EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THREE-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENCE EQUATIONS

Patricia Wong J. Y.
&
Ravi P. Agarwal

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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THREE-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENCE EQUATIONS

PATRICIA J. Y. WONG[1] and RAVI P. AGARWAL[2]
[1]Division of Mathematics, Nanyang Technological University, Singapore 1025
[2]Department of Mathematics, National University of Singapore, Singapore 0511

ABSTRACT: In this paper we shall offer sufficient conditions for the existence and uniqueness of solutions for the three-point boundary value problem

$$\Delta^2 y(n) = f(n, y(n), \Delta y(n)) + e(n), \ n = 0, 1, \cdots, T-1$$

$$y(0) = 0, \qquad y(T+1) = \alpha y(\eta) + b$$

where $1 \le \eta \le T-1$ is a fixed integer and α, b are given finite constants.

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1. INTRODUCTION

Let T be a fixed positive integer. We shall denote $[0, T] = \{0, 1, \cdots, T\}$. Also, the symbols Δ^i and ∇^i denote respectively the ith forward and backward difference operators with stepsize 1.

In this paper we shall consider the three-point boundary value problem

$$\begin{aligned} &\Delta^2 y(n) = f(n, y(n), \Delta y(n)) + e(n), \ n \in [0, T-1] \\ &y(0) = 0, \qquad y(T+1) = \alpha y(\eta) + b \end{aligned} \tag{1.1}$$

where $\eta \in [1, T-1]$ is a fixed integer, α, b are given finite constants and $e(n)$ is defined for $n \in [0, T+1]$. Throughout the paper the function $f : [0, T+1] \times \Re^2 \to \Re$ is assumed to be continuous.

We remark that the continuous analog of a particular case of (1.1)

$$\begin{aligned} &x''(t) = f(t, x(t), x'(t)) + e(t), \ 0 < t < 1 \\ &x(0) = 0, \qquad x(1) = \alpha x(\eta) \end{aligned} \tag{1.2}$$

where $0 < \eta < 1$ is given, has been studied by Gupta [2,3] and Marano [6] when $\alpha = 1$ as well as by Gupta et. al. [4,5] for a general α.

2. EXISTENCE RESULTS

Lemma 2.1. [1, p.24] Suppose that the function $u(n)$ is defined for $n \in [a, b]$. Then, there exists a $c \in [a+1, b-1]$ such that

$$\Delta u(c) \le (\ge) \ \frac{u(b) - u(a)}{b - a} \le (\ge) \ \nabla u(c).$$

Lemma 2.2. [1, p.678] For any function $u(n)$, $n \in [0, M]$ satisfying $u(0) = 0$ the following inequality hold

$$4 \sin^2 \frac{\pi}{2(2M+1)} \sum_{n=1}^{M} u^2(n) \le \sum_{n=0}^{M-1} (\Delta u(n))^2.$$

Theorem 2.1. Suppose that there exist functions $p(n)$, $q(n)$ and $r(n)$ defined on $[0,T+1]$ such that for $n \in [0,T+1]$, $x_1, x_2 \in \Re$,

$$|f(n,x_1,x_2)| \le p(n)|x_1| + q(n)|x_2| + r(n) \tag{2.1}$$

and

$$(T+1-\eta)|\alpha| > (T+1)|\alpha - 1|. \tag{2.2}$$

Let

$$\gamma = \frac{(T+1-\eta)|\alpha|}{(T+1-\eta)|\alpha| - (T+1)|\alpha-1|}. \tag{2.3}$$

If

$$[(T+1)\|p\|_1 + \|q\|_1]\gamma < 1, \tag{2.4}$$

then (1.1) has at least one solution $y(n)$ defined on $[0,T+1]$.

Proof. Let $S = \{y(n) : y(n)$ is defined for $n \in [0,T+1]\}$ and $S_1 = \{y(n) \in S : y(0) = 0,\ y(T+1) = \alpha y(\eta) + b\}$. We define the mappings $L : S_1 \to S$, $N : S \to S$ and $K : S \to S$ respectively by

$$Ly(n) = \Delta^2 y(n), \qquad Ny(n) = f(n, y(n), \Delta y(n))$$

and

$$Ky(n) = \sum_{s=0}^{n-1}(n-1-s)y(s) + \frac{\alpha n}{\theta}\sum_{s=0}^{\eta-1}(\eta-1-s)y(s) - \frac{n}{\theta}\sum_{s=0}^{T}(T-s)y(s) + \frac{bn}{\theta}$$

where $\theta = T+1-\alpha\eta$. It is clear that $\theta \ne 0$ because if $\theta = 0$, i.e., $\alpha = (T+1)/\eta$, then (2.2) is violated.

