On the skewness of Cartesian products with trees

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Abstract

The skewness of a graph $G$ is the minimum number of edges in $G$ whose removal results in a planar graph. It is a parameter that measures how non-planar a graph is, and it also has important applications to VLSI design, but there are few results for skewness of graphs. In this paper, we first prove that the skewness is additive for the Zip product under certain conditions. We then present results on the lower bounds for the skewness of Cartesian products of graphs with trees and paths, respectively. Some exact values of the skewness for Cartesian products of complete graphs with trees, as well as of stars and wheels with paths are obtained by applying these lower bounds.

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1 Introduction

All graphs considered here are simple, finite and undirected. Throughout this paper, let $K_n$, $P_n$, $C_n$ denote the complete graph, path and cycle on $n$ vertices respectively and let $S_m = K_{1,m}$ denote the star graph with $m$ vertices of degree 1 (called the leaves of the star) and one vertex of degree $m$ (the center). We denote by $W_m$ the

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wheel graph on \( m + 1 \) vertices, i.e., the graph obtained from a cycle \( C_m \) by adding a new vertex and edges joining it to all the vertices of the cycle \( C_m \). Please refer to [1] for graph theory notations and terminologies that are not defined here. A **drawing** of a graph \( G = (V, E) \) is a mapping \( D \) that assigns to each vertex in \( V \) a distinct node in the plane and to each edge \( uv \) in \( E \) a continuous arc (i.e., a homeomorphic image of a closed interval) connecting \( D(u) \) and \( D(v) \), not passing through the image of any other node. The nodes and arcs are also called vertices and edges respectively. A graph \( G \) is said to be **planar** if there exists a drawing of \( G \) so that its edges intersect only at their endpoints. Such a drawing of a planar graph \( G \) is called a **plane drawing** or a **plane graph** of \( G \). If a graph \( G \) is not planar, one natural question to ask is: How far away is \( G \) from planarity? For this, various measures for non-planarities of a graph have been proposed, see [14] for a survey. The most celebrated measure is the **crossing number** \( cr(G) \) which asks for the smallest number of edge crossings in any drawing of \( G \). Another measure is the **skewness**, denoted by \( sk(G) \), which is the minimum number of edges in \( G \) whose removal results in a planar graph. This is complementary, and computationally equivalent, to the **Maximum Planar Subgraph Problem** that asks for a planar subgraph of \( G \) with the maximum number of edges. Clearly, \( sk(G) \leq cr(G) \) holds for all graphs \( G \). Although \( sk(G) \) is a lower bound for \( cr(G) \), \( cr(G) - sk(G) \) could be very large. For example, Cimikowski [7] found a family of graphs \( G \) with \( sk(G) = 1 \) but \( cr(G) \) can be arbitrarily large.

As a topological invariant of a graph, skewness is an important research object in topological graph theory, and it also plays important roles in automatic graph drawing and VLSI design [10, 13, 16]. But the results about skewness are quite limited, compared with other topological invariants, e.g., genus, thickness and crossing number. The problem of determining the skewness of a given graph is known to be NP-complete [15, 18]. The skewness of some well-known families of graphs has been calculated (see [4, 5, 7, 9, 12, 17]), such as complete graph \( K_m \), complete bipartite graph \( K_{m,n} \), \( n \)-cube \( Q_n \) and complete \( k \)-partite graphs for \( k \leq 4 \). The only result on skewness for Cartesian products is the skewness of \( C_m \square C_n \) obtained by Mendonça et al. [17], where the Cartesian product \( G_1 \square G_2 \) of two vertex-disjoint graphs \( G_1 \) and \( G_2 \) is defined to be the graph with vertex set \( V(G_1) \times V(G_2) = \{(u,v) : u \in V(G_1), v \in V(G_2)\} \) and edge set \( E(G_1 \square G_2) = \{((x_1,y_1),(x_2,y_2)) : x_1 = x_2 \) and \( y_1y_2 \in E(G_2) \) or \( y_1 = y_2 \) and \( x_1x_2 \in E(G_1)\} \). For more about skewness of a graph, see [6, 8, 14] and the references therein.

Let \( G_i, i = 1, 2 \), be a graph with a vertex \( v_i \in V(G_i) \) whose neighborhood \( N_i = N_{G_i}(v_i) \) has size \( d \). A **zip function** of graphs \( G_1 \) and \( G_2 \) at vertices \( v_1 \) and \( v_2 \) is a bijection \( \sigma : N_1 \rightarrow N_2 \). The **Zip product** \( G_1 \odot_\sigma G_2 \) of graphs \( G_1 \) and \( G_2 \) according
to $\sigma$ is obtained from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ by adding the edges joining $u$ and $\sigma(u)$ for all $u \in N_1$. An example with $d = 4$ and $\sigma(u_j) = w_j$ for $j = 1, 2, 3, 4$ is shown in Figure 1. Applying the Zip product operation, Bokal [2, 3] determined the crossing number of the Cartesian products of various graphs with paths and trees. Particularly, Bokal [2] completely determined the crossing number of $S_m \square P_n$ by using Zip product operation, and hence he completely solved the conjecture proposed by Jendrol’ and Ščerbová [11].

In this article, we apply the same strategy for the crossing number used in [2, 3] to investigate the skewness of graphs. We study graphs produced by the operation of Zip product and of Cartesian product, and obtain the exact values of $sk(G)$ for graphs $G$ in the classes considered in [2, 3]. We first prove that the skewness is additive for the Zip product under certain conditions. We then establish two tight lower bounds for the skewness of the Cartesian product of one graph with a tree and with a path, respectively. Applying these results, we completely determine $sk(K_m \square T)$, where $T$ is a tree with maximum degree $\Delta \leq 2m - 4$, $sk(S_m \square P_n)$ and $sk(W_m \square P_n)$.

