On topological Rudin's Lemma, well-filtered spaces and sober spaces*

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Abstract

Based on the topological Rudin's Lemma, we introduce the notions of Rudin set and well-filtered determined set in a topological space. Using such sets, we formulate and prove some new characterizations of well-filtered spaces and sober spaces. Part of the work was inspired by Xi and Lawson's work on well-filtered spaces. Our study also lead to the definition of a new class of spaces - the strong d-spaces, and some problems whose solutions will strengthen the understanding of the related structures.

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In domain theory, the d-spaces, well-filtered spaces and sober spaces form three of the most important classes of topological spaces. Rudin's Lemma has played a crucial role in studying such spaces. The original application of Rudin's Lemma was in answering some questions on quasicontinuous dcpos. In recent years, however, it has been used to study the various aspects of well-filtered spaces, initiated by Heckmann and Keimel [7]. In this paper, inspired by the topological version of Rudin's Lemma by Heckmann and Keimel, Xi and Lawson's work [16] on well-filtered spaces and our recent work [14] on the well-filtered reflections, we introduce the notions of Rudin set and well-filtered determined set of topological spaces, and use them to establish a series new characterizations of well-filtered spaces and sober spaces. Our study also leads to the definition of new class of spaces - the strong d-spaces, and a number of problems, whose answering will deepen our understanding of the related spaces and structures.

1. Preliminary

For a poset P and $A \subseteq P$, let $\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\}$ and $\uparrow A = \{x \in P : x \geq a \text{ for some } a \in A\}$. For $x \in P$, we write $\downarrow x$ for $\downarrow\{x\}$ and $\uparrow x$ for $\uparrow\{x\}$. A subset A is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). Let $\mathcal{S}^u(X) = \{\uparrow x : x \in X\}$. Define $A^\uparrow = \{x \in P : x \text{ is an upper bound of } A \text{ in } P\}$. Dually, define $A^\downarrow = \{x \in P : x \text{ is a lower bound of A in P}\}$. The set $A^\delta = (A^\uparrow)^\downarrow$ is called the *cut* generated by A. The upper sets form the (*upper*) *Alexandroff topology* $\alpha(P)$. The set of all nonempty upper subsets of P is denoted by $\mathbf{up}(P)$. Let $P^{(<\omega)} = \{F \subseteq P : F \text{ is a nonempty finite set}\}$, $P^{(\leqslant\omega)} = \{F \subseteq P : F \text{ is a nonempty countabel set}\}$ and $\mathbf{Fin}\ P = \{\uparrow F : F \in P^{(<\omega)}\}$. As in [4], the *lower topology*, generated by the complements of the principal filters, is denoted by $\omega(P)$. Dually, one defines the *upper topology* on P, denoted by $\upsilon(P)$.

A nonempty subset D of P is *directed* if every two elements in D have an upper bound in D. The set of all directed sets of P is denoted by $\mathcal{D}(P)$. A subset U of P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a

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topology on P, called the *Scott topology* on P and denoted by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P. The common refinement of $\sigma(P)$ and $\omega(P)$ is called the *Lawson topology* and is denoted by $\lambda(P)$.

A poset P is called an (inf) *semilattice* if for any two elements $a, b \in P$, $inf\{a, b\} = a \wedge b$ exists in P. Dually, P is a *sup semilattice* if for any two elements $a, b \in P$, $sup\{a, b\} = a \vee b$ exists in P. A subset $I \subseteq P$ is called an *ideal* of P if I is a directed and a lower set. Let $\mathrm{Id}(P)$ be the poset (with the set inclusion order) of all ideals of P. Dually, we define the *filters* and denote the poset of all filters of P by $\mathrm{Filt}(P)$. A poset P is called a *directed complete poset*, or *dcpo* for short, if $\bigvee D$ exists in P for any $D \in \mathcal{D}(P)$. For a nonempty subset A of P, let $max(A) = \{a \in A : a \text{ is a maximal element of } A\}$ and $min(A) = \{a \in A : a \text{ is a minimal element of } A\}$.

Definition 1.1. Let P be a poset.

(i) P is called a *complete semilattice* if P is a dcpo and every nonempty subset P has an inf. In particular, a complete semilattice has a smallest element, the infimum of P.

(ii) P is called *bounded complete* if every subset that is bounded above has a sup (i.e., the least upper bound). In particular, a bounded complete poset has a smallest element, the least upper bound of the empty set.

It is easy to see that P is a complete semilattice iff P is a bounded complete dcpo (see, e,g., [4, Proposition O-2.2]).

For a T_0 space X, the *specialization order* $\leq_X$ on X is defined by $x \leq_X y$ iff $x \in \overline{\{y\}}$. In the following, when a T_0 space X is considered as a poset, the order always refers to the specialization order if no other interpretation given. Let $\mathcal{O}(X)$ (resp., $\mathcal{C}(X)$) be the set of all open subsets (resp., closed subsets) of X, and let $\mathcal{S}_c(X) = \{\overline{\{x\}} : x \in X\}$ and $\mathcal{D}_c(X) = \{\overline{D} : D \in \mathcal{D}(X)\}$. A space X is called a *d-space* (or *monotone convergence space*) if X (endowed with the specialization order) is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$ (cf. [4][15]).

Proposition 1.2. *For a T_0 space, the following conditions are equivalent:*

1. *X is a d-space.*
2. *$\mathcal{D}_c(X) = \mathcal{S}_c(X)$.*
3. *For any $D \in \mathcal{D}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{d\in D} \uparrow d \subseteq U$ implies $\uparrow d \subseteq U$ (i.e., $d \in U$) for some $d \in D$.*
4. *For any $D \in \mathcal{D}(X)$ and $A \in \mathcal{C}(X)$, if $D \subseteq A$, then $A \cap \bigcap_{d\in D} \uparrow d \neq \emptyset$.*
5. *For any $D \in \mathcal{D}(X)$, $\overline{D} \cap \bigcap_{d\in D} \uparrow d \neq \emptyset$.*

Proof. (1) $\Leftrightarrow$ (2): If X is a d-space, then for any $D \in \mathcal{D}(X)$, $\overline{D} = \overline{\{sup\, D\}}$, thus (1) $\Rightarrow$ (2). Conversely, if condition (2) holds, then for each $D \in \mathcal{D}(X)$ and $A \in \mathcal{C}(X)$ with $D \subseteq A$, there is $x \in X$ such that $\overline{D} = \overline{\{x\}}$, and consequently, $\bigvee D = x$ and $\bigvee D \in A$ since $\overline{D} \subseteq A$. Thus X is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$, hence X is a d-space.

(1) $\Rightarrow$ (3): Since X is a d-space, $\uparrow \bigvee D = \bigcap_{d\in D} \uparrow d \subseteq U \in \sigma(X)$. Therefore, $\bigvee D \in U$, thus $d \in U$ for some $d \in D$.

(3) $\Rightarrow$ (4): If $A \cap \bigcap_{d\in D} \uparrow d = \emptyset$, then $\bigcap_{d\in D} \uparrow d \subseteq X \setminus A$. By condition (3), $\uparrow d \subseteq X \setminus A$ for some $d \in D$, which contradicts $D \subseteq A$.

(4) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (1): For each $D \in \mathcal{D}(X)$ and $A \in \mathcal{C}(X)$ with $D \subseteq A$, by condition (5), $\overline{D} \cap \bigcap_{d\in D} \uparrow d \neq \emptyset$. Choose one $x \in \overline{D} \cap \bigcap_{d\in D} \uparrow d$. Then $D \subseteq \downarrow x \subseteq \overline{D}$, hence $\overline{D} = \downarrow x$ and $\bigvee D = x$. Therefore, $\bigvee D \in A$ because $\overline{\{\bigvee D\}} = \overline{D} \subseteq A$. Thus X is a d-space. □

Lemma 1.3. *If X is a d-space and A is a nonempty closed subset of X, then $max(A) \neq \emptyset$.*

Proof. By Zorn's Lemma there is a maximal chain C in A. Since X is a d-space, $c = \bigvee C$ exists and $c \in A$. By the maximality of C, we have $c \in max(A)$. □

A nonempty subset A of a T_0 space X is *irreducible* if for any $\{F_1, F_2\} \subseteq \mathcal{C}(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. Denote by $\mathsf{Irr}(X)$ (resp., $\mathsf{Irr}_c(X)$) the set of all irreducible (resp., irreducible closed) subsets of X. Every subset of X that is directed under $\leq_X$ is irreducible. A space X is called *sober*, if for any $F \in \mathsf{Irr}_c(X)$, there is a unique point $a \in X$ such that $F = \overline{\{a\}}$.