We note that N is a bounded mapping and L is one-to-one. Moreover, it follows from Arzela-Ascoli theorem that KN maps a bounded subset of S into a relatively compact subset of S. Thus, $KN : S \to S$ is a compact mapping. Further, it can be easily verified that for $y \in S$, $Ky \in S_1$ and $LKy = y$; and for $y \in S_1$, $KLy = y$.

Now, equation (1.1) can be written in opertor form as $Ly = Ny + e$ which is equivalent to

$$y = KNy + Ke. \tag{2.5}$$

Hence, to prove existence of solutions for (1.1) is the same as showing existence of solutions for (2.5). For this, we apply the Leray-Schauder continuation theorem [7] and it suffices to show that the set of solutions of the family of boundary value problems

$$\begin{gathered}\Delta^2 y(n) = \lambda f(n, y(n), \Delta y(n)) + \lambda e(n),\ n \in [0, T-1],\ 0 \le \lambda \le 1 \\ y(0) = 0, \qquad y(T+1) = \alpha y(\eta) + b\end{gathered} \tag{2.6}$$

is a priori bounded by a constant independent of λ.

Let y be a solution of (2.6) for some λ. We have

$$|y(n)| \le \sum_{s=0}^{n-1}|\Delta y(s)| \le n\|\Delta y\|_\infty \le (T+1)\|\Delta y\|_\infty. \tag{2.7}$$

Next, using Lemma 2.1 we find that there exists a $c \in [\eta+1, T]$ such that

$$\Delta y(c) \le (\ge)\ \frac{y(T+1) - y(\eta)}{T+1-\eta} = \frac{(\alpha-1)y(T+1) + b}{\alpha(T+1-\eta)} \le (\ge)\ \nabla y(c). \tag{2.8}$$

Applying (2.8) we get

$$\Delta y(n) = \sum_{s=c}^{n-1} \Delta^2 y(s) + \Delta y(c) \le (\ge) \sum_{s=c}^{n-1} \Delta^2 y(s) + \frac{(\alpha-1)y(T+1)+b}{\alpha(T+1-\eta)} \equiv A \tag{2.9}$$

and

$$\Delta y(n) = \sum_{s=c-1}^{n-1} \Delta^2 y(s) + \nabla y(c) \ge (\le) \sum_{s=c-1}^{n-1} \Delta^2 y(s) + \frac{(\alpha-1)y(T+1)+b}{\alpha(T+1-\eta)} \equiv B. \tag{2.10}$$

Coupling (2.9) and (2.10) provides

$$B \le (\ge)\ \Delta y(n) \le (\ge)\ A \tag{2.11}$$

which implies

$$\begin{aligned} |\Delta y(n)| &\le \max\{|A|, |B|\} \\ &\le \|\Delta^2 y\|_1 + \frac{|\alpha-1|}{|\alpha|(T+1-\eta)}\|y\|_\infty + \frac{|b|}{|\alpha|(T+1-\eta)} && (2.12) \\ &\le \|\Delta^2 y\|_1 + \frac{(T+1)|\alpha-1|}{|\alpha|(T+1-\eta)}\|\Delta y\|_\infty + \frac{|b|}{|\alpha|(T+1-\eta)} && (2.13) \end{aligned}$$

where we have also used (2.7) in the last inequality. In view of (2.2), it follows from (2.13) that