![Figure 1: A Zip product of $G_1$ and $G_2$](image)

### 2 Preliminaries

The union $G_1 \cup G_2$ of two graphs $G_1$ and $G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, and the join product of two vertex-disjoint graphs $G_1$ and $G_2$, denoted by $G_1 + G_2$, is obtained from $G_1 \cup G_2$ by adding edges joining $u$ and $v$ for all $u \in V(G_1)$ and $v \in V(G_2)$. Let $G + nK_1$ denote the graph $G + N_n$, where $N_n$ is the empty graph of order $n$.

Recall that the skewness of a graph $G$, denoted by $sk(G)$, is defined to be the minimum integer $k \geq 0$ such that $G - E'$ is planar for some subset $E'$ of $E(G)$ with $|E'| = k$. Let $\mathcal{PSK}(G)$ be the family of plane drawings of spanning subgraphs of
G which are planar and of size $|E(G)| - sk(G)$. Clearly, any planar subgraph of $G$ contains at most $|E(G)| - sk(G)$ edges.

The following basic and mostly well-known properties of skewness can be easily obtained by the definition of skewness of a graph or can been found in [6, 8].

**Proposition 1** Let $G_1$ and $G_2$ be two simple graphs.

(i) [6] If $G_1$ and $G_2$ are homeomorphic, then $sk(G_1) = sk(G_2)$;

(ii) [6, 8] If $G_1$ is a subgraph of $G_2$, then $sk(G_1) \leq sk(G_2)$;

(iii) If $G_1$ and $G_2$ are edge-disjoint, then $sk(G_1 \cup G_2) \geq sk(G_1) + sk(G_2)$;

(iv) [6] If $G_1$ is a planar subgraph of graph $G_2$, then $sk(G_2) \leq |E(G_2)| - |E(G_1)|$;

(v) [8] If $G_1$ is connected, then $D$ is connected for each $D \in PSK(G_1)$.

**Proof.** From [6, 8], we only need to prove (iii). By definition, $sk(G_1 \cup G_2)$ is the minimum number of edges in $G_1 \cup G_2$ whose removal results in a planar graph. Then there exist $E'_i \subseteq E(G_i)$ for $i = 1, 2$ such that $|E'_1| + |E'_2| = sk(G_1 \cup G_2)$ and $(G_1 \cup G_2) - (E'_1 \cup E'_2)$ is planar. Note that

$$(G_1 \cup G_2) - (E'_1 \cup E'_2) = (G_1 - E'_1) \cup (G_2 - E'_2).$$

As $(G_1 \cup G_2) - (E'_1 \cup E'_2)$ is planar, $G_i - E'_i$ is planar for $i = 1, 2$, implying that $sk(G_i) \leq |E'_i|$ for $i = 1, 2$. Hence

$$sk(G_1 \cup G_2) \geq sk(G_1) + sk(G_2),$$

as stated. \(\square\)

The following lemmas can be proved by Euler’s polyhedron formula and can be seen in [4, 6, 9, 12].

**Lemma 1** ([9, 12]) $sk(K_m) = \binom{m-3}{2}$ and $sk(K_m,n) = (m-2)(n-2)$.

**Lemma 2** ([4, 6]) Let $G$ be a connected graph on $p$ vertices and $q$ edges with girth $g$. Then,

$$sk(G) \geq \left\lceil q - \frac{g}{g-2}(p-2) \right\rceil.$$
3 Lower bounds for $sk(G_1 \odot_\sigma G_2)$

In this section, we first prove that the skewness is additive for the Zip product under certain conditions. Applying the additivity, we then present two tight lower bounds for the skewness of Cartesian products of graphs with trees and paths, respectively. This proof idea was initially applied to settle the crossing number of Cartesian products of graphs with trees and paths [2, 3], and we notice that the same idea can be used to deal with the similar question about the skewness.

Let $G = (V, E)$ be a graph. By adding an apex to $V_1 \subseteq V$, we mean adding a new vertex $v$ and adding edges joining $v$ to all vertices of $V_1$. For a vertex $x \in V$, we denote by $N_G(x)$ the set of vertices in $G$ which are adjacent to $x$. For a non-empty subset $A \subseteq E$, we denote by $G[A]$ the subgraph of $G$ induced by $A$, i.e., the subgraph of $G$ with vertex set \{ $v \in V : v$ is incident with some edges in $A$ \} and edge set $A$.

For any subset $S \subseteq V(G)$ and $F \subseteq E(G)$, $F$ is called a strict star-set of $S$ in $G$ if $G[F]$ is a star and $S$ is either the vertex set of $G[F]$ or the set of all leaves in $G[F]$. We say that a subset $S$ of $V(G)$ is $k$-strict-star-connected in $G$ if there exist $k$ pairwise different subsets $F_1, F_2, \cdots, F_k \subseteq E(G)$ (possibly with common edges) such that each $F_i$ is a strict star-set of $S$ in $G$. For instance, for the complete bipartite graph $K_{m,n}$ with two partite sets $X$ and $Y$, where $|X| = m$ and $|Y| = n$, $X$ and $Y$ are $n$ and $m$-strict-star-connected in $K_{m,n}$ respectively.