The following two lemmas on irreducible sets are well known that will be used in the sequel.

Lemma 1.4. *Let X be a space and Y a subspace of X. Then the following conditions are equivalent for a subset $A \subseteq Y$:*

(1) *A is an irreducible subset of Y.*
(2) *A is an irreducible subset of X.*
(3) *$\mathrm{cl}_X A$ is an irreducible subset of X.*

Lemma 1.5. *If $f : X \longrightarrow Y$ is continuous and $A \in \mathsf{Irr}(X)$, then $f(A) \in \mathsf{Irr}(Y)$.*

Remark 1.6. If Y is a subspace of a space X and $A \subseteq Y$, then by Lemma 1.4, $\mathsf{Irr}(Y) = \{B \in \mathsf{Irr}(X) : B \subseteq Y\} \subseteq \mathsf{Irr}(X)$ and $\mathsf{Irr}_c(Y) = \{B \in \mathsf{Irr}(X) : B \in \mathcal{C}(Y)\} \subseteq \mathsf{Irr}(X)$. If $Y \in \mathcal{C}(X)$, then $\mathsf{Irr}_c(Y) = \{C \in \mathsf{Irr}_c(X) : C \subseteq Y\} \subseteq \mathsf{Irr}_c(X)$.

For any topological space X, $\mathcal{G} \subseteq 2^X$ and $A \subseteq X$, let $\Diamond_{\mathcal{G}} A = \{G \in \mathcal{G} : G \bigcap A \neq \emptyset\}$ and $\Box_{\mathcal{G}} A = \{G \in \mathcal{G} : G \subseteq A\}$. The sets $\Diamond_{\mathcal{G}} A$ and $\Box_{\mathcal{G}} A$ will be simply written as $\Diamond A$ and $\Box A$, respectively, if there is no confusion. The *upper Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Box_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ as a base and the resulting space is denoted by $P_S(\mathcal{G})$. The *lower Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Diamond U : U \in \mathcal{O}(X)\}$ as a subbase and the resulting space is denoted by $P_H(\mathcal{G})$. If $\mathcal{G} \subseteq \mathsf{Irr}(X)$, then $\{\Diamond_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ is a topology on $\mathcal{G}$. The space $P_H(\mathcal{C}(X) \setminus \{\emptyset\})$ is called the *Hoare power space* or *lower space* of X and is denoted by $P_H(X)$ for short (cf. [13]). Clearly, $P_H(X) = (\mathcal{C}(X) \setminus \{\emptyset\}, \upsilon(\mathcal{C}(X) \setminus \{\emptyset\}))$. So $P_H(X)$ is always sober (see, e.g., [17, Corollary 4.10]). The space $P_H(\mathsf{Irr}_c(X))$, shortly denoted by X^s, with the topological embedding $\eta_X (= x \mapsto \overline{\{x\}}) : X \longrightarrow P_H(\mathsf{Irr}_c(X))$, is the *canonical soberification* of X (cf. [4]).

A subset A of a space X is called *saturated* if A equals the intersection of all open sets containing it (equivalently, A is an upper set with respect to the specialization order). We shall use $\mathsf{K}(X)$ to denote the set of all nonempty compact saturated subsets of X. A space X is called *coherent* if the intersection of any two compact saturated sets is again compact, and *well-filtered* if it is T_0, and for any open set U and filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$. The space $P_S(\mathsf{K}(X))$, denoted shortly by $P_S(X)$, is called the *Smyth power space* or *upper space* of X (cf. [6][13]). The space $P_S(\mathbf{up}(X))$ is called the *Alexandroff power space*. It is easy to see that $P_S(X)$ is a subspace of $P_S(\mathbf{up}(X))$, and the specialization orders on $P_S(\mathbf{up}(X))$ is the *Smyth preorder*, that is, for $K_1, K_2 \in \mathbf{up}(X)$, $K_1 \leq_{P_S(\mathbf{up}(X))} K_2$ iff $K_2 \subseteq K_1$. The *canonical mapping* $\xi_X : X \longrightarrow P_S(X)$, $x \mapsto \uparrow x$, is an order and topological embedding (cf. [6][7][13]). Clearly, $P_S(\mathcal{S}^u(X))$ is a subspace of $P_S(X)$ and X is homeomorphic to $P_S(\mathcal{S}^u(X))$.

By Lemma 1.4 and Lemma 1.5, we have the following corollary.

Corollary 1.7. *Let X be a T_0 space and $\mathcal{A} \subseteq \mathsf{K}(X)$. Then the following three conditions are equivalent:*

(1) *$\mathcal{A} \in \mathsf{Irr}(P_S(X))$.*
(2) *$\mathcal{A} \in \mathsf{Irr}(P_S(\mathbf{up}(X)))$.*
(3) *$\mathrm{cl}_{P_S(\mathbf{up}(X))} \mathcal{A} \in \mathsf{Irr}_c(P_S(\mathbf{up}(X)))$.*

Remark 1.8. Let X be a T_0 space and $\mathcal{A} \subseteq \mathsf{K}(X)$ (resp., $\mathcal{A} \subseteq \mathbf{up}(X)$). Then $\bigcap \mathcal{A} = \bigcap \overline{\mathcal{A}}$, here the closure of $\mathcal{A}$ is taken in $P_S(X)$ (resp., in $P_S(\mathbf{up}(X))$). Clearly, $\bigcap \overline{\mathcal{A}} \subseteq \bigcap \mathcal{A}$. On the other hand, for $K \in \overline{\mathcal{A}}$ and $U \in \mathcal{O}(X)$ with $K \subseteq U$ (that is, $K \in \Box U$), we have $\mathcal{A} \bigcap \Box U \neq \emptyset$, and hence there is a $K_U \in \mathcal{A} \bigcap \Box U$. Therefore $K = \bigcap \{U \in \mathcal{O}(X) : K \subseteq U\} \supseteq \bigcap \{K_U : U \in \mathcal{O}(X) \text{ and } K \subseteq U\} \supseteq \bigcap \mathcal{A}$. It follows that $\bigcap \overline{\mathcal{A}} \supseteq \bigcap \mathcal{A}$. Thus $\bigcap \mathcal{A} = \bigcap \overline{\mathcal{A}}$.

2. Rudin sets and well-filtered determined sets

Rudin's Lemma plays very important roles in domain theory (see [1-7]). Rudin [12] proved her lemma by transfinite methods. Later, Heckmann and Keimel [7] established the following topological variant of Rudin's Lemma.

Lemma 2.1. (Topological Rudin's Lemma) *Let X be a topological space and $\mathcal{A}$ an irreducible subset of the Smyth power space $P_S(X)$. Then every closed set $C \subseteq X$ that meets all members of $\mathcal{A}$ contains an minimal irreducible closed subset A that meets all members of $\mathcal{A}$.*

Applying Lemma 2.1 to the Alexandroff topology on a poset P, one obtains the original Rudin's Lemma.

Corollary 2.2. (Rudin's Lemma) *Let P be a poset, C a nonempty lower subset of P and $\mathcal{F} \in$ **Fin** P a filtered family with $\mathcal{F} \subseteq \Diamond C$. Then there exists a directed subset D of C such that $\mathcal{F} \subseteq \Diamond{\downarrow} D$.*

For a T_0 space X and $\mathcal{K} \subseteq \mathsf{K}(X)$, let $M(\mathcal{K}) = \{A \in \mathcal{C}(X) : K \bigcap A \neq \emptyset$ for all $K \in \mathcal{K}\}$ (that is, $\mathcal{A} \subseteq \Diamond A$) and $m(\mathcal{K}) = \{A \in \mathcal{C}(X) : A$ is a minimal menber of $M(\mathcal{K})\}$.

