EXISTENCE AND UNIQUENESS OF SOLUTIONS
FOR THREE-POINT BOUNDARY VALUE PROBLEMS
FOR SECOND ORDER DIFFERENCE EQUATIONS

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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THREE-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENCE EQUATIONS

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ABSTRACT: In this paper we shall offer sufficient conditions for the existence and uniqueness of solutions for the three-point boundary value problem

\[ \Delta^2 y(n) = f(n, y(n), \Delta y(n)) + e(n), \quad n = 0, 1, \ldots, T - 1 \]

\[ y(0) = 0, \quad y(T + 1) = \alpha y(\eta) + b \]

where \( 1 \leq \eta \leq T - 1 \) is a fixed integer and \( \alpha, b \) are given finite constants.

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1. INTRODUCTION

Let \( T \) be a fixed positive integer. We shall denote \( [0, T] = \{0, 1, \ldots, T\} \). Also, the symbols \( \Delta^i \) and \( \nabla^j \) denote respectively the \( i \)th forward and backward difference operators with stepsize 1.

In this paper we shall consider the three-point boundary value problem

\[ \Delta^2 y(n) = f(n, y(n), \Delta y(n)) + e(n), \quad n \in [0, T - 1] \]

\[ y(0) = 0, \quad y(T + 1) = \alpha y(\eta) + b \]

where \( \eta \in [1, T - 1] \) is a fixed integer, \( \alpha, b \) are given finite constants and \( e(n) \) is defined for \( n \in [0, T + 1] \). Throughout the paper the function \( f: [0, T + 1] \times \mathbb{R}^2 \to \mathbb{R} \) is assumed to be continuous.

We remark that the continuous analog of a particular case of (1.1)

\[ x''(t) = f(t, x(t), x'(t)) + e(t), \quad 0 < t < 1 \]

\[ x(0) = 0, \quad x(1) = \alpha x(\eta) \]

where \( 0 < \eta < 1 \) is given, has been studied by Gupta [2,3] and Marano [6] when \( \alpha = 1 \) as well as by Gupta et. al. [4,5] for a general \( \alpha \).

2. EXISTENCE RESULTS

Lemma 2.1. [1, p.24] Suppose that the function \( u(n) \) is defined for \( n \in [a, b] \). Then, there exists a \( c \in [a + 1, b - 1] \) such that

\[ \Delta u(c) \leq (\geq) \frac{u(b) - u(a)}{b - a} \leq (\geq) \nabla u(c). \]

Lemma 2.2. [1, p.678] For any function \( u(n) \), \( n \in [0, M] \) satisfying \( u(0) = 0 \) the following inequality hold

\[ 4 \sin^2 \frac{\pi}{2(2M + 1)} \sum_{n=1}^{M} u^2(n) \leq \sum_{n=0}^{M-1} (\Delta u(n))^2. \]
Theorem 2.1. Suppose that there exist functions \( p(n), q(n) \) and \( r(n) \) defined on \([0, T+1]\) such that for \( n \in [0, T+1] \), \( x_1, x_2 \in \mathbb{R} \),

\[
|f(n, x_1, x_2)| \leq p(n)|x_1| + q(n)|x_2| + r(n) \tag{2.1}
\]

and

\[
(T + 1 - \eta)|\alpha| > (T + 1)|\alpha| - 1. \tag{2.2}
\]

Let

\[
\gamma = \frac{(T + 1 - \eta)|\alpha|}{(T + 1 - \eta)|\alpha| - (T + 1)|\alpha| - 1}. \tag{2.3}
\]

If

\[
[(T + 1)||p||_1 + ||q||_1] \gamma < 1, \tag{2.4}
\]

then (1.1) has at least one solution \( y(n) \) defined on \([0, T+1]\).

**Proof.** Let \( S = \{y(n) : y(n) \text{ is defined for } n \in [0, T+1]\} \) and \( S_1 = \{y(n) \in S : y(0) = 0, y(T+1) = \alpha y(\eta) + b\} \). We define the mappings \( L : S_1 \rightarrow S, N : S \rightarrow S \) and \( K : S \rightarrow S \) respectively by

\[
Ly(n) = \Delta^2 y(n), \quad Ny(n) = f(n, y(n), \Delta y(n))
\]

and

\[
Ky(n) = \sum_{s=0}^{n-1} (n-1-s)y(s) + \frac{\alpha m}{\theta} \sum_{s=0}^{n-1} (\eta-1-s)y(s) - \frac{n}{\theta} \sum_{s=0}^{T} (T-s)y(s) + \frac{bn}{\theta}
\]

where \( \theta = T + 1 - \alpha \eta \). It is clear that \( \theta \neq 0 \) because if \( \theta = 0 \), i.e., \( \alpha = (T + 1)/\eta \), then (2.2) is violated.

