Existence and uniqueness of solutions of higher order boundary value problems

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1. INTRODUCTION

In this paper we shall provide sufficient criteria for the existence and uniqueness of the solutions of higher order boundary value problems. The obtained results are sharper/general than those known in the literature.

2. SOME INEQUALITIES

We begin with the following result which is a generalization of Wirtinger's inequality.

Lemma 2.1. (Cimmino's Inequality [11, p.69], also see [4]) If $x(t) \in PC^{n,2}[a,b]$ and $x^{(i)}(a) = x^{(i)}(b) = 0,\ 0 \le i \le n-1$ then

$$\int_a^b [D^k x(t)]^2\, dt \le \left(\frac{b-a}{\lambda_{n,k}}\right)^{2n-2k} \int_a^b [D^n x(t)]^2\, dt,\ 0 \le k \le n$$

where $\lambda_{n,n} = 1$ and for $0 \le k \le n-1$, $\lambda_{n,k}$ is the least positive zero of the Wronskian of n linearly independent solutions of the differential equation

$$y^{(2n)} - (-1)^{n+k}\, y^{(2k)} = 0$$

satisfying the partial initial conditions $y^{(i)}(0) = 0,\ 0 \le i \le n-1$.

The first few $\lambda_{n,k}$ are given in the following table (see [4]).

Table 2.1.

$k \backslash n$	1	2	3	4	5
0	π	4.730041	6.283185	7.818707	9.343298
1		2π	7.853205	9.427056	10.995831
2			8.986819	10.535315	12.111801
3				11.526918	13.059858
4					13.975860

Corollary 2.2. If $x(t) \in PC^{2n-j,2}[a,b]$, $0 \le j \le n$ and $x^{(i)}(a) = x^{(i)}(b) = 0$, $0 \le i \le n-1$ then

$$\int_a^b [D^k x(t)]^2\, dt \le \left(\frac{b-a}{\lambda_{n,k}}\right)^{2n-2k} \left(\frac{b-a}{\lambda_{n,j}}\right)^{2n-2j} \int_a^b [D^{2n-j}x(t)]^2\, dt;$$

$$0 \le k \le n,\ 0 \le j \le n.$$

Lemma 2.3. (Hardy - Littlewood's Inequality [11, p.70]) If $x(t) \in C^{(1)}[a,b]$ and either $x(a) = 0$ or $x(b) = 0$ then

$$\int_a^b [x(t)]^2 dt \le \left(\frac{2(b-a)}{\pi}\right)^2 \int_a^b [x'(t)]^2 dt.$$

Lemma 2.4. (Block's Inequality [11, p.85]) If $x(t) \in C^{(1)}[a,b]$ and $x(a) = x(b) = 0$, then

$$\|x\|_\infty \le \frac{\sqrt{b-a}}{2} \left\{\int_a^b [x'(t)]^2 dt\right\}^{1/2}.$$

3. TWO - POINT TAYLOR'S BVP'S

Theorem 3.1. For the differential equation

$$y^{(2m)} = f(t, y, y', \cdots, y^{(m)}),\ m \ge 1 \tag{1}$$

together with the boundary conditions

$$y^{(i)}(0) = y^{(i)}(1) = 0, \ 0 \le i \le m-1 \tag{2}$$

we assume that $f : [0,1] \times R^{m+1} \to R$ satisfies the Caratheodory condition, and the inequality

$$|f(t, y, y', \cdots, y^{(m)})| \le P + \sum_{j=0}^{m} P_j |y^{(j)}| \tag{3}$$

holds, where P, P_j, $0 \le j \le m$ are nonnegative constants. Then, for the existence of a nonzero solution $y(t)$ of the boundary value problem (1), (2) it is necessary that

$$\rho = \sum_{j=0}^{m} \frac{P_j}{\lambda_{m,j}^{m-j} \lambda_{m,0}^{m}} < 1. \tag{4}$$