Definition 2.3. Let X be a T_0 space. A nonempty subset A of X is said to have the *Rudin property*, if there exists a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$ (that is, $\overline{A}$ is a minimal closed set that intersects all members of $\mathcal{K}$). Let $\mathsf{RD}(X) = \{A \in \mathcal{C}(X) : A$ has Rudin property$\}$.

The sets in $\mathsf{RD}(X)$ will also be called *Rudin sets*.

Proposition 2.4. *Let X be a T_0 space and Y a well-filtered space. If $f : X \longrightarrow Y$ is continuous and $A \subseteq X$ is a Rudin set, then there exists a unique $y_A \in X$ such that $\overline{f(A)} = \overline{\{y_A\}}$.*

Proof. Since A has Rudin property, there exists a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$. Let $\mathcal{K}_f = \{\uparrow f(K \cap \overline{A}) : K \in \mathcal{K}\}$. Then $\mathcal{F}_f \subseteq \mathsf{K}(Y)$ is filtered. By the proof of Lemma 2.7, $\overline{f(A)} \in m(\mathcal{K}_f)$. By the well-filteredness of Y, $\bigcap_{K\in\mathcal{K}} \uparrow f(K \cap \overline{A}) \cap \overline{f(A)} \neq \emptyset$. Select a $y_A \in \bigcap_{K\in\mathcal{K}} \uparrow f(K \cap \overline{A}) \cap \overline{f(A)}$, then $\overline{\{y_A\}} \subseteq \overline{f(A)}$ and $K \cap \overline{A} \cap f^{-1}(\overline{\{y_A\}}) \neq \emptyset$ for all $K \in \mathcal{K}$. It follows that $\overline{A} = \overline{A} \cap f^{-1}(\overline{\{y_A\}})$ by the minimality of $\overline{A}$, and consequently, $\overline{f(A)} \subseteq \overline{\{y_A\}}$. Therefore, $\overline{f(A)} = \overline{\{y_A\}}$. The uniqueness of y_A follows from the T_0 separation of Y. $\square$

Definition 2.5. A subset A of a space X is called a *well-filtered determined set* if for any continuous mapping $f : X \longrightarrow Y$ into a well-filtered space Y, there exists a unique $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Denote by $\mathsf{WD}(X)$ the set of all closed well-filtered determined subsets of X.

Obviously, a subset A of a space X is well-filtered determined iff $\overline{A}$ is well-filtered determined.

Proposition 2.6. *Let X be a T_0 space. Then $S_c(X) \subseteq \mathcal{D}_c(X) \subseteq \mathsf{RD}(X) \subseteq \mathsf{WD}(X) \subseteq \mathsf{Irr}_c(X)$.*

Proof. Obviously, $S_c(X) \subseteq \mathcal{D}_c(X)$. Now we prove that the closure of a directed subset D of X is a Rudin set. Let $\mathcal{K}_D = \{\uparrow d : d \in D\}$. Then $\mathcal{K}_D \subseteq \mathsf{K}(X)$ is filtered and $\overline{D} \in M(\mathcal{K}_D)$. If $A \in M(\mathcal{K}_D)$, then $d \in A$ for every $d \in D$, hence $\overline{D} \subseteq A$. So $\overline{D} \in m(\mathcal{K}_D)$. Therefore $\overline{D} \in \mathsf{RD}(X)$. By Proposition 2.4, we have $\mathsf{RD}(X) \subseteq \mathsf{WD}(X)$. Finally, We show $\mathsf{WD}(X) \subseteq \mathsf{Irr}_c(X)$. Let $A \in \mathsf{WD}(X)$. Since $\eta_X : X \longrightarrow X^s$, $x \mapsto \overline{\downarrow x}$, is a continuous mapping into a well-filtered space (X^s is sober), there exists $C \in \mathsf{Irr}_c(X)$ such that $\overline{\eta_X(A)} = \overline{\{C\}}$. Let $U \in \mathcal{O}(X)$. Note that

$$\begin{aligned} A \cap U \neq \emptyset \quad &\Leftrightarrow \quad \eta_X(A) \cap \Diamond U \neq \emptyset \\ &\Leftrightarrow \quad \{C\} \cap \Diamond U \neq \emptyset \\ &\Leftrightarrow \quad C \in \Diamond U \\ &\Leftrightarrow \quad C \cap U \neq \emptyset. \end{aligned}$$

This implies that $A = C$, so $A \in \mathsf{Irr}_c(X)$. $\square$

Lemma 2.7. *Let X, Y be two T_0 spaces and $f : X \longrightarrow Y$ a continuous mapping.*

(1) *If* $A \in \mathsf{RD}(X)$*, then* $\overline{f(A)} \in \mathsf{RD}(Y)$*.*
(2) *If* $A \in \mathsf{WD}(X)$*, then* $\overline{f(A)} \in \mathsf{WD}(Y)$*.*

Proof. (1): Since $A \in \mathsf{RD}(X)$, there exists a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $A \in m(\mathcal{K})$. Let $\mathcal{K}_f = \{\uparrow f(K \cap A) : K \in \mathcal{K}\}$. Then $\mathcal{K}_f \subseteq \mathsf{K}(Y)$ is filtered. For each $K \in \mathcal{K}$, since $K \cap A \neq \emptyset$, we have $\emptyset \neq f(K \cap A) \subseteq \uparrow f(K \cap A) \cap \overline{f(A)}$. So $\overline{f(A)} \in M(\mathcal{K}_f)$. If B is a closed subset of $\overline{f(A)}$ with $B \in M(\mathcal{K}_f)$, then $B \cap \uparrow f(K \cap A) \neq \emptyset$ for every $K \in \mathcal{K}$. So $K \cap A \cap f^{-1}(B) \neq \emptyset$ for all $K \in \mathcal{K}$. It follows that $A = A \cap f^{-1}(B)$ by the minimality of A, and consequently, $\overline{f(A)} \subseteq B$. Therefore, $\overline{f(A)} = B$. Thus $\overline{f(A)} \in \mathsf{RD}(Y)$.

(2): Let Z be a well-filtered space and $g : Y \longrightarrow Z$ a continuous mapping. Since $g \circ f : X \longrightarrow Z$ is continuous and $A \in \mathsf{WD}(X)$, there is $z \in Z$ such that $\overline{g(\overline{f(A)})} = \overline{g \circ f(A)} = \overline{\{z\}}$. Thus $\overline{f(A)} \in \mathsf{WD}(Y)$. □

Lemma 2.8. ([2]) *Let X be a locally hypercompact T_0 space and $A \in \mathsf{Irr}(X)$. Then there exists a directed subset $D \subseteq \downarrow A$ such that $\overline{A} = \overline{D}$.*

Corollary 2.9. *For any locally hypercompact T_0 space X, $\mathsf{Irr}_c(X) = \mathsf{WD}(X) = \mathsf{RD}(X) = \mathcal{D}_c(X)$.*

Proposition 2.10. *For any locally compact T_0 space X, $\mathsf{Irr}_c(X) = \mathsf{WD}(X) = \mathsf{RD}(X)$.*

Proof. Suppose that X is a locally compact T_0 space and $A \in \mathsf{Irr}_c(X)$. Let $\mathcal{K}_A = \{K \in \mathsf{K}(X) : A \cap \operatorname{int} K \neq \emptyset\}$.

Claim 1: $\mathcal{K}_A \neq \emptyset$.

Let $a \in A$. Since X is locally compact, there exists $K \in \mathsf{K}(X)$ such that $a \in \operatorname{int} K$. So $a \in A \cap \operatorname{int} K$ and $K \in \mathcal{K}_A$, implying $\mathcal{K}_A \neq \emptyset$.

Claim 2: $\mathcal{K}_A$ is filtered.

Let $K_1, K_2 \in \mathcal{K}_A$. Then $A \cap \operatorname{int} K_1 \neq \emptyset$ and $A \cap \operatorname{int} K_2 \neq \emptyset$. Since A is irreducible, $A \cap \operatorname{int} K_1 \cap \operatorname{int} K_2 \neq \emptyset$. Let $x \in A \cap \operatorname{int} K_1 \cap \operatorname{int} K_2$. By the local compactness of X again, there exists $K_3 \in \mathsf{K}(X)$ such that $x \in \operatorname{int} K_3 \subseteq K_3 \subseteq \operatorname{int} K_1 \cap \operatorname{int} K_2$. Thus $K_3 \in \mathcal{K}_A$ and $K_3 \subseteq K_1 \cap K_2$. So $\mathcal{K}_A$ is filtered.

