Existence of well-filterifications of T_0 topological spaces

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Abstract. We prove that for every T_0 space X, there is a well-filtered space $W(X)$ and a continuous mapping $\eta_X : X \longrightarrow W(X)$, such that for any well-filtered space Y and any continuous mapping $f : X \longrightarrow Y$ there is a unique continuous mapping $\hat{f} : W(X) \longrightarrow Y$ such that $f = \hat{f} \circ \eta_X$. Such a space $W(X)$ will be called the well-filterification of X. This result gives a positive answer to one of the major open problems on well-filtered spaces. Another result on well-filtered spaces we will prove is that the product of two well-filtered spaces is well-filtered.

Domain theory was contrived by Dana Scott in 1969 as a theory of structures which are suitable for the interpretation of computable functions. Since then, such structures have been extensively studied by people from various different fields. The original links between domain theory and topology is via the Scott topology on posets, which are usually just T_0. One of the early time discoveries on the properties of Scott topology is that the Scott topology of every domain (continuous directed complete poset) is sober. Other such results include that a directed complete poset is a domain if and only if the Scott convergence class is topological [**8**] and a poset is continuous if and only if its Scott open sets form a completely distributive lattice[**1**]. Johnstone constructed the first directed complete poset whose Scott topology is not sober. Then a natural question is: Which properties do the Scott topology have for posets from a given class? The traditional topological properties seemed not helpful for this purpose because they are more suitable for Haudorff spaces. To meet these needs, various non-Hausdorff topological properties have been introduced, which inspired the new interests in non-Hausdorff spaces. These studies further confirmed Scott's claims [**7**]

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"Through a roundabout chain of mathematical events I have become interested in T_0 space, those topological spaces satisfying the weakest separation axiom..." and "What I hope to show in this paper is that from a less geometric point of view T_0-spaces can be not only interesting but also natural." The three non-Hausdorff topological properties heavily studied in domain theory are the sobriety, monotone convergence (or being d-space) and well-filteredness. It has been proved by different authors that every T_0 space has a sobrification and a d-completion [**1**][**2**][**5**], equivalently, the subcategory of all sober spaces and that of all monotone convergent spaces are reflective in the category of all T_0 spaces. However, it is still unknown whether the category of all well-filtered spaces is reflexive in the category of all T_0 spaces. In this paper we give a positive answer to this problem. Our main strategy is to use the criteria for the existence of K-fication of T_0 spaces suggested by Keimel and Lawson in [**5**]. Another problem on well-filteredeness is whether the product of two well-filtered spaces is a well-filtered space. We will also give a positive answer to this problem.

1. K-fication

Assume that a topological property, called K-property, is given. By [**5**], a K-fication of a T_0 space X is a space $F(X)$ with K-property and a continuous mapping $\eta_X : X \longrightarrow F(X)$ which is universal among all continuous mappings from X to spaces with K-property: for any continuous mapping $g : X \longrightarrow Z$ where Z is a space with K-property, there is a unique continuous mapping $\hat{g} : F(X) \longrightarrow Z$ such that $g = \hat{g} \circ \eta_X$. If every T_0 space has a K-fication, then the category of all T_0 spaces with K-property is reflective in the category of all T_0 spaces.

By Keimel and Lawson [**5**], if the K-property satisfies the following four conditions, then every T_0 space has a K-fication:

(K1) Every sober space has K-property.

(K2) If X has K-property and Y is homeomorphic to X, then Y also has K-property.

(K3) If $\{X_i\}_{i\in I}$ is a family of subspaces of a sober space such that each X_i has K-property, then the subspace $\bigcap_{i\in I} X_i$ also has K-property;

(K4) If $f : X \longrightarrow Y$ is a continuous mapping between sober spaces X and Y, then for any subspace Z of Y with K-property, $f^{-1}(Z)$ has K-property.

For a T_0 space (X, τ), the specialization order on X, written $\leq_\tau$ (or just $\leq$), is define as $x \leq_\tau y$ iff $x \in cl(\{y\})$, where cl is the closure operator.

As in a general poset we shall use the following standard notations for any subset A of a T_0 space (X, τ):

$$\uparrow_X A = \{y \in X : \text{there is an } x \in A, x \leq_\tau y\}.$$

$$\downarrow_X A = \{y \in X : \text{there is an } x \in A, y \leq_\tau x\}.$$

For any $x \in X$, $\uparrow_X x = \uparrow_X \{x\}$ and $\downarrow_X x = \downarrow_X \{x\}$.