Further, the following inequalities hold

$$\|y^{(j)}\|_2 \le \frac{1}{\lambda_{m,j}^{m-j} \lambda_{m,0}^{m}} \frac{P}{1-\rho}, \ 0 \le j \le m \tag{5}$$

and

$$\|y^{(j)}\|_\infty \le \frac{1}{2} \frac{1}{\lambda_{m,j+1}^{m-j-1} \lambda_{m,0}^{m}} \frac{P}{1-\rho}, \ 0 \le j \le m-1. \tag{6}$$

Proof. Multiplying both sides of the differential equation (1) by $y^{(2m)}(t)$, integrating the resulting equation with respect to t from 0 to 1, and then using (3), Corollary 2.2 and the Cauchy - Schwartz inequality, we successively obtain

$$\begin{aligned}
\|y^{(2m)}\|_2^2 = \int_0^1 [y^{(2m)}(t)]^2 dt &\le \int_0^1 |f(t, y(t), y'(t), \cdots, y^{(m)}(t))| \, |y^{(2m)}(t)| dt \\
&\le \int_0^1 \left[P + \sum_{j=0}^{m} P_j |y^{(j)}(t)|\right] |y^{(2m)}(t)| dt \\
&\le P\|y^{(2m)}\|_2 + \sum_{j=0}^{m} P_j \|y^{(j)}\|_2 \, \|y^{(2m)}\|_2 \\
&\le P\|y^{(2m)}\|_2 + \sum_{j=0}^{m} \frac{P_j}{\lambda_{m,j}^{m-j} \lambda_{m,0}^{m}} \|y^{(2m)}\|_2^2,
\end{aligned}$$

which in view of $\|y^{(2m)}\|_2 \ne 0$ gives that

$$\|y^{(2m)}\|_2 \le \frac{P}{1-\rho}. \tag{7}$$

Once again from Corollary 2.2, we have

$$\|y^{(j)}\|_2 \le \frac{1}{\lambda_{m,j}^{m-j}\lambda_{m,0}^{m}} \|y^{(2m)}\|_2, \ 0 \le j \le m. \tag{8}$$

A combination of the inequalities (7) and (8) leads to (5). Finally, inequalities (6) are immediate from Lemma 2.4 and (5).

Remark 3.1. For $m = 2$ the inequality (4) reduces to

$$P_0 \frac{1}{\lambda_{2,0}^4} + P_1 \frac{1}{\lambda_{2,1}\lambda_{2,0}^2} + P_2 \frac{1}{\lambda_{2,0}^2} < 1,$$

where $\lambda_{n,k}$ are explicitly given in Table 2.1. This inequality is an improvement over the corresponding Denkowski's [5, (2.3)] inequality

$$P_0 \left(\frac{2}{3\pi + 2\epsilon}\right)^4 + P_1 \frac{1}{\pi}\left(\frac{2}{3\pi + 2\epsilon}\right)^2 + P_2 \left(\frac{2}{3\pi + 2\epsilon}\right)^2 < 1,$$

where ϵ is such that $z_1 = \frac{3}{2}\pi + \epsilon$ is the smallest positive root of the equation $\cosh z = \sec z$.

Corollary 3.2. Assume that the function f satisfies the conditions of Theorem 3.1 and the inequality (4) holds. Then, for the differential equation (1) together with the boundary conditions

$$y^{(i)}(0) = \alpha_i, \ y^{(i)}(1) = \beta_i, \ 0 \le i \le m - 1 \tag{9}$$

there exists at least one solution.

Proof. The proof is similar (but the conclusion is an improvement) to that of Theorem 3.1 in [5] for $m = 2$.