$$\|\Delta y\|_\infty \le \gamma \|\Delta^2 y\|_1 + Q \tag{2.14}$$

where γ is defined in (2.3) and

$$Q = \frac{|b|}{(T+1-\eta)|\alpha| - (T+1)|\alpha-1|}. \tag{2.15}$$

Now, from (2.6) and (2.1) we get

$$\begin{aligned} \|\Delta^2 y\|_1 &\le \|py\|_1 + \|q\Delta y\|_1 + \|r\|_1 + \|e\|_1 \\ &\le \|p\|_1\ \|y\|_\infty + \|q\|_1\ \|\Delta y\|_\infty + \|r\|_1 + \|e\|_1 \\ &\le [(T+1)\|p\|_1 + \|q\|_1]\left[\gamma\|\Delta^2 y\|_1 + Q\right] + \|r\|_1 + \|e\|_1 && (2.16) \end{aligned}$$

where we have used (2.7) and (2.14). Since (2.4) holds, it follows from (2.16) that

$$\|\Delta^2 y\|_1 \le \frac{[(T+1)\|p\|_1 + \|q\|_1]\,Q + \|r\|_1 + \|e\|_1}{1 - [(T+1)\|p\|_1 + \|q\|_1]\,\gamma} \equiv C.$$

Therefore, from (2.7) and (2.14) we find

$$\|y\|_\infty \le (T+1)(\gamma C + Q) \equiv D \tag{2.17}$$

where D is independent of λ. The proof is therefore complete.

Theorem 2.2. Suppose that (2.1) and (2.2) hold. Let

$$\beta^{-1} = 2\left|\sin\frac{\pi}{2[2(T+1)+1]}\right|. \tag{2.18}$$

If

$$\left[\beta\sqrt{T+1}\ \|p\|_2 + \|q\|_1\right]\gamma < 1, \tag{2.19}$$

then (1.1) has at least one solution $y(n)$ defined on $[0, T+1]$.

Proof. Let y be a solution of (2.6) for some λ. As in Theorem 2.1, it suffices to show that y is a priori bounded by a constant independent of λ. Since $y(0) = 0$, from Lemma 2.2 we have

$$\|y\|_2 \le \beta\|\Delta y\|_2 \le \beta\sqrt{T+1}\ \|\Delta y\|_\infty \tag{2.20}$$

where β is defined in (2.18).

It follows from (2.6), (2.1), Swartz's inequality, (2.20) and (2.14) that

$$\begin{aligned}\|\Delta^2 y\|_1 &\le \|p\|_2\ \|y\|_2 + \|q\|_1\ \|\Delta y\|_\infty + \|r\|_1 + \|e\|_1 \\ &\le \left[\beta\sqrt{T+1}\ \|p\|_2 + \|q\|_1\right]\|\Delta y\|_\infty + \|r\|_1 + \|e\|_1 \\ &\le \left[\beta\sqrt{T+1}\ \|p\|_2 + \|q\|_1\right][\gamma\|\Delta^2 y\|_1 + Q] + \|r\|_1 + \|e\|_1\end{aligned}$$

which in view of (2.19) leads to

$$\|\Delta^2 y\|_1 \le \frac{[\beta\sqrt{T+1}\ \|p\|_2 + \|q\|_1]Q + \|r\|_1 + \|e\|_1}{1 - [\beta\sqrt{T+1}\ \|p\|_2 + \|q\|_1]\gamma} \equiv C.$$

Hence, from (2.7) and (2.14) we get (2.17) and this completes the proof.

Theorem 2.3. Suppose that (2.1) and (2.2) hold. If

$$\left[(T+1)\|p\|_1 + \sqrt{T+1}\ \|q\|_2\right]\gamma < 1, \tag{2.21}$$

then (1.1) has at least one solution $y(n)$ defined on $[0, T+1]$.

Proof. Let y be a solution of (2.6) for some λ. It follows from (2.6), (2.1), Swartz's inequality, (2.7) and (2.14) that

$$\begin{aligned}\|\Delta^2 y\|_1 &\le \|p\|_1\ \|y\|_\infty + \|q\|_2\ \|\Delta y\|_2 + \|r\|_1 + \|e\|_1 \\ &\le \left[(T+1)\|p\|_1 + \sqrt{T+1}\ \|q\|_2\right]\|\Delta y\|_\infty + \|r\|_1 + \|e\|_1 \\ &\le \left[(T+1)\|p\|_1 + \sqrt{T+1}\ \|q\|_2\right][\gamma\|\Delta^2 y\|_1 + Q] + \|r\|_1 + \|e\|_1\end{aligned}$$

which in view of (2.21) provides

$$\|\Delta^2 y\|_1 \le \frac{[(T+1)\|p\|_1 + \sqrt{T+1}\ \|q\|_2]Q + \|r\|_1 + \|e\|_1}{1 - [(T+1)\|p\|_1 + \sqrt{T+1}\ \|q\|_2]\gamma} \equiv C.$$

Again, from (2.7) and (2.14) we obtain (2.17) and the proof is complete.