The symbol $\uparrow_X A$ ($\downarrow_X A, \uparrow_X x, \downarrow_X x, resp.$) will be simply written as $\uparrow A$ ($\downarrow A, \uparrow x, \downarrow x, resp.$) if no ambiguous occurs.

By the definition of the specialization order on T_0 space X, for any $x \in X$, $\downarrow x = cl(\{x\})$, the closure of $\{x\}$. Hence $\downarrow x$ is a closed set.

Remark 1. Let X_1 be a subspace of a T_0 space (X, τ). For any $x \in X_1$, the closure $cl_{X_1}\{x\}$ of $\{x\}$ in X_1 equals $X_1 \cap cl_X\{x\}$, where $cl_X\{x\}$ is the closure of $\{x\}$ in X. Hence for any $x, y \in X_1$, $x \leq y$ holds in X_1 if and only if $x \leq y$ holds in X. In other words, the specialization order on X_1 is the restriction of $\leq_\tau$ on X_1.

Hence for any $x \in X_1$, we have $\downarrow_{X_1} x = (\downarrow_X x) \cap X_1$ ($\uparrow_{X_1} x = (\uparrow_X x) \cap X_1$, resp.).

In general, for any $A \subseteq X_1$, $\downarrow_{X_1} A = (\downarrow_X A) \cap X_1$ ($\uparrow_{X_1} A = (\uparrow_X A) \cap X_1$, resp.).

A subset A of a space X is *irreducible* if for any closed sets F_1, F_2 of X, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A T_0 space X is called sober if for any nonempty irreducible closed set F, $F = cl(\{x\})$ for some $x \in X$.

The sobriety satisfies all conditions (K1)-(K4), hence the so-called soberification exists for each T_0 space [**1**].

For any subset A of a space X, the saturation of A, denoted by $sat(A)$, is defined to be

$$sat(A) = \bigcap \{U \in \mathcal{O}(X) : A \subseteq U\},$$

where $\mathcal{O}(X)$ is the set of all open sets of X.

A subset A of a space X is called saturated if $A = sat(A)$.

The saturation of any subset is a saturated set, and the saturation of every compact set is compact[**1**][**3**].

It is a standard fact that for any subset A of a space X[**1**][**3**],

$$sat(A) = \uparrow A.$$

Definition 1. *A T_0 space X is called well-filtered if for any open set U and any filtered family $\mathcal{F}$ of saturated compact subsets of X (for any $F_1, F_2 \in \mathcal{F}$, there exists $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1 \cap F_2$), $\bigcap \mathcal{F} \subseteq U$ implies $F \subseteq U$ for some $F \in \mathcal{F}$.*

Remark 2. (1) Every sober space is well-filtered, and a locally compact space is sober iff it is well-filtered [**1**][**3**].

(2) A T_0 space X is called a monotone convergent space (or d-space), if for any directed subset D of X (with respect to the specialization order on X), $\bigvee D$ exists and D converges (as a net) to $\bigvee D$. Every well-filtered space is monotone convergent. The monotone convergence is a topological property satisfying all conditions (K1)-(K4), thus the d-completion exists for each T_0 spaces [**5**].

In this paper we prove that the well-filtered property satisfies all the conditions (K1)-(K4), hence the well-filterification exists for every T_0 space.

Remark 3. (1) If a space X is well-filtered and $\{F_i\}_{i\in I}$ is a filtered family of (non-empty) saturated compact sets, then $\bigcap\{F_i : i \in I\}$ is a (non-empty) saturated compact set [**3**][**10**].

(2) For any saturated compact set E in a T_0 space, $E =\uparrow C$, where C is a compact set and an anti-chain (with respect to the specialization order). In other words, every element in E is above some minimal element(s) of E. This claim follows from the compactness of E and the Maximal Chain Principle.

For more about sober spaces, well-filtered spaces, d-spaces and saturated sets, we refer the reader to [**1**][**3**][**4**][**6**][**9**][**10**].

