for any nonempty irreducible closed set F, $F = \downarrow b$ for some $b \in Q$. The uniqueness of b follows from the fact that Q is a poset.

Lemma 5. *Let P be a bounded complete algebraic poset and $\hat{P}$ be the dcpo constructed from P in Lemma 1. If Max$(\hat{P})$ is sober then $\Sigma\hat{P}$ is sober.*

Proof. Recall that $\hat{P} = \{(x, e) : x \leq_P e, e \in \mathrm{Max}(P)\}$. Assume that $\mathrm{Max}(\hat{P})$ is sober.

Let $F \subseteq \hat{P}$ be a nonempty irreducible Scott closed set. We need to prove that $F = cl_{\hat{P}}(\{(v, d)\}) = \downarrow(v, d)$ for some $(v, d) \in \hat{P}$. Let $F^* = F \cap \mathrm{Max}(\hat{P})$. Then F^* is a closed subset of $\mathrm{Max}(\hat{P})$. For each $d \in \mathrm{Max}(P)$, put $\hat{P}_d = \{(x, d) : x \leq_P d\}$.

Note that if $(y, e) \leq (x, d)$ and $x \neq d$, then $e = d$ and $y \leq_P x$. We complete the proof by considering two cases.

(i) Assume that $F^* = \emptyset$. Then $F = \bigcup\{F \cap \hat{P}_d : d \in \mathrm{Max}(P)\}$. For any $d \in \mathrm{Max}(P)$, $F \cap \hat{P}_d$ does not contain any maximal element of $\hat{P}$, thus, by the above remark, it is a lower set. If $\mathcal{D} \subseteq F \cap \hat{P}_d$ is a directed set, then $\mathcal{D} = \{(x_i, d) : i \in I\}$ with $\{x_i : i \in I\}$ a directed subset of P. So $\bigvee \mathcal{D} = (\bigvee_P \{x_i : i \in I\}, d)$ is in F because F is Scott closed, so $\bigvee \mathcal{D} \in F \cap \hat{P}_d$. Hence $F \cap \hat{P}_d$ is a Scott closed subset. Since $F \neq \emptyset$, there exists a $d_0 \in \mathrm{Max}(P)$ with $F \cap \hat{P}_{d_0} \neq \emptyset$. Then $F = (F \cap \hat{P}_{d_0}) \cup \bigcup\{F \cap \hat{P}_d : d \in \mathrm{Max}(P), d \neq d_0\}$. Like F, $\bigcup\{F \cap \hat{P}_d : d \in \mathrm{Max}(P), d \neq d_0\}$ is also Scott closed (if $\mathcal{D}$ is a directed set and $\mathcal{D} \subseteq \bigcup\{F \cap \hat{P}_d : d \in \mathrm{Max}(P), d \neq d_0\}$, then $\mathcal{D} \subseteq F \cap \hat{P}_d$ for some $d \in \mathrm{Max}(P)$ with $\neq d_0$). Since F is irreducible and $F \cap \hat{P}_{d_0}$ and $\bigcup\{F \cap \hat{P}_d : d \in \mathrm{Max}(P), d \neq d_0\}$ are disjoint Scott closed sets, we must have $F = F \cap \hat{P}_{d_0}$, which implies $F \subseteq \hat{P}_{d_0}$. Also note that the poset $\hat{P}_{d_0}$ is isomorphic to the subposet $\downarrow d_0$ of P via the mapping $(x, d_0) \mapsto x$. Thus F is homeomorphic to an irreducible Scott closed subset H of $\downarrow d_0$. But $\downarrow d_0$ is an algebraic dcpo, its Scott topology is sober, so $H = \downarrow x_0$ for some $x_0 \leq d_0$, which then implies that $F = \downarrow(x_0, d)$.

(ii) Now assume that $F^* \neq \emptyset$. If we can show that $F = cl_{\hat{P}}(F^*)$, then by Lemma 4, F^* is an irreducible closed set of $\mathrm{Max}(\hat{P})$. So as $\mathrm{Max}(\hat{P})$ is sober and T_1, $F^* = \{(d,d)\}$ for some $d \in \mathrm{Max}(P)$, that further implies that $F = cl_{\hat{P}}(F^*) = cl_{\hat{P}}(\{(d,d)\}) = \downarrow(d,d)$.