Theorem 2.4. Suppose that (2.1) and (2.2) hold. If

$$\gamma\sqrt{T+1}\ [\beta\|p\|_2 + \|q\|_2] < 1, \tag{2.22}$$

then (1.1) has at least one solution $y(n)$ defined on $[0, T+1]$.

Proof. Let y be a solution of (2.6) for some λ. Using Swartz's inequality, (2.20) and (2.14), from (2.6) we find

$$\begin{aligned}
\|\Delta^2 y\|_1 &\leq \|p\|_2\, \|y\|_2 + \|q\|_2\, \|\Delta y\|_2 + \|r\|_1 + \|e\|_1 \\
&\leq [\beta\|p\|_2 + \|q\|_2]\|\Delta y\|_2 + \|r\|_1 + \|e\|_1 \\
&\leq [\beta\|p\|_2 + \|q\|_2]\sqrt{T+1}\,[\gamma\|\Delta^2 y\|_1 + Q] + \|r\|_1 + \|e\|_1.
\end{aligned}$$

Since (2.22) holds, it follows that

$$\|\Delta^2 y\|_1 \leq \frac{[\beta\|p\|_2 + \|q\|_2]Q\sqrt{T+1} + \|r\|_1 + \|e\|_1}{1 - \gamma\sqrt{T+1}\,[\beta\|p\|_2 + \|q\|_2]} \equiv C.$$

As before we obtain (2.17) from (2.7) and (2.14) and this completes the proof.

Theorem 2.5. Suppose that (2.1) and (2.2) hold. If

$$\frac{\gamma(T+1)}{\sqrt{2}}[\beta\|p\|_\infty + \|q\|_\infty] < 1, \tag{2.23}$$

then (1.1) has at least one solution $y(n)$ defined on $[0, T+1]$.

Proof. Let y be a solution of (2.6) for some λ. As in the proof of Theorem 2.1, we have (2.11) which provides

$$\|\Delta y\|_2 \leq \max\{\|A\|_2, \|B\|_2\}. \tag{2.24}$$

To obtain an upper bound for the right side of (2.24), we note that

$$\begin{aligned}
\left|\frac{(\alpha-1)y(T+1)+b}{\alpha(T+1-\eta)}\right| &\leq \left|\frac{\alpha-1}{\alpha(T+1-\eta)}\right| \cdot \left|\sum_{s=0}^{T}\Delta y(s)\right| + \left|\frac{b}{\alpha(T+1-\eta)}\right| \\
&\leq \left|\frac{\alpha-1}{\alpha(T+1-\eta)}\right| \|\Delta y\|_1 + \left|\frac{b}{\alpha(T+1-\eta)}\right| \\
&\leq \sqrt{T+1}\left|\frac{\alpha-1}{\alpha(T+1-\eta)}\right| \|\Delta y\|_2 + \left|\frac{b}{\alpha(T+1-\eta)}\right|. \qquad (2.25)
\end{aligned}$$

Next, using Swartz's inequality we get

$$\begin{aligned}
\left\|\sum_{s=c}^{n-1}\Delta^2 y(s)\right\|_2^2 &= \sum_{n=0}^{T}\left\{\sum_{s=c}^{n-1}\Delta^2 y(s)\right\}^2 \\
&\leq \sum_{n=0}^{T}\left\{\left[\sum_{s=c}^{n-1}\left(\Delta^2 y(s)\right)^2\right]^{1/2}\left[\sum_{s=c}^{n-1}1^2\right]^{1/2}\right\}^2 \\
&= \sum_{n=0}^{T}\left\{\sum_{s=c}^{n-1}\left(\Delta^2 y(s)\right)^2 \cdot |n-c|\right\} \\
&\leq \|\Delta^2 y\|_2^2 \cdot \sum_{n=0}^{T}|n-c| \;\leq\; \frac{1}{2}(T+1)^2\|\Delta^2 y\|_2^2. \qquad (2.26)
\end{aligned}$$