2. Existence of well-filterification

We now verify that the well-filteredness satisfies all the conditions (K1)-(K4) given in [**5**]. The condition (K1) holds due to the fact that every sober space is well-filtered (see Remark 2 (1)). The condition (K2) holds because the well-filteredness is a topological property. Thus we only need to verify the condition (K3) and (K4).

In what follows, all topological spaces to be considered are assumed to be T_0.

Remark 4. If A_1, A_2 and A_3 are subsets of a space X such that $\uparrow A_3 \subseteq\uparrow A_1 \cap \uparrow A_2$, then for any lower set $F \subseteq X$ (i.e. $F =\downarrow F$),

$$\uparrow(F \cap A_3) \subseteq\uparrow(F \cap A_1)\cap \uparrow(F \cap A_2).$$

In fact, let $y \in F \cap A_3$. Then $k_1 \leq y$ for some $k_1 \in A_1$, and $k_2 \leq y$ for some $k_2 \in A_2$. Since $y \in F =\downarrow F$, we have that $k_1, k_2 \in F$. It follows that $k_1 \in F \cap A_1, k_2 \in F \cap A_2$. Hence $y \in\uparrow (F \cap A_1)\cap \uparrow (F \cap A_2)$. Therefore $F \cap A_3 \subseteq\uparrow (F \cap A_1)$ and $F \cap A_3 \subseteq\uparrow (F \cap A_2)$, which then imply that $\uparrow(F \cap A_3) \subseteq\uparrow(F \cap A_1)$ and $\uparrow(F \cap A_3) \subseteq\uparrow(F \cap A_2)$, or

$$\uparrow(F \cap A_3) \subseteq\uparrow(F \cap A_1)\cap \uparrow(F \cap A_2).$$

Remark 5. Let $f : X\longrightarrow Y$ be a continuous mapping between topological spaces.

(1) For any subset $A \subseteq X$, $f(\uparrow_X A) \subseteq \uparrow_Y f(A)$.
(2) If $A, B, C \subseteq X$ and $A \subseteq \uparrow_X B \cap \uparrow_X C$, then

$$\uparrow_Y f(A) \subseteq \uparrow_Y f(B) \cap \uparrow_Y f(C).$$

(3) If $\{\uparrow_X H_i\}_{i\in I}$ is a filtered family of subsets of X, then

$$\{\uparrow_Y f(H_i) : i \in I\}$$

is a filtered family of subsets of Y.

The lemma below will be used to prove several other results.

LEMMA 1. *Let X be a well-filtered space. Then for any filtered family $\{K_i\}_{i\in I}$ of nonempty compact saturated subsets of X, we have*
(1) $\bigcap_{i\in I} K_i = \uparrow C$, where C is a nonempty anti-chain;
(2) for each $a \in C$, $\bigcap_{i\in I} \uparrow(\downarrow a \cap K_i) = \uparrow a$.

PROOF. (1) This follows from (1)(2) of Remark 3.
(2) For each $a \in C$, $a \in C \subseteq \uparrow C = \bigcap_{i\in I} K_i$, so $a \in K_i$ for each $i \in I$. In particular, $\downarrow a \cap K_i \neq \emptyset$ for each $i \in I$.

Now, by Remark 4 and that $\downarrow a$ $(= cl(\{a\}))$ is closed, $\{\uparrow(\downarrow a \cap K_i)\}_{i\in I}$ is a filtered family of nonempty compact saturated sets. Since X is well-filtered, applying (1) to this new family of saturated compact sets, there is a nonempty anti-chain $\hat{C}$ such that

$$\bigcap_{i\in I} \uparrow(\downarrow a \cap K_i) = \uparrow \hat{C}.$$

Note that $a \in \downarrow a \cap K_i (i \in I)$, thus

$$a \in \bigcap_{i\in I} \uparrow(\downarrow a \cap K_i) = \uparrow \hat{C}.$$

Hence $\downarrow a \cap \hat{C} \neq \emptyset$. Take $t \in \downarrow a \cap \hat{C}$. Then

$$t \in \hat{C} \subseteq \uparrow \hat{C} = \bigcap_{i\in I} \uparrow(\downarrow a \cap K_i) \subseteq \bigcap_{i\in I} \uparrow K_i = \bigcap_{i\in I} K_i = \uparrow C.$$

So there is $c \in C$ such that $c \leq t$. Since $t \leq a$, we have $c \leq a$, implying $a = c$ because C is an anti-chain and $a, c \in C$. Hence $a = c = t \in \hat{C}$.