Claim 3: $A \in m(\mathcal{K}_A)$.

Clearly, $\mathcal{K}_A \subseteq \Diamond A$. If B is a proper closed subset of A, then there is $a \in A \setminus B$. Since X is locally compact, there is $K_a \in \mathsf{K}(X)$ such that $a \in \operatorname{int} K_a \subseteq K_a \subseteq X \setminus B$. Then $K_a \in \mathcal{K}_A$ but $K_a \cap B = \emptyset$. Thus A is a minimal closed set that meets all members of $\mathcal{K}_A$, and hence $A \in \mathsf{RD}(X)$. By Proposition 2.6, $\mathsf{Irr}_c(X) = \mathsf{WD}(X) = \mathsf{RD}(X)$. □

3. Some new characterizations of well-filtered spaces and sober spaces

We now prove some new characterizations of well-filtered and sober spaces based on the results in the above section.

Proposition 3.1. *Let X be a T_0 space. Then the following two conditions are equivalent:*

(1) *X is well-filtered;*
(2) $\mathsf{WD}(X) = \mathcal{S}_c(X)$*;*
(3) $\mathsf{RD}(X) = \mathcal{S}_c(X)$*.*

Proof. (1) ⇒ (2): Use the mapping $id_X : X \longrightarrow X$.

(2) ⇒ (3): By Proposition 2.6.

(3) ⇒ (1): Suppose that $\mathcal{K} \subseteq \mathsf{K}(X)$ is filtered, $U \in \mathcal{O}(X)$, and $\bigcap \mathcal{K} \subseteq U$. If $K \not\subseteq U$ for each $K \in \mathcal{K}$, then by Lemma 2.1, the closed set $X \setminus U$ contains an irreducible closed subset A that also meets all members of $\mathcal{K}$ and hence $A \in \mathsf{RD}(X)$. By (2), $A = \overline{\{x\}}$ for some $x \in X$. For each by $K \in \mathcal{K}$, since $K \cap A = K \cap \overline{\{x\}} \neq \emptyset$, we have $x \in K$. So $x \in \bigcap \mathcal{K} \subseteq U \subseteq X \setminus A$, a contradiction. Therefore, $K \subseteq U$ for some $K \in \mathcal{K}$. □

Corollary 3.2. *Every well-filtered space is a d-space.*

Corollary 3.3. *A retract of a well-filtered space is well-filtered.*

Proof. Suppose that Y is a retract of a well-filtered space X. Then there are continuous mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g = id_Y$. Let $B \in \mathsf{RD}(Y)$, then by Lemma 2.7 and Proposition 3.1, there exists a unique $x_B \in X$ such that $\overline{g(B)} = \overline{\{x_B\}}$. Therefore, $\overline{B} = \overline{f \circ g(B)} = \overline{f(\overline{g(B)})} = \overline{f(\overline{\{x_B\}})} = \overline{\{f(x_B)\}}$. By Proposition 3.1, Y is well-filtered. □

By Theorem 2.10 and Proposition 3.1, we get the following well-known result.

Corollary 3.4. ([4][8]) *Every locally compact well-filtered space is sober.*

Example 3.5. Let X be a countably infinite set and X_{cof} the space equipped with the co-finite topology (the empty set and the complements of finite subsets of X are open). Then

(a) $\mathcal{C}(X_{cof}) = \{\emptyset, X\} \cup X^{(<\omega)}$, X_{cof} is T_1 and hence a strong d-space. (see Definition 3.17).
(b) $\mathsf{K}(X_{cof}) = 2^X \setminus \{\emptyset\}$.
(c) X_{cof} is locally compact and first countable.
(d) $\mathcal{S}_c(X_{cof}) = \mathcal{D}_c(X_{cof}) = \{\{x\} : x \in X\}$,
(e) $\mathsf{Irr}_c(X_{cof}) = \mathrm{WD}(X_{cof}) = \mathrm{RD}(X_{cof}) = \{X\} \cup \{\{x\} : x \in X\} \neq \mathcal{D}_c(X_{cof})$. In fact, let $\mathcal{K}_X = \{X \setminus F : F \in X^{(<\omega)}\}$. Then $\mathcal{K}_X \in \mathsf{K}(X_{cof})$ is filtered and $X \in M(\mathcal{K}_X)$. For any $A \in \mathcal{C}(X_{cof})$, if $A \neq X$, then A is finite and hence $A \notin M(\mathcal{K}_X)$ because $A \cap (X \setminus A) = \emptyset$. Thus $X \in \mathrm{RD}(X_{cof})$. Therefore, X_{cof} is a Rudin space and also a well-filtered determined space, but not a DC-space.
(f) X_{cof} is not well-filtered (by Proposition 3.1).

Example 3.6. Let L be the complete lattice constructed by Isbell [9]. Then by [16, Corollary 3.2] (or Corollary 3.16 below), $X = \Sigma L$ is a well-filtered space. Note that ΣL is not sober. Thus by Proposition 3.1, $\mathsf{WD}(X) \neq \mathsf{Irr}_c(X)$ and $\mathsf{RD}(X) \neq \mathsf{Irr}_c(X)$.

Example 3.7. Let X be a uncountably infinite set and X_{coc} the space equipped with the co-countable topology (the empty set and the complements of countable subsets of X are open). Then

(a) $\mathcal{C}(X_{coc}) = \{\emptyset, X\} \cup X^{(\leqslant\omega)}$, X_{coc} is T_1 and hence a strong d-space.
(b) $\mathsf{K}(X_{coc}) = X^{(<\omega)} \setminus \{\emptyset\}$ and $\mathrm{int}\, K = \emptyset$ for all $K \in \mathsf{K}(X_{coc})$.
(c) X_{coc} is not locally compact and not first countable.
(d) $\mathsf{Irr}_c(X_{coc}) = \{X\} \cup \{\{x\} : x \in X\}$, $\mathrm{WD}(X_{coc}) = \mathrm{RD}(X_{coc}) = \mathcal{D}_c(X_{coc}) = \mathcal{S}_c(X_{coc}) = \{\{x\} : x \in X\}$. Therefore, $\mathsf{Irr}_c(X_{coc}) \neq \mathrm{WD}(X_{coc})$, $\mathsf{Irr}_c(X_{coc}) \neq \mathrm{RD}(X_{coc})$, $\mathsf{Irr}_c(X_{coc}) \neq \mathcal{D}_c(X_{coc})$, and $\mathsf{Irr}_c(X_{coc}) \neq \mathcal{S}_c(X_{coc})$.
(e) X_{coc} is well-filtered (by Proposition 3.1), but it not sober.

Theorem 3.8. *For any T_0 space X, the following conditions are equivalent:*

(1) *X is well-filtered.*
(2) *For each $(A, K) \in \mathsf{WD}(X) \times \mathbf{up}(X)$, $max(A) \neq \emptyset$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$.*
(3) *For each $(A, K) \in \mathsf{RD}(X) \times \mathbf{up}(X)$, $max(A) \neq \emptyset$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$.*
(4) *For each $(A, K) \in \mathsf{WD}(X) \times \mathsf{K}(X)$, $max(A) \neq \emptyset$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$.*
(5) *For each $(A, K) \in \mathsf{RD}(X) \times \mathsf{K}(X)$, $max(A) \neq \emptyset$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$.*

Proof. (1) ⇒ (2): Suppose that X is well-filtered and $(A, K) \in \mathsf{Irr}_c(X) \times \mathbf{up}(X)$. Then by Proposition 3.1, there is $x \in X$ such that $A = \overline{\{x\}}$, and hence $max(A) = \{x\} \neq \emptyset$. Now we show that $\downarrow (A \cap K) = \downarrow (\downarrow x \cap K)$ is closed. If $\downarrow (\downarrow x \cap K) \neq \emptyset$ (i.e., $\downarrow x \cap K \neq \emptyset$), then $x \in K$ since K is saturated (that is, K is an upper set). It follows that $\downarrow (\downarrow x \cap K) = \downarrow x \in \mathcal{C}(X)$.