Similarly, it can be verified that

$$\left\| \sum_{s=c-1}^{n-1} \Delta^2 y(s) \right\|_2 \le \frac{T+1}{\sqrt{2}} \, \|\Delta^2 y\|_2. \tag{2.27}$$

Using (2.25), (2.26), (2.27) and Swartz's inequality, it follows from (2.24) that

$$\|\Delta y\|_2 \le \frac{T+1}{\sqrt{2}} \, \|\Delta^2 y\|_2 + \left[\sqrt{T+1} \, \left| \frac{\alpha - 1}{\alpha(T+1-\eta)} \right| \, \|\Delta y\|_2 + \left| \frac{b}{\alpha(T+1-\eta)} \right| \right] \sqrt{T+1}$$

or

$$\|\Delta y\|_2 \le \frac{\gamma(T+1)}{\sqrt{2}} \, \|\Delta^2 y\|_2 + Q\sqrt{T+1}. \tag{2.28}$$

Now, using (2.20) and (2.28) from (2.6) we get

$$\begin{aligned}
\|\Delta^2 y\|_2 &\le \|py\|_2 + \|q\Delta y\|_2 + \|r\|_2 + \|e\|_2 \\
&\le \|p\|_\infty \, \|y\|_2 + \|q\|_\infty \, \|\Delta y\|_2 + \|r\|_2 + \|e\|_2 \\
&\le [\beta\|p\|_\infty + \|q\|_\infty] \|\Delta y\|_2 + \|r\|_2 + \|e\|_2 \\
&\le [\beta\|p\|_\infty + \|q\|_\infty] \left[\frac{\gamma(T+1)}{\sqrt{2}} \, \|\Delta^2 y\|_2 + Q\sqrt{T+1} \right] + \|r\|_2 + \|e\|_2
\end{aligned}$$

which in view of (2.23) implies

$$\|\Delta^2 y\|_2 \le \frac{[\beta\|p\|_\infty + \|q\|_\infty] Q\sqrt{T+1} + \|r\|_2 + \|e\|_2}{1 - \frac{\gamma(T+1)}{\sqrt{2}} [\beta\|p\|_\infty + \|q\|_\infty]} \equiv C.$$

Hence, it follows from (2.7), (2.14) and Swartz's inequality that

$$\begin{aligned}
\|y\|_\infty \le (T+1)\,(\gamma\|\Delta^2 y\|_1 + Q) &\le (T+1)\left(\gamma\sqrt{T} \, \|\Delta^2 y\|_2 + Q\right) \\
&\le (T+1)\left(\gamma\sqrt{T} \, C + Q\right) \equiv D
\end{aligned}$$

where D is independent of λ. This completes the proof.

3. UNIQUENESS RESULTS

Theorem 3.1. Suppose that there exist nonnegative constants c, d such that for $n \in [0, T+1]$, $x_1, x_2, y_1, y_2 \in \Re$,

$$|f(n, y_1, y_2) - f(n, x_1, x_2)| \le c|y_1 - x_1| + d|y_2 - x_2|. \tag{3.1}$$

Further, suppose that (2.2) holds. If

$$\frac{\gamma(T+1)}{\sqrt{2}} \, (c\beta + d) < 1, \tag{3.2}$$

then (1.1) has a unique solution $y(n)$ defined on $[0, T+1]$.

Proof. The existence of a solution for (1.1) follows from Theorem 2.5. Let y_1 and y_2 be two solutions of (1.1). Then, we have

$$\begin{aligned} \Delta^2(y_1 - y_2)(n) &= f(n, y_1(n), \Delta y_1(n)) - f(n, y_2(n), \Delta y_2(n)), \ n \in [0, T-1] \\ (y_1 - y_2)(0) = 0, &\qquad (y_1 - y_2)(T+1) = \alpha(y_1 - y_2)(\eta). \end{aligned} \tag{3.3}$$