We now show that $\hat{C} = \{a\}$. Assume, on the contrary that there is $a' \in \hat{C} - \{a\}$. Then, as $\hat{C}$ is an anti-chain, we have

$$\uparrow \hat{C} \subseteq (X - \downarrow a) \cup (X - \downarrow a'),$$

here $(X - \downarrow a) \cup (X - \downarrow a') = (X - cl\{a\}) \cup (X - cl\{a'\})$ is an open set.

Since X is well-filtered, $\{\uparrow(\downarrow a \cap K_i) : i \in I\}$ is a filtered family of compact saturated sets of X and

$$\bigcap_{i\in I} \uparrow(\downarrow a \cap K_i) = \uparrow \hat{C} \subseteq (X - \downarrow a) \cup (X - \downarrow a'),$$

there is $i_0 \in I$ such that $\downarrow a \cap K_{i_0} \subseteq (X - \downarrow a) \cup (X - \downarrow a')$. Since $\downarrow a \cap K_{i_0}$ and $X - \downarrow a$ are disjoint, $\downarrow a \cap K_{i_0} \subseteq X - \downarrow a'$, implying $(\downarrow a \cap K_{i_0}) \cap \downarrow a' = \emptyset$. However $a' \in \uparrow(\downarrow a \cap K_{i_0})$, so $\downarrow a' \cap (\downarrow a \cap K_{i_0}) \neq \emptyset$. This contradiction shows that $\hat{C} = \{a\}$, thus

$$\bigcap_{i\in I} \uparrow(\downarrow a \cap K_i) = \uparrow \hat{C} = \uparrow a,$$

as desired.

□

Remark 6. (1) For any open set U of a space X, $U = sat(U) = \uparrow U$, and for any closed set F of X, $F = \downarrow F$ (see [**1**][**3**]).

(2) Let A and B be subsets of a space X. Then $sat(A) \subseteq sat(B)$ if and only if every open neighbourhood U of B (i.e. $B \subseteq U$) contains A.

The following result is a direct corollary of the general Topological Rudin's Lemma given by Keimel and Heckmann in [**4**] (see Lemma 3. 1 of [**4**]).

Lemma 2. *Let X be a topological space and $\mathcal{F}$ a filtered family of compact subsets of X (for any $F_1, F_2 \in \mathcal{F}$, there is $F \in \mathcal{F}$ such that $F \subseteq \uparrow F_1 \cap \uparrow F_2$). Any closed set $C \subseteq X$ that meets all members of $\mathcal{F}$ contains an irreducible closed subset A that still meets all members of $\mathcal{F}$.*

In addition, this irreducible closed set A can be taken as a minimal one: if $A' \subseteq A$ is a proper closed subset of A, then $A' \cap F = \emptyset$ for some $F \in \mathcal{F}$.

Lemma 3. *Let W be a subspace of a sober space X. Assume that $\{K_i\}_{i\in I}$ is a filtered family of compact saturated subsets of W and U is an open set of X such that (i) $\bigcap_{i\in I} K_i \subseteq U$ and (ii) $K_i \not\subseteq U (\forall i \in I)$. Then there is $e \in (X - W) \cap (X - U)$ such that*

$$\bigcap_{i\in I} \uparrow(\downarrow e \cap K_i) = \uparrow e.$$

Proof. Note that each K_i is also a compact subset of X. In addition, as $\{K_i : i \in I\}$ is a filtered family of compacts sets in the subspace W, it is also a filtered family of compact sets in X.

Now the closed set $U^c = X - U$ has a nonempty intersection with each $K_i (i \in I)$. By Lemma 2, there is a minimal irreducible closed set $F \subseteq U^c$

such that $F \cap K_i \neq \emptyset$ $(i \in I)$. Since X is sober, $F = cl\{e\} = \downarrow e$ for some $e \in X$.