We note that \( N \) is a bounded mapping and \( L \) is one-to-one. Moreover, it follows from Arzela-Ascoli theorem that \( KN \) maps a bounded subset of \( S \) into a relatively compact subset of \( S \). Thus, \( KN : S \rightarrow S \) is a compact mapping. Further, it can be easily verified that for \( y \in S \), \( Ky \in S_1 \) and \( LKy = y \); and for \( y \in S_1 \), \( KLy = y \).

Now, equation (1.1) can be written in operator form as

\[
Ly(n) = Ny + e \quad \text{which is equivalent to}
\]

\[
y = KNy + Ke. \tag{2.5}
\]

Hence, to prove existence of solutions for (1.1) is the same as showing existence of solutions for (2.5). For this, we apply the Leray-Schauder continuation theorem [7] and it suffices to show that the set of solutions of the family of boundary value problems

\[
\Delta^2 y(n) = \lambda f(n, y(n), \Delta y(n)) + \lambda e(n), \quad n \in [0, T-1], \quad 0 \leq \lambda \leq 1
\]

\[
y(0) = 0, \quad y(T+1) = \alpha y(\eta) + b \tag{2.6}
\]

is a priori bounded by a constant independent of \( \lambda \).

Let \( y \) be a solution of (2.6) for some \( \lambda \). We have

\[
|y(n)| \leq \sum_{s=0}^{n-1} |\Delta y(s)| \leq n ||\Delta y||_\infty \leq (T+1)||\Delta y||_\infty. \tag{2.7}
\]

Next, using Lemma 2.1 we find that there exists a \( c \in [\eta + 1, T] \) such that

\[
\Delta y(c) \leq (\geq) \frac{y(T+1) - y(\eta)}{T+1 - \eta} = \frac{\alpha - 1}{\alpha(T+1 - \eta)} \leq (\geq) \nabla y(c). \tag{2.8}
\]
Applying (2.8) we get
\[
\Delta y(n) = \sum_{s=c}^{n-1} \Delta^2 y(s) + \Delta y(c) \leq (\geq) \sum_{s=c}^{n-1} \Delta^2 y(s) + \frac{(\alpha - 1)y(T + 1) + b}{\alpha(T + 1 - \eta)} \equiv A \quad (2.9)
\]
and
\[
\Delta y(n) = \sum_{s=c-1}^{n-1} \Delta^2 y(s) + \nabla y(c) \geq (\leq) \sum_{s=c-1}^{n-1} \Delta^2 y(s) + \frac{(\alpha - 1)y(T + 1) + b}{\alpha(T + 1 - \eta)} \equiv B. \quad (2.10)
\]
Coupling (2.9) and (2.10) provides
\[
B \leq (\geq) \Delta y(n) \leq (\geq) A \quad (2.11)
\]
which implies
\[
|\Delta y(n)| \leq \max\{|A|, |B|\} \\
\leq \|\Delta^2 y\|_1 + \frac{|\alpha - 1|}{\alpha(T + 1 - \eta)} \|y\|_\infty + \frac{|b|}{|\alpha|(T + 1 - \eta)} \quad (2.12)
\]
\[
\leq \|\Delta^2 y\|_1 + \frac{(T + 1)|\alpha - 1|}{|\alpha|(T + 1 - \eta)} \|\Delta y\|_\infty + \frac{|b|}{|\alpha|(T + 1 - \eta)} \quad (2.13)
\]
where we have also used (2.7) in the last inequality. In view of (2.2), it follows from (2.13) that
\[
\|\Delta y\|_\infty \leq \gamma \|\Delta^2 y\|_1 + Q \quad (2.14)
\]
where \(\gamma\) is defined in (2.3) and
\[
Q = \frac{|b|}{(T + 1 - \eta)|\alpha| - (T + 1)|\alpha - 1|}. \quad (2.15)
\]
Now, from (2.6) and (2.1) we get
\[
\|\Delta^2 y\|_1 \leq \|p y\|_1 + \|q \Delta y\|_1 + \|r\|_1 + \|e\|_1 \\
\leq \|p\|_1 \|y\|_\infty + \|q\|_1 \|\Delta y\|_\infty + \|r\|_1 + \|e\|_1 \\
\leq [(T + 1)\|p\|_1 + \|q\|_1] \left[\gamma \|\Delta^2 y\|_1 + Q\right] + \|r\|_1 + \|e\|_1 \quad (2.16)
\]
where we have used (2.7) and (2.14). Since (2.4) holds, it follows from (2.16) that
\[
\|\Delta^2 y\|_1 \leq \frac{[(T + 1)\|p\|_1 + \|q\|_1] Q + \|r\|_1 + \|e\|_1}{1 - [(T + 1)\|p\|_1 + \|q\|_1] \gamma} \equiv C.
\]
Therefore, from (2.7) and (2.14) we find
\[
\|y\|_\infty \leq (T + 1)(\gamma C + Q) \equiv D \quad (2.17)
\]
where \(D\) is independent of \(\lambda\). The proof is therefore complete.