Corollary 3.3. Assume that for a fixed $(y_0, \cdots, y_m) \in R^{m+1}$ the function $f(t, y_0, \cdots, y_m)$ is measurable with respect to $t \in [0, 1]$, and for all $(y_0, \cdots, y_m)$, $(z_0, \cdots, z_m) \in R^{m+1}$ satisfies the Lipschitz condition

$$|f(t, y_0, \cdots, y_m) - f(t, z_0, \cdots, z_m)| \le \sum_{i=0}^{m} P_i |y_i - z_i|, \tag{10}$$

where the nonnegative constants P_i, $0 \le i \le m$ satisfy the inequality (4). Further, let $f(t, 0, \cdots, 0)$ be sommable on $[0, 1]$. Then, the boundary value problem (1), (9) has a unique solution.

Proof. The proof is similar (but the conclusion is an improvement) to that of Theorem 3.2 in [5] for $m = 2$. For an arbitrary m this result also generalizes the work of Herold [9,10].

Corollary 3.4. Assume that the function f is continuous in $[0,1] \times R^{m+1}$ and satisfies the Lipschitz condition (10). Further, let the inequality (4) holds. Then,

(i) for each sufficiently large n there exists a unique solution $v^n = (v_0^n, \cdots, v_n^n)$ of the discrete boundary value problem

$$(\nabla\Delta)^{m-1} \nabla \Delta v_i^n = h_n^{2m} f\left(t_i^n, v_i^n, \frac{\Delta v_i^n}{h_n}, \frac{\nabla\Delta v_i^n}{h_n^2}, \frac{\Delta \nabla \Delta v_i^n}{h_n^3}, \cdots\right), \tag{11}$$

$$m \le i \le n-m$$

$$\Delta^i v_0^n = h_n^i \alpha_i, \ \nabla^i v_n^n = h_n^i \beta_i, \ 0 \le i \le m-1 \tag{12}$$

where Δ and ∇ are the usual forward and the backward differences and $t_i^n = ih_n$, $h_n = 1/n$,

(ii) $\lim_{n\to\infty} |v_i^n - y(t_i^n)| = 0$ $(i \in N)$, where $y(t)$ is the solution of the boundary value problem (1), (9). Further, this convergence is uniform.

Proof. The proof is similar (but the conclusion is an improvement) to that of Theorem 4.1 in [5] for $m = 2$.

4. REFLECTION OF THE ARGUMENT BVP'S

Theorem 4.1. For the differential equation

$$y^{(2m)}(t) = f(t, y(t), y(-t), y'(t), y'(-t), \cdots, y^{(m)}(t), y^{(m)}(-t)), \ m \ge 1 \tag{13}$$

together with the boundary conditions

$$y^{(i)}(-1) = y^{(i)}(1) = 0, \ 0 \le i \le m-1 \tag{14}$$

we assume that $f : [-1,1] \times R^{2m+2} \to R$ satisfies the Carathèodory condition, and the inequality

$$|f(t, y_0, z_0, \cdots, y_m, z_m)| \le P + \sum_{j=0}^{m} (P_j|y_j| + Q_j|z_j|) \tag{15}$$

holds, where P, P_j, Q_j, $0 \le j \le m$ are nonnegative constants. Then, for the existence of a nonzero solution $y(t)$ of the boundary value problem (13), (14) it is necessary that

$$\rho_1 = \sum_{j=0}^{m} \frac{2^{2m-j}(P_j + Q_j)}{\lambda_{m,j}^{m-j} \lambda_{m,0}^{m}} < 1. \tag{16}$$

Further, the following inequalities hold

$$\|y^{(j)}\|_2 \le \frac{2^{2m-j+1/2}}{\lambda_{m,j}^{m-j}\lambda_{m,0}^{m}}\ \frac{P}{1-\rho_1},\ 0 \le j \le m \tag{17}$$

and

$$\|y^{(j)}\|_\infty \le \frac{2^{2m-j-1}}{\lambda_{m,j+1}^{m-j-1}\lambda_{m,0}^{m}}\ \frac{P}{1-\rho_1},\ 0 \le j \le m-1. \tag{18}$$

Proof. The proof is similar to that of Theorem 3.1 except that now we also need to use the obvious equality $\int_{-1}^{1}|y^{(k)}(-t)|^2dt = \int_{-1}^{1}|y^{(k)}(t)|^2dt$.