Claim 1. $e \notin W$. As a mater of fact, if $e \in W$, then as $\downarrow e \cap K_i \neq \emptyset$, there is $k \in \downarrow e \cap K_i$ such that $k \leq e$ holds in X. Therefore $k \leq e$ holds in the subspace W as well by Remark 1. Hence $e \in \uparrow_W K_i = K_i$ because K_i is saturated in W. It follows that $e \in K_i$ for each $i \in I$. Then

$$e \in \bigcap_{i \in I} K_i \subseteq U,$$

which contradicts the assumption that $e \in \downarrow e = F \subseteq U^c$.

Claim 2. $\bigcap_{i \in I} \uparrow(\downarrow e \cap K_i) = \uparrow e$.

Clearly $\uparrow e \subseteq \bigcap_{i \in I} \uparrow(\downarrow e \cap K_i)$ holds. Note that, as an intersection of saturated sets $\uparrow(\downarrow e \cap K_i)(i \in I)$, $\bigcap_{i \in I} \uparrow(\downarrow e \cap K_i)$ is a saturated set. Hence, by (2) of Remark 6, in order to show that $\bigcap_{i \in I} \uparrow(\downarrow e \cap K_i) \subseteq \uparrow e$ holds, we only need to verify that every open neighbourhood of e must contain $\bigcap_{i \in I} \uparrow(\downarrow e \cap K_i)$ (note that $\uparrow e = sat(\{e\})$).

Let V be any open set of X containing e. As $\downarrow e$ is a closed set, each $\downarrow e \cap K_i$ is compact (the intersection of a compact set and a closed set is compact). In addition, by Remark 4, we have that $\{\downarrow e \cap K_i : i \in I\}$ is a filtered family of sets in X.

If $V^c \cap \downarrow e \cap K_i \neq \emptyset$ for all $i \in I$, then by Lemma 2, there is a minimal irreducible closed set G of X such that $G \subseteq V^c$ and $G \cap \downarrow e \cap K_i \neq \emptyset$ for all $i \in I$. Then $G = \downarrow e'$ for some $e' \in X$ $(e' \in G \subseteq V^c)$ because X is sober. Now $\downarrow e' \cap \downarrow e \cap K_i \neq \emptyset$ for all $i \in I$, so $\downarrow e' \cap \downarrow e = \downarrow e$ due to the minimality of $\downarrow e$.

On the other hand, $(\downarrow e' \cap \downarrow e) \cap \downarrow e \cap K_i \neq \emptyset$ for all $i \in I$, so $\downarrow e' \cap \downarrow e = \downarrow e'$ due to the minimality of $\downarrow e'$. It thus follows that $e = e'$. But $e' \notin V$ and $e \in V$, this contradiction shows that there is $i \in I$ such that $\downarrow e \cap K_i \subseteq V$, hence $\bigcap_{i \in I} \uparrow(\downarrow e \cap K_i) \subseteq V$. All these then show that $\bigcap_{i \in I} \uparrow(\downarrow e \cap K_i) \subseteq \uparrow e$.

Therefore $\bigcap_{i \in I} \uparrow(\downarrow e \cap K_i) = \uparrow e$.

The combination of Claim 1 and Claim 2 completes the proof. $\square$

Now we prove that the well-filteredness satisfies condition (K4).

LEMMA 4. *Let $f : (X, \tau) \longrightarrow (Y, \mu)$ be a continuous mapping between sober spaces. Then for any well-filtered subspace Z of Y, $f^{-1}(Z)$ is a well-filtered subspace of X.*

PROOF. Let $\{K_i\}_{i \in I} \subseteq f^{-1}(Z)$ be a filtered family of compact saturated subsets of $f^{-1}(Z)$ and $\bigcap\{K_i : i \in I\} \subseteq U$ with U an open set of X. We show that $K_i \subseteq U$ holds for some $i \in I$.

Assume that $K_i \nsubseteq U$ for every $i \in I$. Then by Lemma 3, there is $e \in (X - f^{-1}(Z)) \cap (X - U)$ such that

$$\bigcap_{i\in I}\uparrow(\downarrow e\cap K_i)=\uparrow e.$$

Now we verify that the following equation holds in Y:

$$\bigcap_{i\in I}\uparrow_Y f(\downarrow e\cap K_i)=\uparrow_Y f(e).$$

Note that every continuous mapping preserves the specialization order. Hence it follows from (2) of Remark 5 easily that

$$\bigcap_{i\in I}\uparrow_Y f(\downarrow e\cap K_i)\supseteq\uparrow_Y f(e).$$

Now let $V\subseteq Y$ be open and $f(e)\in V$. Then $e\in f^{-1}(V)$, and

$$\bigcap_{i\in I}\uparrow(\downarrow e\cap K_i)=\uparrow e\subseteq f^{-1}(V).$$