**Theorem 2.2.** Suppose that (2.1) and (2.2) hold. Let
\[
\beta^{-1} = 2 \left| \sin \frac{\pi}{2(2(T + 1) + 1)} \right|. \quad (2.18)
\]
If
\[ \left[ \beta \sqrt{T+1} \|p\|_2 + \|q\|_1 \right] \gamma < 1, \tag{2.19} \]
then (1.1) has at least one solution \( y(n) \) defined on \([0, T+1]\).

**Proof.** Let \( y \) be a solution of (2.6) for some \( \lambda \). As in Theorem 2.1, it suffices to show that \( y \) is a priori bounded by a constant independent of \( \lambda \). Since \( y(0) = 0 \), from Lemma 2.2 we have
\[ \|y\|_2 \leq \beta \|\Delta y\|_2 \leq \beta \sqrt{T+1} \|\Delta y\|_\infty, \tag{2.20} \]
where \( \beta \) is defined in (2.18).

It follows from (2.6), (2.1), Swartz’s inequality, (2.20) and (2.14) that
\[ \|\Delta^2 y\|_1 \leq \|p\|_2 \|y\|_2 + \|q\|_1 \|\Delta y\|_\infty + \|r\|_1 + \|e\|_1 \]
\[ \leq \left[ \beta \sqrt{T+1} \|p\|_2 + \|q\|_1 \right] \|\Delta y\|_\infty + \|r\|_1 + \|e\|_1 \]
\[ \leq \left[ \beta \sqrt{T+1} \|p\|_2 + \|q\|_1 \right] \left[ \gamma \|\Delta^2 y\|_1 + Q \right] + \|r\|_1 + \|e\|_1 \]
which in view of (2.19) leads to
\[ \|\Delta^2 y\|_1 \leq \frac{\left[ \beta \sqrt{T+1} \|p\|_2 + \|q\|_1 \right] Q + \|r\|_1 + \|e\|_1}{1 - \left[ \beta \sqrt{T+1} \|p\|_2 + \|q\|_1 \right] \gamma} \equiv C. \]

Hence, from (2.7) and (2.14) we get (2.17) and this completes the proof.

**Theorem 2.3.** Suppose that (2.1) and (2.2) hold. If
\[ \left( (T+1)\|p\|_1 + \sqrt{T+1} \|q\|_2 \right) \gamma < 1, \tag{2.21} \]
then (1.1) has at least one solution \( y(n) \) defined on \([0, T+1]\).

**Proof.** Let \( y \) be a solution of (2.6) for some \( \lambda \). It follows from (2.6), (2.1), Swartz’s inequality, (2.7) and (2.14) that
\[ \|\Delta^2 y\|_1 \leq \|p\|_1 \|y\|_\infty + \|q\|_2 \|\Delta y\|_2 + \|r\|_1 + \|e\|_1 \]
\[ \leq \left[ (T+1)\|p\|_1 + \sqrt{T+1} \|q\|_2 \right] \|\Delta y\|_\infty + \|r\|_1 + \|e\|_1 \]
\[ \leq \left[ (T+1)\|p\|_1 + \sqrt{T+1} \|q\|_2 \right] \left[ \gamma \|\Delta^2 y\|_1 + Q \right] + \|r\|_1 + \|e\|_1 \]
which in view of (2.21) provides
\[ \|\Delta^2 y\|_1 \leq \frac{\left[ (T+1)\|p\|_1 + \sqrt{T+1} \|q\|_2 \right] Q + \|r\|_1 + \|e\|_1}{1 - \left[ (T+1)\|p\|_1 + \sqrt{T+1} \|q\|_2 \right] \gamma} \equiv C. \]

Again, from (2.7) and (2.14) we obtain (2.17) and the proof is complete.