When asking for the skewness of a graph, it is natural to first decompose the graph to obtain simpler instances. It is trivial that connected components can be considered separately. Regarding small edge cuts, Farr and Eades [8] presented some expressions for the skewness of a graph $G$ in terms of the skewness of various derived graphs which are obtained by cutting $G$ at a cut set consisting of at most four edges (provided one exists). Specifically, they obtained the following result which is now described in terms of Zip product.

**Lemma 3 ([8])** Let $G_1$ and $G_2$ be vertex-disjoint connected graphs. Assume that $v_i$ is a vertex in $G_i$ with $d_{G_i}(v_i) = m \leq 3$ and $N_i = N_{G_i}(v_i)$ for $i = 1, 2$. Then,

$$sk(G_1 \odot_\sigma G_2) \geq \max\{sk(G_1) + sk(G_2 - v_2), sk(G_1 - v_1) + sk(G_2)\}$$

holds for any bijection $\sigma : N_1 \rightarrow N_2$.

Unfortunately, there exist graphs violating the decomposability of Lemma 3 for $m \geq 4$ (a counterexample will be given in Remark 1(b)). We show in Theorem 1 that the property of Lemma 3 still holds under certain conditions for $m \geq 4$. 
Theorem 1 Let $G_1$ and $G_2$ be vertex-disjoint connected graphs. Assume that $d_{G_i}(v_i) = d_{G_2}(v_2)$, where $v_i$ is a vertex in $G_i$ for $i = 1, 2$. If $N_1$ is 1-strict-star-connected in $G_1 - v_1$, then

$$sk(G_1 \circ_\sigma G_2) \geq sk(G_1 - v_1) + sk(G_2)$$

for any bijection $\sigma : N_1 \rightarrow N_2$, where $N_i = N_{G_i}(v_i)$ for $i = 1, 2$.

Proof. Let $G$ denote the graph $G_1 \circ_\sigma G_2$, $N_1 = \{x_i : i = 1, 2, \ldots, m\}$, $N_2 = \{y_i : i = 1, 2, \ldots, m\}$ and let $E(N_1, N_2) = \{x_iy_i : i = 1, 2, \ldots, m\}$ be the set of edges in $G$ whose ends are in $N_1$ and $N_2$ respectively. As $N_1$ is 1-strict-star-connected in $G_1 - v_1$, there exists a star set $F_1$ of $N_1$ in $G_1 - v_1$. Let $x_0$ be the center of the star $T_1$ which is the subgraph of $G_1 - v_1$ induced by $F_1$. Note that $x_0$ may be a vertex in $N_1$.

For any $D \in \mathcal{PSK}(G)$, let $B(D) = E(D) \cap E(N_1, N_2)$. We call an edge in $B(D)$ a red edge, and a vertex, with which are incident red edges, a red vertex. $D - B(D)$ is considered as the plane drawing obtained from $D$ by removing all edges in $B(D)$. Since $G$ is connected (and also $D$), it is not difficult to prove Claim 1.

Claim 1 For any $D \in \mathcal{PSK}(G)$, each component of $D - B(D)$ contains at least one red vertex.

Proof of Claim 1. Since all edges in $B(D)$ are red edges and $D$ is connected, each component of $D - B(D)$ has some vertices incident with edges in $B(D)$. By definition, each vertex in a component of $D - B(D)$ incident with edges in $B(D)$ is a red vertex. Thus the claim holds. \qed

Let $D \in \mathcal{PSK}(G)$. For any two components $A_1, A_2$ of $D - B(D)$, $A_2$ lies in some face of $A_1$ in the plane drawing $D - B(D)$. Write $A_1 \geq_D A_2$ (or $A_2 \leq_D A_1$) if $A_2$ is within an internal face of $A_1$ in the plane drawing $D - B(D)$. Thus the relation $\geq_D$ depends on $D$. Note that $A_1 \not\geq_D A_2$ means that $A_2$ is not within an internal face of $A_1$ in the plane drawing $D - B(D)$. Clearly $A \not\geq_D A$ for any component $A$ of $D - B(D)$, $A_1 \geq_D A_2$ implies that $A_2 \not\leq_D A_1$, and $A_1 \geq_D A_2$ and $A_2 \geq_D A_3$ imply that $A_1 \geq_D A_3$.

From now on, we assume that $D$ is a member in $\mathcal{PSK}(G)$ such that $c(D - B(D)) \leq c(D' - B(D'))$ holds for all $D' \in \mathcal{PSK}(G)$, where $c(H)$ is the number of components in a graph $H$. Note that such $D$ may not be unique.

For $i = 1, 2$, let $D_i$ be the plane drawing obtained from $D$ by removing all vertices in $V_{3-i}$, where $V_i = V(G_i - v_i)$. Thus $D_i$ is actually the subgraph of $D$ induced by $V_i$. 


Clearly, the components of $D - B(D)$ consists of components of $D_1$ and components of $D_2$.

**Claim 2** $D_1$ is connected.

**Proof of Claim 2.** Suppose that $D_1$ is disconnected. Let $A$ be the component of $D_1$ which contains $x_0$. Let $A'$ be a component of $D_1$ distinct from $A$. We can choose a suitable $D \in \mathcal{PSK}(G)$ with the minimum value of $c(D - B(D))$ such that $A' \not\preceq_D A$. Thus either $A \succeq_D A'$ or $A'$ lies in the external face of $A$.