As X is well-filtered (every sober space is well-filtered), there exists $i_0\in I$ such that

$$\downarrow e\cap K_{i_0}\subseteq f^{-1}(V),$$

implying $f(\downarrow e\cap K_{i_0})\subseteq V$. So

$$\bigcap_{i\in I}\uparrow_Y f(\downarrow e\cap K_i)\subseteq\uparrow_Y f(\downarrow e\cap K_{i_0})\subseteq\uparrow_Y V=V.$$

By (2) of Remark 6,

$$\bigcap_{i\in I}\uparrow_Y f(\downarrow e\cap K_i)\subseteq\uparrow_Y f(e).$$

Therefore

$$\bigcap_{i\in I}\uparrow_Y f(\downarrow e\cap K_i)=\uparrow_Y f(e).$$

Since $e\in X-f^{-1}(Z)$, $f(e)\notin Z$ and hence

$$\bigcap_{i\in I}(\uparrow_Y f(\downarrow e\cap K_i))\cap Z=\uparrow_Y f(e)\cap Z\subseteq Z-\downarrow_Y f(e).$$

By Remark 5, $\{\uparrow_Y(f(\downarrow e\cap K_i):i\in I\}$ is a filtered family of subsets of Y, then $\{\uparrow_Y f(\downarrow e\cap K_i))\cap Z:i\in I\}$ is a filtered family of saturated compact subsets of the subspace Z. As Z is well-filtered, there is $i_0\in I$ such that

$$\uparrow_Y(f(\downarrow e\cap K_{i_0}))\cap Z\subseteq Z-\downarrow_Y f(e).$$

But this is impossible. In fact, choose one $u\in\downarrow e\cap K_{i_0}$. Then $u\in K_{i_0}\subseteq f^{-1}(Z)$, implying $f(u)\in\uparrow_Y(f(\downarrow e\cap K_{i_0}))\cap Z$. On the other hand, $u\leq e$ implies $f(u)\leq f(e)$, so $f(u)\in\downarrow_Y f(e)$, thus $f(u)\notin Z-\downarrow_Y f(e)$.

This contradiction shows that there exists $i \in I$ such that $K_i \subseteq U$. Hence $f^{-1}(Z)$ is well-filtered.

□

Next we verify that the well-filteredness satisfies condition (K3).

LEMMA 5. *Let $\{X_i\}_{i\in I}$ be a family of well-filtered subspaces of a sober space X. Then $\bigcap_{i\in I} X_i$ is a well-filtered subspace.*

PROOF. We only need to consider the case when $\bigcap_{i\in I} X_i \neq \emptyset$.

Let $\{K_i\}_{i\in I}$ be a filtered family of compact saturated subsets of the subspace $\bigcap_{i\in I} X_i$ and U be an open set of X such that $\bigcap_{i\in I} K_i \subseteq U$. If $K_i \nsubseteq U$ for all $i \in I$, then by Lemma 3, there is $e \notin \bigcap_{i\in I} X_i$ such that

$$\bigcap_{i\in I} \uparrow_X (\downarrow_X e \cap K_i) = \uparrow_X e.$$

Thus there is i_0 such that $e \notin X_{i_0}$. Note that $K_i \subseteq X_{i_0}$ for each $i \in I$.

$$\bigcap_{i\in I} \uparrow_{X_{i_0}} (\downarrow_X e \cap K_i) = \bigcap_{i\in I} \uparrow_X(\downarrow_X e \cap K_i) \cap X_{i_0} = \uparrow_X e \cap X_{i_0} \subseteq X_{i_0} - \downarrow_X e.$$

Since X_{i_0} is well-filtered, there is $i' \in I$ such that $\downarrow_X e \cap K_{i'} \subseteq X_{i_0} - \downarrow_X e$. By the assumption, $\downarrow_X e \cap K_{i'} \neq \emptyset$. Choose $u \in \downarrow_X e \cap K_{i'}$. Then $u \in \downarrow_X e$, thus $u \notin X_{i_0} - \downarrow_X e$. This contradicts $\downarrow_X e \cap K_{i'} \subseteq X_{i_0} - \downarrow_X e$. This contradiction shows that there must be K_i such that $K_i \subseteq U$, hence $\bigcap_{i\in I} X_i$ is well-filtered. □

Now all conditions (K1)-(K4) are satisfied by the well-filteredness, therefore we have the following result.