**Theorem 2.4.** Suppose that (2.1) and (2.2) hold. If
\[ \gamma \sqrt{T+1} \left[ \beta \|p\|_2 + \|q\|_2 \right] < 1, \tag{2.22} \]
then (1.1) has at least one solution \( y(n) \) defined on \([0, T+1]\).
Proof. Let \( y \) be a solution of (2.6) for some \( \lambda \). Using Swartz's inequality, (2.20) and (2.14), from (2.6) we find
\[
\| \Delta^2 y \|_1 \leq \|p\|_2 \|y\|_2 + \|q\|_2 \|\Delta y\|_2 + \|r\|_1 + \|e\|_1 \\
\leq [\beta \|p\|_2 + \|q\|_2] \|\Delta y\|_2 + \|r\|_1 + \|e\|_1 \\
\leq [\beta \|p\|_2 + \|q\|_2] \sqrt{T+1} [\gamma \|\Delta^2 y\|_1 + Q] + \|r\|_1 + \|e\|_1.
\]
Since (2.22) holds, it follows that
\[
\| \Delta^2 y \|_1 \leq \frac{[\beta \|p\|_2 + \|q\|_2] \sqrt{T+1} [\gamma \|\Delta^2 y\|_1 + Q] + \|r\|_1 + \|e\|_1}{1 - \gamma \sqrt{T+1}} = C.
\]
As before we obtain (2.17) from (2.7) and (2.14) and this completes the proof.

Theorem 2.5. Suppose that (2.1) and (2.2) hold. If
\[
\frac{\gamma}{\sqrt{2}} \frac{(T+1)}{\sqrt{2}} [\beta \|p\|_\infty + \|q\|_\infty] < 1, \tag{2.23}
\]
then (1.1) has at least one solution \( y(n) \) defined on \([0, T+1]\).

Proof. Let \( y \) be a solution of (2.6) for some \( \lambda \). As in the proof of Theorem 2.1, we have (2.11) which provides
\[
\| \Delta y \|_2 \leq \max\{\|A\|_2, \|B\|_2\}. \tag{2.24}
\]
To obtain an upper bound for the right side of (2.24), we note that
\[
\left| \frac{(\alpha - 1) y(T + 1) + b}{\alpha(T + 1 - \eta)} \right| \leq \left| \frac{(\alpha - 1)}{\alpha(T + 1 - \eta)} \right| \sum_{s=0}^{T} \Delta y(s) + \left| \frac{b}{\alpha(T + 1 - \eta)} \right| \\
\leq \left| \frac{(\alpha - 1)}{\alpha(T + 1 - \eta)} \right| \| \Delta y \|_1 + \left| \frac{b}{\alpha(T + 1 - \eta)} \right| \\
\leq \sqrt{T+1} \left| \frac{(\alpha - 1)}{\alpha(T + 1 - \eta)} \right| \| \Delta y \|_2 + \left| \frac{b}{\alpha(T + 1 - \eta)} \right|. \tag{2.25}
\]
Next, using Swartz's inequality we get
\[
\left\| \sum_{s=c}^{n-1} \Delta^2 y(s) \right\|_2^2 = \sum_{n=0}^{T} \left\{ \sum_{s=c}^{n-1} \Delta^2 y(s) \right\}^2 \\
\leq \sum_{n=0}^{T} \left\{ \left[ \sum_{s=c}^{n-1} (\Delta^2 y(s))^2 \right]^{1/2} \left[ \sum_{s=c}^{n-1} 1^2 \right]^{1/2} \right\}^2 \\
= \sum_{n=0}^{T} \left\{ \sum_{s=c}^{n-1} (\Delta^2 y(s))^2 \cdot |n - c| \right\} \\
\leq \| \Delta^2 y \|_2^2 \cdot \sum_{n=0}^{T} |n - c| \leq \frac{1}{2} (T+1)^2 \| \Delta^2 y \|_2^2. \tag{2.26}
\]
Similarly, it can be verified that
\[
\left\| \sum_{s=c-1}^{n-1} \Delta^2 y(s) \right\|_2 \leq \frac{T+1}{\sqrt{2}} \left\| \Delta^2 y \right\|_2. \tag{2.27}
\]