Let $S$ be the set of red vertices on the boundary of the external face of $A'$. By Claim 1, $A'$ contains at least one red vertex. As $A' \not\preceq_D A$, $S \neq \emptyset$. We now construct a plane drawing $D''$ from $D$ by the following steps:

(i) remove all red edges of $D$ which are incident with vertices in $S$ and lie in the external face of $A'$. Thus, we get a disconnected plane drawing $D'$;

(ii) let $A''$ be the subgraph of $D$ which is obtained from $D$ by removing all vertices of $D$ in the external face of $A'$. Observe that $A''$ is a component of $D'$ and recall that $A'$ is a subgraph of $D'$. By the definition of $A''$, $A'$ is a subgraph of $A''$. Move component $A''$ of $D'$ to a face of $D'$ whose boundary contains vertex $x_0$;

(iii) add new edges joining $x_0$ to all vertices in $S$ and get a plane drawing $D''$.

It is not difficult to observe that $D''$ is a plane drawing of some spanning subgraph of $G$ with

$$|E(D'')| = |E(D')| + |S| \geq |E(D)| - |S| + |S| = |E(D)|.$$ 

By the definition of $\mathcal{PSK}(G)$, we have $D'' \in \mathcal{PSK}(G)$. Thus, $|E(D'')| = |E(D)|$. However, $c(D - B(D)) = c(D'' - B(D'')) + 1$, contradicting the assumption of the minimality on $c(D - B(D))$. \qed

Now we continue to prove the Theorem. By Claim 2, $D_1$ is connected. Note that $D_2$ may be disconnected. By contracting all edges in $D_1$, one can easily get a plane drawing $P$ of some spanning subgraph of $G_2$. Observe that

$$|E(P)| = |E(D_2)| + |B(D)|.$$ 

By the definition of skewness, we have

$$sk(G_2) \leq |E(G_2)| - |E(P)| = |E(G_2)| - |E(D_2)| - |B(D)|.$$  \hspace{1cm} (3.1)
Since \( D_1 \) is a plane drawing of some spanning subgraph of \( G_1 - v_1 \), we have
\[
\text{sk}(G_1 - v_1) \leq |E(G_1 - v_1)| - |E(D_1)| = |E(G_1)| - |E(D_1)| - m. \tag{3.2}
\]

Considering that \( D \in \mathcal{PSK}(G) \), it follows that
\[
\text{sk}(G_1 \circ_\sigma G_2) = \sum_{i=1}^{2}(|E(G_i - v_i)| - |E(D_i)|) + |E(N_1, N_2)| - |B(D)|
\]
\[
= \sum_{i=1}^{2}(|E(G_i)| - |E(D_i)|) - |B(D)| - m. \tag{3.3}
\]

By (3.1), (3.2) and (3.3), we have
\[
\text{sk}(G_1 \circ_\sigma G_2) \geq \text{sk}(G_1 - v_1) + \text{sk}(G_2),
\]
as desired, and the proof is completed. \( \square \)

**Remark 1**

(a) The lower bound of Theorem 1 is tight for some families of graphs. For example, if \( G_i \) is the graph obtained from the complete graph \( K_m \) by adding an apex \( v_i \) to \( V(K_m) \) for \( i = 1, 2 \), then it follows from Theorem 1 that
\[
\text{sk}(G_1 \circ_\sigma G_2) \geq \text{sk}(G_1 - v_1) + \text{sk}(G_2) = \text{sk}(K_m) + \text{sk}(K_{m+1})
\]
for any bijection \( \sigma : N_{G_1}(v_1) \rightarrow N_{G_2}(v_2) \). Since \( G_1 \circ_\sigma G_2 \cong K_m \square T_2 \) for any bijection \( \sigma \), where \( T_2 \) is a tree of order 2, the equality holds by Theorem 2 in Section 4.

(b) The condition “1-strict-star-connected” in the theorem is necessary. For example, if \( G_1 \) is the graph obtained by adding an apex \( v_1 \) to all vertices of \( P_4 \), and \( G_2 \) is the graph obtained by adding an apex \( v_2 \) to all vertices of degree two of \( K_{2,4} \), then \( N_{G_1}(v_1) \) is not 1-strict-star-connected in \( G_1 - v_1 \) and
\[
\text{sk}(G_1 - v_1) + \text{sk}(G_2) = \text{sk}(P_4) + \text{sk}(K_{3,4}) = 2.
\]
However, the plane drawing shown in Figure 2 implies that
\[
\text{sk}(G_1 \circ_\sigma G_2) \leq 1
\]
holds for any bijection \( \sigma : N_{G_1}(v_1) \rightarrow N_{G_2}(v_2) \).

A vertex in \( G \) is called a dominating vertex in \( G \) if it is adjacent to all other vertices in \( G \).
Corollary 1 Let $G$ be a graph with a dominating vertex, and $T_n$ be a tree with $n$ vertices. Then, for $n \geq 1$,

$$sk(G \square T_n) \geq sk(G) + (n - 1)sk(G + K_1).$$

Proof. Let $G_n$ denote the graph $G \square T_n$. We denote the vertices of $T_n$ by $t_1, t_2, \ldots, t_n$. The result is trivial for $n = 1$. Suppose now that the statement is true for $n = k$ ($k \geq 1$), and we consider the case that $n = k + 1$.

Without loss of generality, assume that $t_{k+1}$ is a leaf of $T_{k+1}$ and $t_k t_{k+1} \in E(T_{k+1})$. Note that the vertex set of $G_{k+1}$ is $\{(u, t_i) : u \in V(G), i = 1, 2, \ldots, k + 1\}$ and $G_k = G \square T_k$ is the subgraph of $G_{k+1}$ induced by $\{(u, t_i) : u \in V(G), i = 1, 2, \ldots, k\}$.