Clearly $F = cl_{\hat{P}}(F^*) \cup \bigcup\{F \cap \hat{P}_d : (d,d) \notin F^*\}$. In addition, $\bigcup\{F \cap \hat{P}_d : (d,d) \notin F^*\}$ is a Scott closed subset of $\hat{P}$. Since F is irreducible, $F = cl_{\hat{P}}(F^*)$ or $F = \bigcup\{F \cap \hat{P}_d : (d,d) \notin F^*\}$. But, as $F^* \neq \emptyset$ implies $F \neq \bigcup\{F \cap \hat{P}_d : (d,d) \notin F^*\}$, hence $F = cl_{\hat{P}}(F^*)$.

All these show that $\Sigma\hat{P}$ is sober.

$\square$

From the above two results we obtain the following.

Theorem 3. *A topological space X has a dcpo model whose Scott topology is sober if and only if X is T_1 and sober.*

We call a dcpo P sober, if its Scott topology is sober. Johnstone first constructed a non-sober dcpo in [**7**], then Isbell gave a non-sober complete lattice in [**6**]. Finding a non-sober dcpo is surprisingly not easy (as far as the authors know, up-to-date, only three such dcpos have been constructed).

Now if X is any T_1 and non-sober space, then the dcpo model constructed for X in Theorem 2 is non-sober. In the next section, we shall construct a concrete new non-sober dcpo.

3. Dcpo models of some special spaces

In the above section, we constructed a dcpo model for each T_1 space. As the elements of these dcpo models are pairs of filters of open sets, they are generally not convenient to handle. In this section, we shall construct the dcpo models for some special spaces in a more direct way.

Let ω_1 be the first non-countable ordinal and $W = [0, \omega_1)$ be the set of all ordinals less than ω_1. Thus W consists of all finite and infinite countable ordinals. Let $W_1 = [0, \omega_1]$.

Remark 4. The following facts are well known (see, for example, [**2**]).

1) $|W| = \aleph_1$.

2) For any countable subset $D \subseteq W$, $\sup D \in W$, here the $\sup D$ is taken with respect to the usual linear order on ordinals.

3) For any $\alpha \in W$, $\{\beta : \beta \leq \alpha\}$ is a finite or countably infinite subset of W.

Let $Y = W \times \{\omega_1\}$ and τ be the co-countable topology on Y, that is $U \in \tau$ if and only if either $U = \emptyset$ or $Y - U$ is a finite or countably infinite set. We now construct a simpler dcpo model for (Y, τ).

Let $P = W \times W_1$ and define the partial order $\leq$ on P by $(u, v) \leq (u', v')$ iff
(i) either $u = u'$ and $v \leq v'$; or
(ii) $v' = \omega_1$ and $v \leq u'$.

By the definition of the partial order on P, a subset $E \subseteq P$ is a directed set iff either E has a largest element or there is a $u \in W$ such that $E = \{(u, v_i) : i \in I\}$, and $\{v_i : i \in I\} \subseteq W$ is a directed subset of W (it is actually a chain). Thus it follows that $(P, \leq)$ is a dcpo.

In addition, $\mathrm{Max}(P) = W \times \{\omega_1\} = Y$. Note that the definition of P is similar to the dcpo X defined by Johnstone in Exercise 1.9 in [**8**].

LEMMA 6. *(1) For any finite or countably infinite subset $A \subseteq W \times \{\omega_1\}$, there is a Scott closed set F of P such that $A = F \cap (W \times \{\omega_1\})$.*

(2) If F is a Scott closed subset of P and $(W \times \{\omega_1\}) \cap F$ is uncountable, then $F = P$ (which then implies $F \cap (W \times \{\omega_1\}) = W \times \{\omega_1\}$).

PROOF. (1) Without loss of generality, we assume that $A = \{(u_i, \omega_1) : i \in \mathbb{N}\}$ is a countably infinite subset of $W \times \{\omega_1\}$. Then $\{u_i : i \in \mathbb{N}\}$ is a countable subset of W, so $\{u_i : i \in \mathbb{N}\}$ has a supremum in W, say u_0.

Consider the set $F = \downarrow A \cup \downarrow \{(u, u_0) : u \in W\}$. Clearly F is a lower set of P. Using the features of directed subsets of P, we can verify that F is a Scott closed set of P. Furthermore, $F \cap (W \times \{\omega_1\}) = \{u_i : i \in I\} = A$ (note that $(u, u_0) \notin W \times \{\omega_1\}$).