(2) ⇒ (3) and (4) ⇒ (5): By Proposition 2.6.

(2) ⇒ (4) and (3) ⇒ (5): Trivial.

(5) ⇒ (1): Suppose that $\mathcal{K} \subseteq \mathsf{K}(X)$ is filtered, $U \in \mathcal{O}(X)$, and $\bigcap \mathcal{K} \subseteq U$. If $K \not\subseteq U$ for each $K \in \mathcal{K}$, then by Lemma 2.1, $X \setminus U$ contains a minimal irreducible closed subset A that still meets all members of $\mathcal{K}$, and hence $A \in \mathsf{RD}(X)$. For any $\{K_1, K_2\} \subseteq \mathcal{K}$, we can find $K_3 \in \mathcal{K}$ with $K_3 \subseteq K_1 \cap K_2$. It follows that $\downarrow (A \cap K_1) \in \mathcal{C}(X)$ and $\emptyset \neq A \cap K_3 \subseteq \downarrow (A \cap K_1) \cap K_2 \neq \emptyset$. By (5) and the minimality of A, we have $\downarrow (A \cap K_1) = A$ for all $K_1 \in \mathcal{K}$. Select an $x \in max(A)$. Then for each $K \in \mathcal{K}$, $x \in \downarrow (A \cap K)$, and consequently, there is $a_k \in A \cap K$ such that $x \leq a_k$. By the maximality of x we have $x = a_k$. Therefore, $x \in K$ for all $K \in \mathcal{K}$, and so $x \in \bigcap \mathcal{K} \subseteq U \subseteq X \setminus A$, a contradiction. Thus X is well-filtered. □

Note that if X is a d-space, then by Lemma 1.3, $max(A) \neq \emptyset$ for every closed set A of X. Therefore, by Theorem 3.8, we obtain the following corollary.

Corollary 3.9. *For a d-space X, the following conditions are equivalent:*

(1) *X is well-filtered.*
(2) *For each $(A, K) \in \mathsf{WD}(X) \times \mathbf{up}(X)$, $\downarrow (A \cap K) \in \mathcal{C}(X)$.*
(3) *For each $(A, K) \in \mathsf{RD}(X) \times \mathbf{up}(X)$, $\downarrow (A \cap K) \in \mathcal{C}(X)$.*
(4) *For each $(A, K) \in \mathsf{WD}(X) \times \mathsf{K}(X)$, $\downarrow (A \cap K) \in \mathcal{C}(X)$.*
(5) *For each $(A, K) \in \mathsf{RD}(X) \times \mathsf{K}(X)$, $\downarrow (A \cap K) \in \mathcal{C}(X)$.*

Corollary 3.10. ([16]) *Let X be a d-space such that $\downarrow (A \cap K)$ is closed for all $A \in \mathcal{C}(X)$ and $K \in \mathsf{K}(X)$. Then X is well-filtered.*

Definition 3.11. Let P be a poset equipped with a topology. The partial order is said to be *upper semiclosed* if each $\uparrow x$ is closed.

Definition 3.12. A topological space X with a partially order is called *upper semicompact*, if $\uparrow x$ is compact for any $x \in X$, or equivalently, if $\uparrow x \cap A$ is compact for any $x \in X$ and $A \in \mathcal{C}(X)$. X is called *weakly upper semicompact* if $\uparrow x \cap A$ is compact for any $x \in X$ and $A \in \mathsf{Irr}_c(X)$.

Lemma 3.13. ([4]) *Let X be a topological space with an upper semiclosed partial order. If A is a compact subset of X, then $\downarrow A$ is Scott closed.*

Lemma 3.14. *Let X be a T_0-space such that ΣX is a d-space. For $A \in \mathcal{C}(X)$ and $K \in \mathsf{K}(\Sigma X)$, if $\downarrow(\uparrow x \cap A) \in \mathcal{C}(\Sigma X)$ for all $x \in X$, then $\downarrow(K \cap A) = \bigcup_{k \in K} \downarrow(\uparrow k \cap A) \in \mathcal{C}(\Sigma X)$.*

Proof. Since ΣX is a d-space, X is a dcpo. Let $D \in \mathcal{D}(X)$ such that $D \subseteq \downarrow(K \cap A)$. If $\bigvee D \notin \downarrow(K \cap A)$, then for each $k \in K$, $\bigvee D \notin \downarrow(\uparrow k \cap A)$, and hence $\bigcap_{d \in D} \uparrow d \cap \downarrow(\uparrow k \cap A) = \emptyset$. For each $k \in K$, since ΣX is a d-space and $\downarrow(\uparrow k \cap A) \in \sigma(X)$, by Proposition 1.2, there is a $d_k \in D$ such that $\uparrow d_k \cap \uparrow k \cap A = \emptyset$, and consequently, $k \in X \setminus \downarrow(\uparrow d_k \cap A)$ and $\downarrow(\uparrow d_k \cap A) \in \mathcal{C}(\Sigma X)$. By the compactness of K in ΣX, there exists a finite subset $\{d_{k_1}, ..., d_{k_n}\} \subseteq D$ such that $K \subseteq \bigcup_{i=1}^{n} (X \setminus \downarrow(\uparrow d_{k_i} \cap A))$. By the directness of D, there is a d_0 such that $\uparrow d_0 \subseteq \bigcap_{i=1}^{n} \uparrow d_{k_i}$. It follows that $K \subseteq X \setminus \downarrow(\uparrow d_o \cap A)$, which contradicts $d_o \in \downarrow(K \cap A)$, hence $\bigvee D \in \downarrow(K \cap A)$. $\square$

By Corollary 3.9, Lemma 3.13 and Lemma 3.14, we get the following corollaries.

Corollary 3.15. *For a dcpo P, if $(P, \lambda(P))$ is weakly upper semicompact, then $(P, \sigma(P))$ is well-filtered.*

Corollary 3.16. ([16]) *For a dcpo P, if $(P, \lambda(P))$ is upper semicompact (in particular, if P is bounded complete), then $(P, \sigma(P))$ is well-filtered.*

In order to reveal finer links between d-spaces and well-filtered spaces, we introduce another class of T_0 spaces.

Definition 3.17. A T_0 space X is called a strong d-space if for any $D \in \mathcal{D}(X)$, $x \in X$ and $U \in \mathcal{O}(X)$, $\bigcap_{d \in D} \uparrow d \cap \uparrow x \subseteq U$ implies $\uparrow d \cap \uparrow x \subseteq U$ for some $d \in D$.

Clearly, X is a strong d-space iff for any $D \in \mathcal{D}(X)$, $x \in X$ and $A \in \mathcal{C}(X)$, if $\uparrow d \cap \uparrow x \cap A \neq \emptyset$ for all $d \in D$, then $\bigcap_{d \in D} (\uparrow d \cap \uparrow x) \cap A \neq \emptyset$. Also it is easy to verify that every coherent well-filtered space is a strong d-space.

Proposition 3.18. *For a T_0 space X, the following two conditions are equivalent:*

(1) *X is a strong d-space.*
(2) *For any $D \in \mathcal{D}(X)$, $\uparrow F \in \mathbf{Fin}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{d \in D} \uparrow d \cap \uparrow F \subseteq U$ implies $\uparrow d \cap \uparrow F \subseteq U$ for some $d \in D$.*

Proof. (1) $\Rightarrow$ (2): Let $D \in \mathcal{D}$, $\uparrow F \in \mathbf{Fin}(X)$ and $U \in \mathcal{O}(X)$ such that $\bigcap_{d\in D} \uparrow d \cap \uparrow F \subseteq U$. Then for each $u \in F$, $\bigcap_{d\in D} \uparrow d \cap \uparrow u \subseteq U$, and hence $\uparrow d_u \cap \uparrow u \subseteq U$ for some $d_u \in D$. Since F is finite and D is a direct subset of X, there is a $d_0 \in D$ such that $\uparrow d_0 \subseteq \bigcap_{u\in F} \uparrow u$. It follows that $\uparrow d_0 \cap \uparrow F \subseteq U$.