Let $\hat{G}$ be the graph obtained from $G$ by adding an apex $w$ to all vertices of $G$, and let $\hat{G}_k$ be the graph obtained from $G_k$ by adding an apex $w_k$ to all vertices of $\{(u, t_k) : u \in V(G)\}$. Then, $G_{k+1} = \hat{G}_k \circ \sigma \hat{G}$, where $\sigma$ maps $(u, t_k)$ to $u$ for all $u \in V(G)$. Since $G$ has a dominating vertex, $N_{\hat{G}_k}(w_k)$ is 1-strict-star-connected in $\hat{G}_k - w_k$. Theorem 1 then implies that

$$sk(\hat{G}_k \circ \sigma \hat{G}) \geq sk(\hat{G}_k - w_k) + sk(\hat{G}) = sk(G_k) + sk(G + K_1).$$

As $G \square T_{k+1} = \hat{G}_k \circ \sigma \hat{G}$ and $G_k = G \square T_k$, we have

$$sk(G \square T_{k+1}) \geq sk(G \square T_k) + sk(G + K_1).$$

By the induction hypothesis, $sk(G \square T_k) \geq sk(G) + (k - 1)sk(G + K_1)$. So,

$$sk(G \square T_{k+1}) \geq sk(G) + (k - 1)sk(G + K_1) + sk(G + K_1) = sk(G) + k \times sk(G + K_1).$$

Thus, the result holds for $n = k + 1$. □
Remark 2  The lower bounds of Corollary 1 are tight for some families of graphs, such as complete graph $K_m$, which can be seen in Section 4.

Corollary 2  Let $G$ be a graph with a dominating vertex. Then, for $n \geq 1$,

$$sk(G \square P_n) \geq \left\lfloor \frac{n-1}{2} \right\rfloor sk(G + 2K_1) + \frac{1 + (-1)^n}{2} sk(G + K_1) + \left\lfloor \frac{n+1}{2} \right\rfloor sk(G).$$

Proof. Let $t_it_i+1 \in E(P_n)$ for all $i = 1, 2, \cdots, n-1$, where $t_1, t_2, \cdots, t_n$ are vertices of $P_n$. For any $i = 1, 2, \cdots, n$, let $P_i$ be the subgraph of $P_n$ induced by \{t_1, t_2, \cdots, t_i\}, let $H_i$ denote the graph $G \square P_i$ and let $\hat{H}_i$ be the graph obtained from $H_i$ by adding an apex $w_i$ to all vertices of $\{(u, t_i) : u \in V(G)\}$.

Claim 3  $sk(\hat{H}_2) \geq sk(G) + sk(G + 2K_1)$.

Proof of Claim 3. Let $G_1$ be the graph obtained from $G$ by adding an apex $v_1$ to all $u \in V(G)$, and let $G_2$ be the graph obtained from $G$ by adding two apices $v_2$ and $v_3$ to all $u \in V(G)$. Thus, we have $\hat{H}_2 = G_1 \circ\sigma G_2$, where $\sigma: N_{G_1}(v_1) \to N_{G_2}(v_2)$ maps a vertex $u \in N_{G_1}(v_1)$ to its counterpart in $N_{G_2}(v_2)$. As $G$ contains a dominating vertex, $N_{G_1}(v_1)$ is 1-strict-star-connected in $G_1 - v_1$. Note that $G_1 - v_1 \cong G$ and $G_2 \cong G + 2K_1$, it follows from Theorem 1 that

$$sk(\hat{H}_2) = sk(G_1 \circ\sigma G_2) \geq sk(G_1 - v_1) + sk(G_2) = sk(G) + sk(G + 2K_1),$$

as stated. \qed

Now we continue to prove Corollary 2 by induction on $n$. The case $n = 1$ is trivial. Corollary 1 implies that the result also holds for the case $n = 2$. Suppose now that the corollary is true for $n \leq k$ ($k \geq 2$), and we consider the case that $n = k + 1$.

It is not difficult to verify that $H_{k+1} = \hat{H}_{k-1} \circ\sigma \hat{H}_2$, where $\sigma$ is the bijection from $N_1 = N_{\hat{H}_{k-1}}(w_{k-1})$ to $N_2 = N_{\hat{H}_2}(w_2)$ defined by $\sigma((u, t_{k-1})) = (u, t_2)$ for all $u \in V(G)$. Note that $N_1$ is 1-strict-star-connected in $\hat{H}_{k-1} - w_{k-1}$. Thus, it follows from Theorem 1 and Claim 3 that

$$sk(H_{k+1}) = sk(\hat{H}_{k-1} \circ\sigma \hat{H}_2) \geq sk(\hat{H}_{k-1} - w_{k-1}) + sk(\hat{H}_2) \geq sk(H_{k-1}) + sk(G + 2K_1) + sk(G).$$
By the induction hypothesis, we have
\[ sk(H_k) \geq \left\lfloor \frac{k-2}{2} \right\rfloor sk(G + 2K_1) + \frac{1 + (-1)^{k-1}}{2} sk(G + K_1) + \left\lfloor \frac{k}{2} \right\rfloor sk(G). \]
So, the above inequalities imply that
\[ sk(H_{k+1}) \geq \left\lfloor \frac{k-2}{2} \right\rfloor sk(G + 2K_1) + \frac{1 + (-1)^{k-1}}{2} sk(G + K_1) + \left\lfloor \frac{k}{2} \right\rfloor sk(G) + sk(G + 2K_1) + sk(G) \]
\[ = \left\lfloor \frac{k}{2} \right\rfloor sk(G + 2K_1) + \frac{1 + (-1)^{k+1}}{2} sk(G + K_1) + \left\lfloor \frac{k+2}{2} \right\rfloor sk(G). \]
Thus, the result holds for \( n = k + 1 \).