Using (2.25), (2.26), (2.27) and Swartz’s inequality, it follows from (2.24) that
\[
\| \Delta y \|_2 \leq \frac{T+1}{\sqrt{2}} \| \Delta^2 y \|_2 + \left[ \sqrt{T+1} \left| \frac{\alpha - 1}{\alpha(T + 1 - \eta)} \right| \| \Delta y \|_2 + \left| \frac{b}{\alpha(T + 1 - \eta)} \right| \right] \sqrt{T + 1}
\]
or
\[
\| \Delta y \|_2 \leq \frac{\gamma(T + 1)}{\sqrt{2}} \| \Delta^2 y \|_2 + Q \sqrt{T + 1}. \tag{2.28}
\]

Now, using (2.20) and (2.28) from (2.6) we get
\[
\| \Delta^2 y \|_2 \leq \| p y \|_2 + \| q \Delta y \|_2 + \| r \|_2 + \| e \|_2
\]
\[
\leq \| p \|_{\infty} \| y \|_2 + \| q \|_{\infty} \| \Delta y \|_2 + \| r \|_2 + \| e \|_2
\]
\[
\leq \| p \|_{\infty} + \| q \|_{\infty} \| \Delta y \|_2 + \| r \|_2 + \| e \|_2
\]
\[
\leq \left[ \| p \|_{\infty} + \| q \|_{\infty} \right] \left[ \frac{\gamma(T + 1)}{\sqrt{2}} \| \Delta^2 y \|_2 + Q \sqrt{T + 1} \right] + \| r \|_2 + \| e \|_2
\]
which in view of (2.23) implies
\[
\| \Delta^2 y \|_2 \leq \frac{[\| p \|_{\infty} + \| q \|_{\infty}]Q \sqrt{T + 1} + \| r \|_2 + \| e \|_2}{1 - \frac{\gamma(T + 1)}{\sqrt{2}}} \equiv C.
\]

Hence, it follows from (2.7), (2.14) and Swartz’s inequality that
\[
\| y \|_{\infty} \leq (T + 1) \left( \gamma \| \Delta^2 y \|_2 + Q \right) \leq (T + 1) \left( \gamma \sqrt{T} \| \Delta^2 y \|_2 + Q \right)
\]
\[
\leq (T + 1) \left( \gamma \sqrt{T} \right) (C + Q) \equiv D
\]
where D is independent of \( \lambda \). This completes the proof.

3. UNIQUENESS RESULTS

Theorem 3.1. Suppose that there exist nonnegative constants \( c, d \) such that for \( n \in [0, T + 1] \), \( x_1, x_2, y_1, y_2 \in \mathbb{R} \),
\[
|f(n, y_1, y_2) - f(n, x_1, x_2)| \leq c|y_1 - x_1| + d|y_2 - x_2|.
\tag{3.1}
\]

Further, suppose that (2.2) holds. If
\[
\frac{\gamma(T + 1)}{\sqrt{2}} (c\beta + d) < 1,
\tag{3.2}
\]
then (1.1) has a unique solution \( y(n) \) defined on \([0, T + 1]\).
Proof. The existence of a solution for (1.1) follows from Theorem 2.5. Let \( y_1 \) and \( y_2 \) be two solutions of (1.1). Then, we have

\[
\Delta^2(y_1 - y_2)(n) = f(n, y_1(n), \Delta y_1(n)) - f(n, y_2(n), \Delta y_2(n)), \quad n \in [0, T - 1]
\]

\[
(y_1 - y_2)(0) = 0, \quad (y_1 - y_2)(T + 1) = \alpha(y_1 - y_2)(\eta).
\]