(2) $\Rightarrow$ (1): Trivial. $\square$

Definition 3.19. Let P be a dcpo. A subset $U \subseteq P$ is called *strongly Scott open* if (i) $U = \uparrow U$, and (ii) for any $D \in \mathcal{D}(P)$ and $x \in P$, $\bigcap_{d\in D} \uparrow d \cap \uparrow x \subseteq U$ (that is, $\uparrow \bigvee D \cap \uparrow x \subseteq U$) implies $\uparrow d \cap \uparrow x \subseteq U$ for some $d \in D$. Let $\sigma^s(P)$ denote the set of all strongly Scott open subsets of P.

Clearly, if $U, V \in \sigma^s(P)$, then $U \cap V \in \sigma^s(P)$. The topology generated by $\sigma^s(P)$ (as a base) is called the *strong Scott topology* on P and denote it by $\sigma_s(P)$. The space $(P, \sigma_s(P))$ is called the *strong Scott space* of P, and will be denote by $\Sigma_s P$.

For any x, y in a poset P, if $\uparrow x \cap \downarrow y \neq \emptyset$, then $\downarrow(\uparrow x \cap \downarrow y) = \downarrow y$, and whence for any nonempty finite subset F of P, $\downarrow(\uparrow x \cap \downarrow F) = \emptyset$ or $\downarrow(\uparrow x \cap \downarrow F) = \downarrow F_x$, where $F_x = \{u \in F : \uparrow x \cap \downarrow u \neq \emptyset\}$. Now we check $P \setminus \downarrow F \in \sigma^s(P)$. For any $D \in \mathcal{D}(P)$ and $x \in X$, if $\bigcap_{d\in D} \uparrow d \cap \uparrow x \subseteq P \setminus \downarrow F$, then $\bigcap_{d\in D} \uparrow d \cap \uparrow x \cap \downarrow F = \emptyset$, and whence $\bigcap_{d\in D} \uparrow d \cap \downarrow(\uparrow x \cap \downarrow F) = \emptyset$, or equivalently, $\bigcap_{d\in D} \uparrow d \subseteq P \setminus \downarrow(\uparrow x \cap \downarrow F) \in \upsilon(P) \subseteq \sigma(P)$. Therefore, $\uparrow d \subseteq P \setminus \downarrow(\uparrow x \cap \downarrow F)$ for some $d \in D$, and so $\uparrow d \cap \uparrow x \subseteq P \setminus \downarrow F$. Thus we have the following relations:

$$\upsilon(P) \subseteq \sigma_s(P) \subseteq \sigma(P).$$

Therefore, if $(P, \upsilon(P))$ is not a strong d-space, then any space (P, τ) with $\upsilon(P) \subseteq \tau \subseteq \alpha(P)$ is not a strong d-space. In particular, $\Sigma_s P$ and ΣP are not strong d-spaces. For two topologies τ_1 and τ_2 on P with $\upsilon(P) \subseteq \tau_1 \subseteq \tau_2$, if (P, τ_2) is a strong d-space, then (P, τ_1) is also a strong d-space.

Remark 3.20. (1) Every T_1 space is a strong d-space.
(2) If a dcpo P is a sup semilattice, then $\sigma_s(P) = \sigma(P)$. In this case, $(P, \sigma(P))$ is a strong d-space.
(3) For a dcpo P, if $\sigma_s(P) = \sigma^s(P)$, then $\Sigma_s P$ is a strong d-space.

Proposition 3.21. *For a T_0 space, consider the following two conditions:*

(1) *X is a strong d-space.*
(2) *X is a d-space, and $\mathcal{O}(X) \subseteq \sigma_s(X)$.*

Then (1) $\Rightarrow$ (2)*, and the two conditions are equivalent if X (with the specialization order) is a sup semilattice.*

Proof. (1) $\Rightarrow$ (2): Let $D \in \mathcal{D}(X)$ and $U \in \mathcal{O}(X)$ with $\bigcap_{d\in D} \uparrow d \subseteq U$. Then for any $c \in D$,

$$\bigcap_{d\in D} \uparrow d \cap \uparrow c = \bigcap_{d\in D} \uparrow d \subseteq U,$$

and hence $\uparrow d \cap \uparrow c \subseteq U$ for some $d \in D$. Take one $e \in D$ with $d \leq e$ and $e \leq c$. Then $\uparrow e \subseteq \uparrow d \cap \uparrow c \subseteq U$. By Proposition 1.2, X is a d-space. Now we prove that $\mathcal{O}(X) \subseteq \sigma_s(X)$. Suppose $U \in \mathcal{O}(X)$, $x \in X$ and $D \in \mathcal{D}(X)$ such that $\bigcap_{d\in D} \uparrow d \cap \uparrow x \subseteq U$. Since X is a strong d-space, $\uparrow d \cap \uparrow x \subseteq U$ for some $d \in D$. Thus U is strongly Scott open.

(2) $\Rightarrow$ (1): Suppose that X is a sup semilattice. For any $D \in \mathcal{D}$, $x \in X$ and $U \in \mathcal{O}(X)$, if $\bigcap_{d\in D} \uparrow d \cap \uparrow x \subseteq U$, then $\bigcap_{d\in D} \uparrow(d \vee x) \subseteq U$ and $\{d \vee x : d \in D\} \in \mathcal{D}(X)$. By Proposition 1.2, $\uparrow d \cap \uparrow x = \uparrow d \vee x \subseteq U$ for some $d \in D$. Thus X is a strong d-space. $\square$

Definition 3.22. A poset P is said to have *property D* if for any nonempty subset $\{x_i : i \in I\} \subseteq P$ that has a lower bound (that is $\bigcap_{i\in I} \downarrow x_i \neq \emptyset$), $\bigcap_{i\in I} \downarrow x_i \in \mathrm{Id}(P)$.

Clearly, every bounded complete poset has property D. For a dcpo P, P satisfies property D iff every nonempty subset $\{x_i : i \in I\} \subseteq P$ that has a lower bound has the greatest lower bound (that is, $\bigcap_{i\in I} \downarrow x_i$ is a principal ideal of P).

Lemma 3.23. *For a poset P, the following two conditions are equivalent:*

(1) $(P, \upsilon(P))$ *is a strong d-space;*
(2) *P is a dcpo, and for any $\{F_i : i \in I\} \subseteq P^{(<\omega)}$ and $x \in P$, $\downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i) \in \mathcal{C}(\Sigma P)$.*

Proof. (1) $\Rightarrow$ (2): Suppose that $(P, \upsilon(P))$ is a strong d-space. Then $(P, \upsilon(P))$ is a d-space, and hence P is a dcpo. For $\{F_i : i \in I\} \subseteq P^{(<\omega)}$ and $x \in P$, we show that $\downarrow\bigcap_{i\in I}(\uparrow x \cap \downarrow F_i) \in \mathcal{C}(\Sigma P)$. For any $D \in \mathcal{D}(P)$ with $D \subseteq \downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i)$, if $\bigvee D \notin \downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i)$, then $\uparrow\bigvee D \cap \uparrow x \cap \bigcap_{i\in I} \downarrow F_i = \bigcap_{d\in D} \uparrow d \cap \uparrow x \cap \bigcap_{i\in I} \downarrow F_i = \emptyset$, and whence $\bigcap_{d\in D} \uparrow d \cap \uparrow x \subseteq P \setminus \bigcap_{i\in I} \downarrow F_i \in \upsilon(P)$. Since $(P, \upsilon(P))$ is a strong d-space, there is $d \in D$ such that $\uparrow d \cap \uparrow x \subseteq P \setminus \bigcap_{i\in I} \downarrow F_i$, which is a contradiction with $d \in \downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i)$. Therefore, $\bigvee D \in \downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i)$. Thus $\downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i) \in \mathcal{C}(\Sigma P)$.

(2) $\Rightarrow$ (1): For any $D \in \mathcal{D}(P)$, $x \in P$ and $U \in \upsilon(P)$ such that $\bigcap_{d\in D} \uparrow d \cap \uparrow x \subseteq U$, if $U = P$, then $\uparrow d \cap \uparrow x \subseteq U$ for all $d \in D$; if U is a proper υ-open subset of P, then there is a family $\{F_i : i \in I\} \subseteq P^{(<\omega)}$ such that $U = P \setminus \bigcap_{i\in I} \downarrow F_i$. Therefore, $\uparrow\bigvee D = \bigcap_{d\in D} \uparrow d \subseteq P \setminus \downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i) \in \sigma(P)$ by condition (2). It follows that $\uparrow d \subseteq P \setminus \downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i)$ for some $d \in D$, and whence $\uparrow d \cap \uparrow x \subseteq P \setminus \bigcap_{i\in I} \downarrow F_i$, proving that $(P, \upsilon(P))$ is a strong d-space. $\square$

Similarly, we have the following result.