**Remark 3** The lower bound of Corollary 2 is tight when \( G \) is a graph in some families of graphs such as star \( S_m \) and wheel \( W_m \). This will be given in Section 5 and 6, respectively.

### 4 \( sk(K_m \Box T_n) \)

In this section, we give an explicit expression for \( sk(K_m \Box T_n) \), where \( T_n \) is any tree of order \( n \), under the condition that the maximum degree of \( T_n \) is not more than \( 2m - 4 \). It is not difficult to verify that \( sk(K_m \Box T_n) = 0 \) when \( m \leq 2 \).

Let \( F(D) \) denote the set of faces of a plane drawing \( D \). For each face \( f \in F(D) \), its **degree** in \( D \), denoted by \( d_D(f) \), is defined to be the number of edges in \( D \) which are on the boundary of \( f \), where each bridge of \( D \) on the boundary of \( f \) is counted twice. Recall that \( \mathcal{PSK}(G) \) denotes the family of plane drawings of spanning subgraphs of \( G \) which are planar and of size \( |E(G)| - sk(G) \). Thus, \( |E(D)| = |E(G)| - sk(G) \) for each \( D \in \mathcal{PSK}(G) \).

**Theorem 2** Let \( m \) be an integer with \( m \geq 3 \) and let \( T_n \) be a tree with \( n \) vertices and maximum degree \( \Delta \leq 2m - 4 \). Then,
\[ sk(K_m \Box T_n) = (n - 1) \binom{m-2}{2} + \binom{m-3}{2}. \]

**Proof.** Let \( D \in \mathcal{PSK}(K_m) \). Since \( sk(K_m) = \binom{m-3}{2} \) by Lemma 1, we have \( |E(D)| = |E(K_m)| - sk(K_m) = 3m - 6 \). Euler’s polyhedron formula implies that \( |F(D)| = 2m - 4 \). Thus, it follows that
\[ 6m - 12 = 3|F(D)| \leq \sum_{f \in F(D)} d_D(f) = 2|E(D)| = 6m - 12, \]
which further implies that \( d_D(f) = 3 \) for each face \( f \) in \( D \). We denote the faces of \( D \) by \( f_1, f_2, \ldots, f_{2m-4} \). Let \( t_1, t_2, \ldots, t_n \) be an ordering of vertices of \( T_n \) by a depth-first search. Let \( l_i = d_{T_n}(t_i) \) for \( 1 \leq i \leq n \) and let \( e_i = t_i t_j \) for \( 2 \leq i \leq n \), where \( j_i \in \{1, 2, \ldots, i-1\} \).

By the given condition, \( \Delta = \max_{1 \leq i \leq n} l_i \leq 2m - 4 \). As \( T \) is a tree, \( E(T) \) can be partitioned into at most \( \Delta \) subsets which are matchings of \( T \). Thus there exists a mapping \( \psi : E(T_n) \rightarrow \{1, 2, \ldots, \Delta\} \) such that \( \psi(e_i) \neq \psi(e_j) \) whenever edges \( e_i \) and \( e_j \) are incident with a common vertex.

Using the above setup, we construct a plane drawing of some spanning subgraph of \( K_m \square T_n \). For \( i = 1, 2, \ldots, n \), let \( D_i \) be a copy of \( D \) such that when \( i \geq 2 \), the external face of \( D_i \) is face \( f_{\psi(e_i)} \). Let \( \Theta_1 = D_1 \) and for \( i = 2, 3, \ldots, n \), let \( \Theta_i \) be the plane drawing obtained from \( \Theta_{i-1} \) and \( D_i \) by the two steps below:

(i) inserting \( D_i \) into face \( f_{\psi(e_i)} \) of \( D_j \) in \( \Theta_{i-1} \);

(ii) for each vertex \( u \) on the boundary of the external face of \( D_i \), adding an edge joining \( u \) to its counterpart on the boundary of face \( f_{\psi(e_i)} \) of \( D_j \) in \( \Theta_{i-1} \).

As each face of \( D \) is a triangle, we have \(|E(\Theta_n)| = n|E(D)| + 3(n-1)\). Since \( \Theta_n \) is a plane drawing of some spanning subgraph of \( K_m \square T_n \),

\[
sk(K_m \square T_n) \leq |E(K_m \square T_n)| - |E(\Theta_n)| \\
= n|E(K_m)| + m(n-1) - n|E(D)| - 3(n-1) \\
\leq n \times sk(K_m) + (n-1)(m-3) \\
= n \left( \binom{m-3}{2} \right) + (n-1)(m-3) \\
= (n-1) \left( \binom{m-2}{2} \right) + \left( \binom{m-3}{2} \right).
\]

On the other hand, since \( K_m \) contains a dominating vertex and \( K_m + K_1 \cong K_{m+1} \), Corollary 1 implies that

\[
sk(K_m \square T_n) \geq sk(K_m) + (n-1)sk(K_{m+1}) \\
= (n-1) \left( \binom{m-2}{2} \right) + \left( \binom{m-3}{2} \right).
\]

This completes the proof. \( \square \)
In this section, we give an explicit expression for \( sk(S_m \square P_n) \). Clearly \( sk(S_m \square P_n) = 0 \) when \( m \leq 1 \).