Remark 4.1. Results corresponding to Corollaries 3.3 and 3.4 for the problem (13), (14) can be stated rather easily.

Remark 4.2. For $m = 1$ inequality (16) reduces to $P_0 + Q_0 < \pi^2/4$ which is an improvement over the corresponding condition $P_0 + Q_0 < 2$ (their Theorem 3.5) by Wiener and Aftabizadeh [12]. However, it is the same as given by Gupta [6,7].

5. LIDSTONE BVP'S

Theorem 5.1. For the differential equation

$$y^{(2m)} = f(t, y, y', \cdots, y^{(2m-1)}),\ m \ge 1 \tag{19}$$

together with the boundary conditions

$$y^{(2i)}(a) = y^{(2i)}(b) = 0,\ 0 \le i \le m-1 \tag{20}$$

we assume that $f : [a,b] \times R^{2m} \to R$ satisfies the Caratheodory condition, and the inequality

$$|f(t, y_0, y_1, \cdots, y_{2m-1})| \le P + \sum_{j=0}^{2m-1} P_j|y_j| \tag{21}$$

holds, where $P,\ P_j,\ 0 \le j \le 2m-1$ are nonnegative constants. Then, for the existence of a nonzero solution $y(t)$ of the boundary value problem (19), (20) it is necessary that

$$\rho_2 = \sum_{j=0}^{2m-1} P_j\left(\frac{b-a}{\pi}\right)^{2m-j} < 1. \tag{22}$$

Further, the following inequalities hold

$$\|y^{(j)}\|_2 \le \frac{(b-a)^{2m-j+1/2}}{\pi^{2m-j}}\,\frac{P}{1-\rho_2},\ 0 \le j \le 2m-1 \tag{23}$$

and

$$\|y\|_\infty \le \frac{1}{2}\,\frac{(b-a)^{2m}}{\pi^{2m-1}}\,\frac{P}{1-\rho_2}. \tag{24}$$

Proof. The proof is similar to that of Theorem 3.1 except that now we need to use the inequalities

$$\int_a^b |y^{(k)}(t)|^2 dt \le \left(\frac{b-a}{\pi}\right)^{4m-2k} \int_a^b |y^{(2m)}(t)|^2 dt,\ 0 \le k \le 2m-1$$

which are immediate from a repeated application of Corollary 2.2.

Remark 5.1. Results corresponding to Corollaries 3.3 and 3.4 for the problem (19), (20) can be stated rather easily.

Remark 5.2. For $m = 2$ and $a = 0,\ b = 1$ the inequality (22) reduces to

$$P_0 + \pi P_1 + \pi^2 P_2 + \pi^3 P_3 \ < \ \pi^4, \tag{25}$$

which is an improvement over the corresponding condition

$$P_0 + \pi P_1 + \pi P_2 + \pi^2 P_3 \ < \ \pi^3$$

by Gupta [8, (2.14)]. Further, for the differential equation $y^{(4)} = f(t, y, y'')$ our inequality (25) reduces to

$$P_0 + \pi^2 P_2 \ < \ \pi^4,$$

which is sharper than

$$P_0 + 8P_2 \ < \ 64$$

obtained by Aftabizadeh [1, (4.9)].

Remark 5.3. Several constructive methods for the boundary value problem (19), (20) are available in Agarwal and Wong [3,4].