Lemma 3.24. *For a poset P, the following two conditions are equivalent:*

(1) *ΣP is a strong d-space;*
(2) *P is a dcpo, and for any $A \in \mathcal{C}(\Sigma P)$ and $x \in P$, $\downarrow(\uparrow x \cap A) \in \mathcal{C}(\Sigma P)$.*

Corollary 3.25. *For a dcpo P satisfying property D (in particular, P is a complete semilattice), $(P, \upsilon(P))$ is a strong d-space.*

Proof. For any $\{F_i : i \in I\} \subseteq P^{(<\omega)}$ and $x \in P$, we show that $\downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i) = \bigcap_{i\in I} \downarrow(\uparrow x \cap \downarrow F_i)$. Obviously, $\downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i) \subseteq \bigcap_{i\in I} \downarrow(\uparrow x \cap \downarrow F_i)$. Conversely, if $y \in \bigcap_{i\in I} \downarrow(\uparrow x \cap \downarrow F_i)$, then for each $i \in I$, there exists $u_i \in \uparrow x \cap \downarrow F_i$, and hence there is $t_i \in F_i$ such that $u_i \leq t_i$. It follows that $x, y \in \bigcap_{i\in I} \downarrow t_i$. Since P satisfies property D, there is a $z \in \bigcap_{i\in I} \downarrow t_i \subseteq \bigcap_{i\in I} \downarrow F_i$ such that $y \leq z$ and $x \leq z$, and whence $z \in \uparrow x \cap \bigcap_{i\in I} \downarrow F_i$ and $y \in \downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i)$. Thus $\downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i) = \bigcap_{i\in I} \downarrow(\uparrow x \cap \downarrow F_i)$. For any $s, t \in P$, if $\uparrow s \cap \downarrow t \neq \emptyset$, then $t \in \uparrow s \cap \downarrow t$, and hence $\downarrow(\uparrow s \cap \downarrow t) = \downarrow t$. Therefore, for each $i \in I$, $\downarrow(\uparrow x \cap \downarrow F_i) = \bigcup_{t\in F_i} \downarrow(\uparrow x \cap \downarrow t) = \downarrow F_i^x$, where $F_i^x = \{t \in F_i : \uparrow x \cap \downarrow t \neq \emptyset\}$. It follows that $\downarrow(\uparrow x \cap \bigcap_{i\in I} \downarrow F_i) = \bigcap_{i\in I} \downarrow(\uparrow x \cap \downarrow F_i) \in \mathcal{C}((P, \upsilon(P))) \subseteq \mathcal{C}(\Sigma P)$. By lemma 3.23, $(P, \upsilon(P))$ is a strong d-space. $\square$

Remark 3.26. For a dcpo P, consider the following three conditions:

(1) P has property D.
(2) For any $\{F_i : i \in I\} \subseteq P^{(<\omega)}$ and $x \in P$, $\downarrow(\bigcap_{i\in I}(\uparrow x \cap \downarrow F_i)) = \bigcap_{i\in I} \downarrow(\uparrow x \cap \downarrow F_i)$ (note $\bigcap_{i\in I} \downarrow(\uparrow x \cap \downarrow F_i)$ is always υ-closed).
(3) $(P, \upsilon(P))$ is a d-space.

Then by Lemma 3.23 and the proof of Corollary 3.25, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proposition 3.27. *If X is a d-space and $\downarrow(\uparrow x \cap A) \in \mathcal{C}(X)$ for all $x \in X$ and $A \in \mathcal{C}(X)$, then X is a strong d-space.*

Proof. Suppose that $D \in \mathcal{D}$, $x \in X$ and $U \in \mathcal{O}(X)$ such that $\bigcap_{d\in D} \uparrow d \cap \uparrow x \subseteq U$. Let $A = X \setminus U$. Then $A \in \mathcal{C}(X)$. If for any $d \in D$, $\uparrow d \cap \uparrow x \not\subseteq U$, then $\uparrow d \cap \downarrow(\uparrow x \cap A) \neq \emptyset$. Since X is a d-space and $\downarrow(\uparrow x \cap A) \in \mathcal{C}(X)$, by Proposition 1.2, we have $\bigcap_{d\in D} \uparrow d \cap \downarrow(\uparrow x \cap A) \neq \emptyset$, and hence $\bigcap_{d\in D} \uparrow d \cap \uparrow x \cap A \neq \emptyset$, a contradiction. Thus $\uparrow d \cap \uparrow x \subseteq U$ for some $d \in D$. $\square$

By Theorem 3.8, Corollary 3.9, Lemma 3.13 and Proposition 3.27, we get the following corollaries.

Corollary 3.28. *For any T_0 space X, if for each $B \in \mathsf{WD}(X)$ and $(A, K) \in \mathcal{C}(X) \times \mathsf{K}(X)$, $max(B) \neq \emptyset$ and $\downarrow(A \cap K) \in \mathcal{C}(X)$, then X is a well-filtered strong d-space.*

Corollary 3.29. *For any T_0 space X, if for each $B \in \mathrm{RD}(X)$ and $(A, K) \in \mathcal{C}(X) \times \mathsf{K}(X)$, $max(B) \neq \emptyset$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$, then X is a well-filtered strong d-space.*

Corollary 3.30. *Let X be a d-space such that $\downarrow (A \cap K)$ is closed for all $A \in \mathcal{C}(X)$ and $K \in \mathsf{K}(X)$. Then X is a well-filtered strong d-space.*

Corollary 3.31. *For a dcpo P, if $(P, \lambda(P))$ is upper semicompact, then $(P, \sigma(P))$ is a strong d-space.*

The following example shows that for a dcpo P, $(P, \upsilon(P))$ and $(P, \sigma(P))$ need not be strong d-spaces, although they are always d-spaces.

Example 3.32. (Johnstone space) Let $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ with ordering defined by $(j, k) \leq (m, n)$ iff $j = m$ and $k \leq n$, or $n = \omega$ and $k \leq m$ (see [11]). Then $\mathbb{J}$ is a dcpo, and hence the Johnstone space $\Sigma\,\mathbb{J}$ is a d-space. However, $\Sigma\,\mathbb{J}$ is not well-filtered (see [5, Exercise 8.3.9]), and hence non-sober. Clearly, $\bigcap_{n\in\mathbb{N}}(\uparrow(1, n) \cap \uparrow(2, 1)) = \emptyset$, but $\uparrow(1, n) \cap \uparrow(2, 1) = \{(m, \omega) : n \leq m\} \neq \emptyset$ for all n. Hence $(\mathbb{J}, \upsilon(\mathbb{J}))$ and $\Sigma\,\mathbb{J}$ are not strong d-spaces.

The following example shows that even for a continuous dcpo P, if ΣP is not coherent, ΣP and $\Sigma_s P$ may not be strong d-spaces.

Example 3.33. Let $C = \{a_1, a_2, ..., a_n, ...\} \cup \{\omega_0\}$ and $P = C \cup \{b\} \cup \{\omega_1, ..., \omega_n, ...\}$ with the order generated by

(a) $a_1 < a_2 < ... < a_n < a_{n+1} < ...$;
(b) $a_n < \omega_0$ for all $n \in \mathbb{N}$;
(c) $b < \omega_n$ and $a_m < \omega_n$ for all $n, m \in \mathbb{N}$ with $m \leq n$.