**Theorem 3** For \( m \geq 2 \) and \( n \geq 1 \),

\[
sk(S_m \square P_n) = (m - 2) \left\lfloor \frac{n - 1}{2} \right\rfloor.
\]

**Proof.** Let \( V(S_m \square P_n) = \bigcup_{i=1}^{n} \{ x_j^i : 0 \leq j \leq m \} \) and,

\[
E(S_m \square P_n) = \left( \bigcup_{i=1}^{n} \{ x_j^i : 1 \leq j \leq m \} \right) \cup \left( \bigcup_{i=0}^{m} \{ x_j^i x_j^{i+1} : 1 \leq i \leq n - 1 \} \right).
\]

Let \( SP_{m,n} \) be the spanning subgraph of \( S_m \square P_n \) with edge set

\[
E(SP_{m,n}) = \left( \bigcup_{i=1}^{n} \{ x_j^i x_j^{i+1} : 1 \leq j \leq m \} \right)
\]

\[
\cup \left( \bigcup_{j=0}^{m} \{ x_j^i x_j^{i+1} : 1 \leq i \leq n - 1 \text{ and } i \equiv 1 (\text{mod } 2) \} \right)
\]

\[
\cup \left( \bigcup_{j=0}^{2} \{ x_j^i x_j^{i+1} : 1 \leq i \leq n - 1 \text{ and } i \equiv 0 (\text{mod } 2) \} \right).
\]

Figure 3 illustrates a plane drawing of the graph \( SP_{m,6} \). This drawing can be naturally extended to a plane drawing of \( SP_{m,n} \) for any \( n \geq 2 \). Thus, it follows from the definition of skewness that

\[
sk(S_m \square P_n) \leq |E(S_m \square P_n)| - |E(SP_{m,n})|
\]

\[
= |E(S_m \square P_n)| - (|E(S_m \square P_n)| - (m - 2) \left\lfloor \frac{n - 1}{2} \right\rfloor)
\]

\[
= (m - 2) \left\lfloor \frac{n - 1}{2} \right\rfloor.
\]

On the other hand, \( S_m + 2K_1 \) contains \( K_{3,m} \) as a subgraph, which implies that \( sk(S_m + 2K_1) \geq sk(K_{3,m}) = m - 2 \). Thus, it follows from Corollary 2 that

\[
sk(S_m \square P_n) \geq (m - 2) \left\lfloor \frac{n - 1}{2} \right\rfloor.
\]

Therefore, the proof is completed.

**Remark 4** By Theorem 3, we know that \( S_m \square P_n \) is planar if and only if \( m \leq 2 \) or \( n \leq 2 \).
Figure 3: A plane drawing of the graph $SP_{m,6}$ if the dashed edges are ignored. The vertical small vertices refer to arbitrary number of vertices.

6 $sk(W_m \Box P_n)$

We first determine $sk(C_m + P_n)$, and then apply this result to determine $sk(W_m \Box P_n)$.

Lemma 4 For $m \geq 3$ and $n \geq 2$,

$$sk(C_m + P_n) = (m - 2)(n - 2) + 1.$$ 

Proof. Figure 4 illustrates a plane triangulation $D$ of some spanning subgraph of $C_m + P_n$, which implies that $2|E(D)| = 3|F(D)|$. By Euler's polyhedron formula, $|E(D)| = 3|V(C_m + P_n)| - 6$. Thus, it follows that

$$sk(C_m + P_n) \leq |E(C_m + P_n)| - |E(D)| = |E(C_m + P_n)| - 3|V(C_m + P_n)| + 6 = (m - 2)(n - 2) + 1.$$ 

On the other hand, by Lemma 2,

$$sk(C_m + P_n) \geq |E(C_m + P_n)| - 3|V(C_m + P_n)| + 6 = (m - 2)(n - 2) + 1,$$

which implies the claim. $\square$

Theorem 4 For $m \geq 3$ and $n \geq 1$,

$$sk(W_m \Box P_n) = (m - 2)\left\lfloor \frac{n - 1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.$$
Figure 4: A plane triangulation of some spanning subgraph of $C_m + P_n$ if the dashed edges are ignored. The vertical small circle vertices and level small square vertices refer to arbitrary number of vertices.

**Proof.** The vertex set and edge set of $W_m \square P_n$ can be expressed as

\[
V(W_m \square P_n) = \bigcup_{i=1}^{n} \{x_i^j : 0 \leq j \leq m\}
\]

and

\[
E(W_m \square P_n) = \left( \bigcup_{i=1}^{n} \{x_i^0 x_i^j, x_j^i x_{j+1}^i : 1 \leq j \leq m\} \right) \cup \left( \bigcup_{j=0}^{m} \{x_j^i x_{j+1}^i : 1 \leq i \leq n - 1\} \right),
\]

where $x_{m+1}^i$ represents the vertex $x_i^1$. Let $WP_{m,n}$ be the spanning subgraph of $W_m \square P_n$ with edge set

\[
E(WP_{m,n}) = \left( \bigcup_{i=1}^{n} \{x_i^0 x_i^j, x_j^i x_{j+1}^i : 1 \leq j \leq m\} \right) \cup \left( \bigcup_{j=1}^{m} \{x_j^i x_{j+1}^i : 1 \leq i \leq n - 1 \text{ and } i \equiv 1(\text{mod } 2)\} \right) \cup \left( \bigcup_{j=0}^{2} \{x_j^i x_{j+1}^i : 1 \leq i \leq n - 1 \text{ and } i \equiv 0(\text{mod } 2)\} \right).
\]