Corollary 5.2. For the differential equation

$$y^{(4)} = A(t)y''' + B(t)y'' + C(t)y + D(t) \tag{26}$$

together with the boundary conditions

$$y(-a) = y''(-a) = y(a) = y''(a) = 0, \tag{27}$$

where the functions $A(t)$, $B(t)$, $C(t)$, $D(t)$ are continuous and $|A(t)| \le A_0$, $|B(t)| \le B_0$, $|C(t)| \le C_0$ and $|D(t)| \le D_0$ there exists a nonzero solution $y(t)$ provided

$$\rho_3 = \left(\frac{2a}{\pi}\right) A_0 + \left(\frac{2a}{\pi}\right)^2 B_0 + \left(\frac{2a}{\pi}\right)^4 C_0 < 1. \tag{28}$$

Further, the following inequalities hold

$$\|y^{(j)}\|_2 \le \frac{(2a)^{4-j+1/2}}{\pi^{4-j}} \frac{D_0}{1-\rho_3}, \; 0 \le j \le 3 \tag{29}$$

and

$$\|y\|_\infty \le \frac{1}{2} \frac{(2a)^4}{\pi^3} \frac{D_0}{1-\rho_3}. \tag{30}$$

Remark 5.4. Imposing stringent conditions on the functions $A(t)$, $B(t)$, $C(t)$, and $D(t)$ a corresponding result for the boundary value problem (26), (27) has been obtained by Wiener and Aftabizadeh [12]. We also note that for the linear equation (26) subject to the boundary conditions

$$y(-a) = \alpha_0, \; y''(-a) = \alpha_2, \; y(a) = \beta_0, \; y''(a) = \beta_2 \tag{31}$$

the uniqueness implies existence type of arguments can be applied [2]. Therefore, if the inequality (28) holds then the boundary value problem (26), (31) has a unique solution.

Remark 5.5. Since reflection of the argument boundary value problem

$$y''(t) = a(t)\, y(-t) + b(t), \qquad y(-a) = y(a) = 0 \tag{32}$$

can be transformed to (26), (27) with

$$A(t) = \frac{2a'(t)}{a(t)}, \qquad B(t) = \frac{1}{2}a'(t) - \frac{1}{4}a^2(t),$$

$$C(t) = a(t)a(-t), \qquad D(t) = a(t)\left[b(-t) + \left(\frac{b(t)}{a(t)}\right)''\right]$$

if the conditions of Corollary 5.2 are satisfied then in view of Remark 5.4 the problem (32) has a unique solution.

6. MIXED BVP'S

Theorem 6.1. For the differential equation (19) together with the boundary conditions

$$y^{(2i)}(a) = y^{(2i+1)}(b) = 0, \; 0 \le i \le m-1 \tag{33}$$

we assume that $f : [a,b] \times R^{2m} \to R$ satisfies the Caratheodory condition, and the inequality (21) holds, where P, P_j, $0 \le j \le 2m-1$ are nonnegative constants. Then, for the existence of a nonzero solution $y(t)$ of the boundary value problem (19), (33) it is necessary that

$$\rho_4 = \sum_{j=0}^{2m-1} P_j \left(\frac{2(b-a)}{\pi}\right)^{2m-j} < 1. \tag{34}$$

Further, the following inequalities hold

$$\|y^{(j)}\|_2 \le \left(\frac{2}{\pi}\right)^{2m-j} (b-a)^{2m-j+1/2} \frac{P}{1-\rho_4}, \; 0 \le j \le 2m-1. \tag{35}$$

Proof. The proof is similar to that of Theorem 3.1 except that now we need to use the inequalities

$$\int_a^b |y^{(k)}(t)|^2 dt \le \left(\frac{2(b-a)}{\pi}\right)^{4m-2k} \int_a^b |y^{(2m)}(t)|^2 dt, \; 0 \le k \le 2m-1$$

which are immediate from a repeated application of Lemma 2.3.

Remark 6.1. Results corresponding to Corollaries 3.3 and 3.4 for the problem (19), (33) can be stated rather easily.

Remark 6.2. For $m = 2$ and $a = 0$, $b = 1$ the inequality (34) reduces to

$$16P_0 + 8\pi P_1 + 4\pi^2 P_2 + 2\pi^3 P_3 \; < \; \pi^4, \tag{36}$$

which is the same as obtained by Gupta [8, (2.25)].

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