Using (3.1), (2.20) and (2.28) (with $b = 0$), it follows from (3.3) that

$$\begin{aligned} \|\Delta^2 y_1 - \Delta^2 y_2\|_2 &\le c\|y_1 - y_2\|_2 + d\|\Delta y_1 - \Delta y_2\|_2 \\ &\le \frac{\gamma(T+1)}{\sqrt{2}}(c\beta + d)\|\Delta^2 y_1 - \Delta^2 y_2\|_2 \end{aligned}$$

which in view of (3.2) gives rise to

$$\|\Delta^2 y_1 - \Delta^2 y_2\|_2 = 0. \tag{3.4}$$

Now, using (2.20), (2.28) (with $b = 0$) and (3.4), we have

$$\|y_1 - y_2\|_2 \le \beta\|\Delta y_1 - \Delta y_2\|_2 \le \beta\, \frac{\gamma(T+1)}{\sqrt{2}}\|\Delta^2 y_1 - \Delta^2 y_2\|_2 = 0$$

which implies $\|y_1 - y_2\|_2 = 0$ and hence

$$y_1(n) = y_2(n), \ 0 \le n \le T+1. \tag{3.5}$$

Theorem 3.2. Suppose that (3.1) and (2.2) hold. If

$$\left[(T+1)(T+2)c + \sqrt{(T+1)(T+2)}\ d\right]\gamma < 1, \tag{3.6}$$

then (1.1) has a unique solution $y(n)$ defined on $[0, T+1]$.

Proof. The existence of a solution for (1.1) follows from Theorem 2.3. If y is a solution of (1.1), then we have

$$|y(n)| \le \sum_{s=0}^{n-1} |\Delta y(s)| \le \|\Delta y\|_1 \tag{3.7}$$

which also implies

$$\|y\|_1 \le (T+2)\ \|\Delta y\|_1. \tag{3.8}$$

Using (3.7), it follows from (2.12) that

$$|\Delta y(n)| \le \|\Delta^2 y\|_1 + \frac{|\alpha - 1|}{|\alpha|(T+1-\eta)}\ \|\Delta y\|_1 + \frac{|b|}{|\alpha|(T+1-\eta)}$$

which on summing from 0 to T gives

$$\|\Delta y\|_1 \le \gamma(T+1)\|\Delta^2 y\|_1 + (T+1)Q. \tag{3.9}$$

Now, to show uniqueness once again let y_1 and y_2 be two solutions of (1.1). Using (3.1), (3.8) and (3.9) (with $b = 0$), it follows from (3.3) that

$$\begin{aligned} \|\Delta^2 y_1 - \Delta^2 y_2\|_1 &\le c\|y_1 - y_2\|_1 + d\|\Delta y_1 - \Delta y_2\|_1 \\ &\le [c(T+2) + d]\gamma(T+1)\|\Delta^2 y_1 - \Delta^2 y_2\|_1 \end{aligned}$$

which in view of (3.6) provides

$$\|\Delta^2 y_1 - \Delta^2 y_2\|_1 = 0. \tag{3.10}$$

Next, using (3.8), (3.9) (with $b = 0$) and (3.10), we get

$$\|y_1 - y_2\|_1 \le (T+2)\|\Delta y_1 - \Delta y_2\|_1 \le \gamma(T+1)(T+2)\|\Delta^2 y_1 - \Delta^2 y_2\|_1 = 0$$

which implies $\|y_1 - y_2\|_1 = 0$ and hence (3.5) follows. This completes the proof.

Example 3.1. Consider the boundary value problem

$$\Delta^2 y(n) = \frac{2}{2n+1}\,\Delta y(n) + 2n + 3, \quad y(0) = 0, \quad y(10) = 4y(5), \quad n \in [0,8].$$

The general solution is given by

$$y(n) = c_1 + c_2 n^2 + \frac{1}{6}\,n(n-1)(4n+1).$$

We see that the boundary conditions lead to some inconsistency and so this problem has no solution. In fact, (2.2) is not satisfied and this illustrates Theorems 2.1-2.5.

Example 3.2. The boundary value problem

$$\Delta^2 y(n) = \frac{y(n)}{100(n+10)} + \frac{\Delta y(n)}{10(n+200)} + e(n), \quad y(0) = 0, \quad y(7) = 3y(2) + b, \quad n \in [0,5]$$

where b and $e(n)$ are arbitrary but fixed, satisfies Theorems 3.1-3.2. Hence, a unique solution exists.

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