Figure 5 illustrates a plane drawing of the graph $WP_{m,6}$. This plane drawing can be naturally extended to a plane drawing of the graph $WP_{m,n}$ for $n \geq 2$. Thus, it follows from the definition of skewness that

\[
sk(W_m \square P_n) \leq |E(W_m \square P_n)| - |E(WP_{m,n})|
\]

\[
= |E(W_m \square P_n)| - (|E(W_m \square P_n)| - (m - 2)\left[\frac{n-1}{2}\right] - \left\lfloor\frac{n}{2}\right\rfloor)
\]

\[
= (m - 2)\left[\frac{n-1}{2}\right] + \left\lfloor\frac{n}{2}\right\rfloor,
\]

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and that the theorem is true if the equality holds.

It is not difficult to verify that \( W_m + K_1 \cong C_m + P_2 \) and \( W_m + 2K_1 \cong C_m + P_3 \),
and therefore it follows from Lemma 4 that \( sk(W_m + K_1) = sk(C_m + P_2) = 1 \) and
\( sk(W_m + 2K_1) = sk(C_m + P_3) = m - 1 \). Corollary 2 implies that
\[
sk(W_m \Box P_n) \geq \left\lceil \frac{n-1}{2} \right\rceil sk(W_m + 2K_1) + \frac{1 + (-1)^n}{2} sk(W_m + K_1)
\]
\[
= \left\lceil \frac{n-1}{2} \right\rceil (m-1) + \frac{1 + (-1)^n}{2}
\]
\[
= (m-2)\left\lceil \frac{n-1}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil .
\]
Therefore, the proof is done. \( \Box \)

7 Conclusion

By definition, \( sk(G) \leq cr(G) \) holds for any connected graph \( G \). In general, it is
unknown how large is the gap between \( sk(G) \) and \( cr(G) \) for a graph \( G \). Now we can
compare these two numbers for the two families of graphs \( \{S_m \Box P_n : m \geq 1, n \geq 1\} \)
and \( \{W_m \Box P_n : m \geq 3, n \geq 1\} \). Bokal [2, 3] showed that
\[
cr(S_m \Box P_n) = (n-2)\left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{m-1}{2} \right\rceil , \quad n \geq 2, m \geq 1
\]
and
\[
cr(W_m \Box P_n) = (n-2)(\left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{m-1}{2} \right\rceil + 1) + 2, \quad n \geq 2, m \geq 3.
\]
By Theorems 3 and 4, we have
\[
\lim_{m \to \infty} sk(S_m \Box P_n)/cr(S_m \Box P_n) = \lim_{m \to \infty} sk(W_m \Box P_n)/cr(W_m \Box P_n) = 0,
\]
\[
\lim_{n \to \infty} \frac{sk(S_m \Box P_n)}{cr(S_m \Box P_n)} = \frac{(m - 2)}{2\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m - 1}{2} \right\rfloor},
\]
and
\[
\lim_{n \to \infty} \frac{sk(W_m \Box P_n)}{cr(W_m \Box P_n)} = \frac{(m - 1)}{2\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m - 1}{2} \right\rfloor + 2}.
\]

Note that the graph \(S_m \Box P_n\) is of order \((m + 1)n\), and so
\[
\lim_{n \to \infty} \frac{cr(S_m \Box P_n) - sk(S_m \Box P_n)}{|V(S_m \Box P_n)|} = \frac{2\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m - 1}{2} \right\rfloor - (m - 2)}{2(m + 1)}.
\]

Thus \(cr(G) - sk(G)\) can be a very large number, and it is even possible that \(cr(G) - sk(G) \geq |V(G)|\).

The two numbers \(sk(S_m \Box P_n)\) and \(sk(W_m \Box P_n)\) are determined in Theorems 3 and 4 respectively by applying Theorem 1 and its corollaries. We wonder if \(sk(S_m \Box T_n)\) and \(sk(W_m \Box T_n)\) can be determined by the same approach, where \(T_n\) is a tree of order \(n\). Applying Corollary 1, we can first get lower bounds for these two numbers:
\[
sk(S_m \Box T_n) \geq sk(S_m) + (n - 1)sk(S_m + K_1) = 0
\]
and
\[
sk(W_m \Box T_n) \geq sk(W_m) + (n - 1)sk(W_m + K_1) = (n - 1).
\]

Clearly, both results are not helpful for finding explicit expressions of \(sk(S_m \Box T_n)\) and \(sk(W_m \Box T_n)\). It seems that these two numbers depend on the structure of \(T_n\), not just on \(n\). For example, if \(T_n\) is a star, then \(sk(S_m \Box T_n)\) and \(sk(W_m \Box T_n)\) may not be equal to \(sk(S_m \Box P_n)\) and \(sk(W_m \Box P_n)\) respectively.

Note that the basic condition for applying Theorem 1 and its corollaries to determine \(sk(K_m \Box T_n)\), \(sk(S_m \Box P_n)\) and \(sk(W_m \Box P_n)\) is that \(K_m\), \(S_m\) and \(W_m\) all contain dominating vertices. Can this condition be replaced by a weaker one so that \(sk(G \Box P_n)\) or even \(sk(G \Box T_n)\) can also be determined for a graph \(G\) satisfying this weaker condition?

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