Then P is a dcpo and $D \in \mathcal{D}(P)$ iff $D \subseteq C$ or D has a largest element (that is $\downarrow D$ is a principal ideal of P), and hence $x \ll x$ for all $x \in P \setminus \{\omega_0\}$. Therefore, P is a continuous dcpo and ΣP is sober. $\uparrow a_1, \uparrow b \in \mathsf{K}(\Sigma\ )$, but $\uparrow a_1 \cap \uparrow b = \{\omega_1, \omega_2, ..., \omega_n, ...\}$ is not Scott compact (note $\{\omega_n\} \in \sigma(P)$ for all $n \in \mathbb{N}$). Thus ΣP is not coherent. For any $m \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} \uparrow a_n \cap \uparrow b = \emptyset \subseteq \{\omega_m\} \in \sigma(P)$, but $\uparrow a_n \cap \uparrow b = \{\omega_n, \omega_{n+1}, ...\} \nsubseteq \{\omega_m\}$ for any $n \in \mathbb{N}$. Thus ΣP is not a strong d-space. Since $\bigcap_{n\in\mathbb{N}} \uparrow a_n \cap \uparrow b = \emptyset$, but $\uparrow a_n \cap \uparrow b = \{\omega_n, \omega_{n+1}, ...\} \neq \emptyset$, $(P, \upsilon(P))$ is not a strong d-space, and hence $\Sigma_s P$ is not a strong d-space because $\upsilon(P) \subseteq \sigma_s(P)$.

Remark 3.34. Let P be the continuous domain in Example 3.33. Then $\downarrow(\uparrow b \cap P) = P \setminus \{\omega_0\}$ is not Scott closed since $\{\omega_0\} \notin \sigma(P)$ (or equivalently, ω_0 is not a compact element of P). So as a sufficient condition for a d-space being well-filtered, the condition that $\downarrow (A \cap K)$ is closed for all $A \in \mathcal{C}(X)$ and $K \in \mathsf{K}(X)$ seems a little too strong (see Corollary 3.9, Corollary 3.10 and Corollary 3.30).

Lemma 3.35. ([7]) *For a T_0 space X, the following conditions are equivalent:*

(1) *X is sober.*
(2) *For any $\mathcal{A} \subseteq \mathsf{Irr}(P_S(X))$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{A} \subseteq U$ implies $K \subseteq U$ for some $K \in U$.*
(3) *$P_S(X)$ is sober.*

Remark 3.36. By Remark 1.8 and Lemma 3.35, we have that X is sober iff for any $\mathcal{A} \subseteq \mathsf{Irr}_c(P_S(X))$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{A} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{A}$.

For a T_0 space and $A \subseteq X$, define $\Psi_{\mathbf{up}(X)}(A) = \{K \in \mathbf{up}(X) : K \cap A \neq \emptyset\}$ and $\Psi_{\mathsf{K}(X)}(A) = \{K \in \mathsf{K}(X) : K \cap A \neq \emptyset\}$.

The following theorem provides new characterizations of sober spaces similar to that for well-filtered spaces given in Theorem 4.8.

Theorem 3.37. *The following conditions are equivalent for a T_0 space X:*

(1) *X is sober.*
(2) *For any $(A, K) \in \mathsf{Irr}_c(X) \times \mathbf{up}(X)$, $\Psi_{\mathbf{up}(X)}(A) \in \mathrm{Filt}(\mathbf{up}(X))$, $max(A) \neq \emptyset$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$.*
(3) *For any $(A, K) \in \mathsf{Irr}_c(X) \times \mathsf{K}(X)$, $\Psi_{\mathsf{K}(X)}(A) \in \mathrm{Filt}(\mathsf{K}(X))$, $max(A) \neq \emptyset$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$.*

Proof. We only prove the equivalence of (1) and (3). The proof of the equivalence of (1) and (3) is similar.

(1) ⇒ (3): Suppose that X is sober and $(A, K) \in \mathsf{Irr}_c(X) \times \mathsf{K}(X)$. Then there is an $x \in X$ such that $A = \overline{\{x\}}$, and hence $max(A) = \{x\} \neq \emptyset$. Clearly, $\Psi_{\mathsf{K}(X)}(A) = \{K \in \mathsf{K}(X) : K \cap \downarrow x \neq \emptyset\} = \uparrow_{\mathsf{K}(X)} \uparrow x$ is a principal filter of $\mathsf{K}(X)$. Now we show that $\downarrow (A \cap K)$ is a closed subset of X. Obviously, if $\downarrow x \cap K = \emptyset$, then $\downarrow (A \cap K) = \downarrow (\downarrow x \cap K) = \emptyset$; if $\downarrow x \cap K \neq \emptyset$ (in this case $x \in K$ since K is an upper set), then $\downarrow (A \cap K) = \downarrow (\downarrow x \cap K) = \downarrow x$. Thus $\downarrow (A \cap K) = \downarrow x \in \mathcal{C}(X)$.

(3) ⇒ (1): Suppose $\mathcal{A} \subseteq \mathsf{Irr}(P_S(X))$ and $U \in \mathcal{O}(X)$ such that $\bigcap \mathcal{A} \subseteq U$. If $K \not\subseteq U$ for all $K \in U$, then by Lemma 2.1, $X \setminus U$ contains a minimal irreducible closed subset A that still meets all members of $\mathcal{A}$. For any $\{K_1, K_2\} \subseteq \mathcal{A}$, since $\Psi_{\mathsf{K}(X)}(A) \in \mathrm{Filt}(\mathsf{K}(X))$, there is a $K_3 \in \Psi_{\mathsf{K}(X)}(A)$ with $K_3 \subseteq K_1 \cap K_2$. It follows that $\emptyset \neq A \cap K_3 \subseteq \downarrow (A \cap K_1) \cap K_2 \neq \emptyset$. By (3) and the minimality of A, we have $\downarrow (A \cap K_1) = A$ for all $K_1 \in \mathcal{A}$. Select an $x \in max(A)$. Then for each $K \in \mathcal{A}$, $x \in \downarrow (A \cap K)$, and consequently, there is $a_k \in A \cap K$ such that $x \leq a_k$. By the maximality of x, we have $x = a_k$. Therefore, $x \in K$ for all $K \in \mathcal{A}$, and whence $x \in \bigcap \mathcal{A} \subseteq U \subseteq X \setminus A$, a contradiction. By Lemma 3.35, X is sober. □

By Lemma 1.3 and Theorem 3.37, we get the following corollary.

Corollary 3.38. *For a d-space X, the following conditions are equivalent:*

(1) *X is sober.*
(2) *For each $(A, K) \in \mathsf{Irr}_c(X) \times \mathbf{up}(X)$, $\Psi_{\mathbf{up}(X)}(A) \in \mathrm{Filt}(\mathbf{up}(X))$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$.*
(3) *For each $(A, K) \in \mathsf{Irr}_c(X) \times \mathsf{K}(X)$, $\Psi_{\mathsf{K}(X)}(A) \in \mathrm{Filt}(\mathsf{K}(X))$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$.*

4. Conclusion

In this paper, based on topological Rudin's Lemma, we introduced two new classes of subsets lying between the classes of all closures of directed subsets and that of irreducible closed sets. Using such subsets, we obtained several new characterizations of sober spaces and well-filtered spaces, which improve and generalize the related results of Xi and Lawson [16].

Our study also leads to the definition of a new class of spaces — strong d-spaces and a new topology — the strong Scott topology which may deserve further investigation.

We now close our paper with some problems.

Problem 4.1. Does $\mathsf{RD}(X) = \mathsf{WD}(X)$ hold for every T_0 space X?

Problem 4.2. Does $\mathsf{Irr}_c(X) = \mathsf{WD}(X)$ hold for every core compact T_0 space X?

Problem 4.3. Are the following conditions equivalent for any T_0 space X?

(a) X is sober.
(b) For each $(A, K) \in \mathsf{Irr}_c(X) \times \mathbf{up}(X)$, $max(A) \neq \emptyset$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$.
(c) For each $(A, K) \in \mathsf{Irr}_c(X) \times \mathsf{K}(X)$, $max(A) \neq \emptyset$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$.

Problem 4.4. Are the following conditions equivalent for any d-space X?

(a) X is sober.
(b) For each $(A, K) \in \mathsf{Irr}_c(X) \times \mathbf{up}(X)$, $\downarrow (A \cap K) \in \mathcal{C}(X)$.
(c) For each $(A, K) \in \mathsf{Irr}_c(X) \times \mathsf{K}(X)$, $\downarrow (A \cap K) \in \mathcal{C}(X)$.

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