Using (3.1), (2.20) and (2.28) (with \( b = 0 \)), it follows from (3.3) that

\[
\|\Delta^2 y_1 - \Delta^2 y_2\| \leq c\|y_1 - y_2\|_2 + d\|\Delta y_1 - \Delta y_2\|_2
\]

\[
\leq \frac{\gamma(T + 1)}{\sqrt{2}} (c\beta + d)\|\Delta^2 y_1 - \Delta^2 y_2\|_2
\]

which in view of (3.2) gives rise to

\[
\|\Delta^2 y_1 - \Delta^2 y_2\|_2 = 0.
\]

Now, using (2.20), (2.28) (with \( b = 0 \)) and (3.4), we have

\[
\|y_1 - y_2\|_2 \leq \beta\|\Delta y_1 - \Delta y_2\|_2 \leq \beta \frac{\gamma(T + 1)}{\sqrt{2}}\|\Delta^2 y_1 - \Delta^2 y_2\|_2 = 0
\]

which implies \( \|y_1 - y_2\|_2 = 0 \) and hence

\[
y_1(n) = y_2(n), \quad 0 \leq n \leq T + 1.
\]

Theorem 3.2. Suppose that (3.1) and (2.2) hold. If

\[
\left[ (T + 1)(T + 2)c + \sqrt{(T + 1)(T + 2)} \right] \gamma < 1,
\]

then (1.1) has a unique solution \( y(n) \) defined on \([0, T + 1]\).

Proof. The existence of a solution for (1.1) follows from Theorem 2.3. If \( y \) is a solution of (1.1), then we have

\[
|y(n)| \leq \sum_{s=0}^{n-1} |\Delta y(s)| \leq \|\Delta y\|_1
\]

which also implies

\[
\|y\|_1 \leq (T + 2) \|\Delta y\|_1.
\]

Using (3.7), it follows from (2.12) that

\[
|\Delta y(n)| \leq \|\Delta^2 y\|_1 + \frac{|\alpha - 1|}{|\alpha|(T + 1 - \eta)} \|\Delta y\|_1 + \frac{|b|}{|\alpha|(T + 1 - \eta)}
\]

which on summing from 0 to \( T \) gives

\[
\|\Delta y\|_1 \leq \gamma(T + 1)\|\Delta^2 y\|_1 + (T + 1)Q.
\]

Now, to show uniqueness once again let \( y_1 \) and \( y_2 \) be two solutions of (1.1). Using (3.1), (3.8) and (3.9) (with \( b = 0 \)), it follows from (3.3) that

\[
\|\Delta^2 y_1 - \Delta^2 y_2\|_1 \leq c\|y_1 - y_2\|_1 + d\|\Delta y_1 - \Delta y_2\|_1
\]

\[
\leq [c(T + 2) + d]\gamma(T + 1)\|\Delta^2 y_1 - \Delta^2 y_2\|_1
\]
which in view of (3.6) provides
\[ \|\Delta^2 y_1 - \Delta^2 y_2\|_1 = 0. \]

(3.10)

Next, using (3.8), (3.9) (with \( b = 0 \)) and (3.10), we get
\[ \|y_1 - y_2\| \leq (T + 2)\|\Delta y_1 - \Delta y_2\| \leq \gamma(T + 1)(T + 2)\|\Delta^2 y_1 - \Delta^2 y_2\|_1 = 0 \]
which implies \( \|y_1 - y_2\|_1 = 0 \) and hence (3.5) follows. This completes the proof.

Example 3.1. Consider the boundary value problem
\[ \Delta^2 y(n) = \frac{2}{2n + 1} \Delta y(n) + 2n + 3, \quad y(0) = 0, \quad y(10) = 4y(5), \quad n \in [0, 8]. \]

The general solution is given by
\[ y(n) = c_1 + c_2 n^2 + \frac{1}{6} n(n - 1)(4n + 1). \]

We see that the boundary conditions lead to some inconsistency and so this problem has no solution. In fact, (2.2) is not satisfied and this illustrates Theorems 2.1-2.5.

Example 3.2. The boundary value problem
\[ \Delta^2 y(n) = \frac{y(n)}{100(n + 10)} + \frac{\Delta y(n)}{10(n + 200)} + e(n), \quad y(0) = 0, \quad y(7) = 3y(2) + b, \quad n \in [0, 5] \]
where \( b \) and \( e(n) \) are arbitrary but fixed, satisfies Theorems 3.1-3.2. Hence, a unique solution exists.

REFERENCES