(2) Now assume that $F \subseteq P$ is a Scott closed set and $B = F \cap (W \times \{\omega_1\})$ is uncountable. Let $B = \{(u_j, \omega_1) : j \in J\}$. Then $\{u_j : j \in J\}$ has no upper bound in W. Let $u \in W$. For any $v \in W$, there is $(u_j, \omega_1) \in B$ such that $v \leq u_j$, thus $(u, v) \leq (u_j, \omega_1) \in F$. Since F is a lower set, $(u, v) \in F$. Therefore $H = \{(u, v) \in P : v \in W\} \subseteq F$. Since F is Scott closed and H is a chain, $\bigvee_P H = (u, \omega_1) \in F$. Therefore $F = P$. $\square$

By the above lemma we deduce the following.

PROPOSITION 2. *The dcpo P defined above is a dcpo model of the space of set $Y = W \times \{\omega_1\}$ with the co-countable topology.*

As $W \times \{\omega_1\}$ with the co-countable topology is non-sober, its dcpo model P is non-sober in the Scott topology. This gives another example of non-sober dcpo.

REMARK 5. (1) In general, let $\aleph$ be a cardinal and $\omega_\aleph$ be the smallest ordinal whose cardinal is $\aleph$. Let $K = \{\alpha : \alpha \text{ is an ordinal and } \alpha < \omega_\aleph\} \times \{\omega_\aleph\}$.

The co-$\aleph$ topology μ on K is the topology consisting of the empty set and the sets whose complements have cardinal less than $\aleph$. Then we can construct a dcpo model of K with the co-$\aleph$ topology, in a similar way as for constructing the above P.

(2) In particular, consider the countable cardinal $\aleph_0$. The smallest countable ordinal is ω_0 and $\{\alpha : \alpha \text{ is an ordinal with } \alpha < \omega_0\} = \mathbb{N}$, the set of all natural numbers. Now, let $Q = \mathbb{N} \times (\mathbb{N} \cup \{\omega_0\})$ and define the partial order $\leq$ on Q by $(m, n) \leq (m', n')$ iff either $m = m'$ and $n \leq n'$, or $n' = \omega_0$ and $n \leq m'$. If we replace ω_0 by ∞, then Q equals the dcpo X constructed by Johnstone (see Exercise 1.9 in [**8**]). Just as for Proposition 2, we can show that Q is a dcpo model of $\mathbb{N} \times \{\omega_0\}$ with the co-finite topology.

A dcpo model P of a T_1 space X is said to satisfy the Lawson condition if X is homeomorphic to $\mathrm{Max}(P)$ with the subspace topology inherited from the Lawson topology on P (see [**5**] for the definition of Lawson topology and related results). A main problem here is: which spaces have a dcpo model satisfying the Lawson condition? There have been some results on special posets. For example, Liang and Keimel proved that a space has a continuous poset model satisfying the Lawson condition iff it is Tychnov [**10**], Lawson proved that a space has a continuous dcpo model satisfying Lawson condition that has a countable base iff the space is Polish [**9**]. In [**16**](see also [**4**]), it was proved that a space has an algebraic poset model satisfying the Lawson condition iff it is zero-dimensional.

THEOREM 4. *If a space is zero dimensional then it has a dcpo model satisfying the Lawson condition.*

PROOF. Assume that X is zero dimensional. Then there is a bounded complete algebraic poset such that the set $\mathrm{Max}(P)$ with the inherited Lawson topology on P is homeomorphic to X (See Theorem 3 of [**16**]). Then the poset $\hat{P}$ as constructed in the proof of Lemma1 is a dcpo. One can easily verify that $\mathrm{Max}(P)$ and $\mathrm{Max}(\hat{P})$ with the inherited Lawson topology are homeomorphic. Thus $\hat{P}$ is a dcpo model of X satisfying the Lawson condition. □

In this paper we proved that every T_1 space has a dcpo model and the space is sober if and only if it has a sober dcpo model. One immediate application is in constructing new examples of dcpo whose Scott topology is not sober. It is hoped that more general properties of maximal point spaces of continuous dcpos can be uncovered so that one can finally characterize the spaces that have a continuous dcpo model.

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