THEOREM 1. *For any T_0 space X, there is a well-filtered space $W(X)$ and a continuous mapping $\eta_X : X \longrightarrow W(X)$ which is universal from X to well-filtered spaces.*

COROLLARY 1. *The category of all well-filtered spaces is reflective in the category of all T_0 spaces.*

3. The product of two well-filtered spaces is well-filtered

It is well-known that the product of two sober spaces is sober [**1**]. However it is still unknown whether the product of two well-filtered spaces is well-filtered.

PROPOSITION 1. *If X and Y are well-filtered spaces, then the product space $X \times Y$ is well-filtered.*

PROOF. Let $\{K_i\}_{i\in I}$ be a filtered family of compact saturated subsets of $X\times Y$ and $W\subseteq X\times Y$ open such that

$$\bigcap_{i\in I} K_i\subseteq U.$$

Assume that $K_i\cap U^c\neq\emptyset$ for all $i\in I$. Then there is a minimal closed set $F\subseteq X\times Y$, $F\subseteq U^c$ such that

$$F\cap K_i\neq\emptyset(\forall i\in I).$$

Then

$$\bigcap_{i\in I}\uparrow_X p_X(K_i\cap F)=\uparrow_X C_X,\ \bigcap_{i\in I}\uparrow_X p_Y(K_i\cap F)=\uparrow_Y C_Y,$$

where $C_X\subseteq X$ and $C_Y\subseteq Y$ are nonempty anti-chains, and $p_X: X\times Y\longrightarrow X$ and $p_Y: X\times Y\longrightarrow Y$ are the projection mappings.

Choose an element $x_0\in C_X$ and an element $y_0\in C_Y$. We have the following

$$(\downarrow_X x_0\times Y)\cap(F\cap K_i)\neq\emptyset(\forall i\in I).$$

In fact, for each $i\in I$, $x_0\in\uparrow_X p_X(K_i\cap F)$, so there exists $(u_1,u_2)\in K_i\cap F$ with $x_0\geq u_1$. Hence $(u_1,u_2)\in(\downarrow_X x_0\times Y)\cap(K_i\cap F)$.

Similarly, $(X\times\downarrow_Y y_0)\cap F\cap K_i\neq\emptyset(\forall i\in I)$.

By the minimality of F, we have $F\subseteq\downarrow_X x_0\times Y$, as well as $F\subseteq X\times\downarrow_Y y_0$. Therefore

$$F\subseteq(\downarrow_X x_0\times Y)\cap(X\times\downarrow_Y y_0)=\downarrow_X x_0\times\downarrow_Y y_0.$$

Since $F\cap K_i\neq\emptyset$, $(\downarrow_X x_0\times\downarrow_Y y_0)\cap K_i\neq\emptyset$ holds for each $i\in I$. Since each K_i is saturated , we have $(x_0,y_0)\in K_i(\forall i\in I)$. Thus $(x_0,y_0)\in\bigcap_{i\in I}K_i\subseteq U$. There are open sets $U_1\subseteq X, U_2\subseteq Y$ such that $(x_0,y_0)\in U_1\times U_2\subseteq U$.

Applying Lemma 1 to $\{p_X(K_i\cap F): i\in I\}$ and $\{p_Y(K_i\cap F): i\in I\}$ we have

$$\bigcap_{i\in I}\uparrow_X(\downarrow_X x_0\cap p_X(K_i\cap F))=\uparrow_X x_0,\bigcap_{i\in I}\uparrow_Y(\downarrow_Y y_0\cap p_Y(K_i\cap F))=\uparrow_Y y_0.$$

As X and Y are well-filtered, and $\{K_i: i\in I\}$ is filtered, there is a K_{i_0} such that

$$\downarrow_X x_0\cap p_X(K_{i_0}\cap F)\subseteq U_1,\downarrow_Y y_0\cap p_Y(K_{i_0}\cap F)\subseteq U_2.$$

Thus

$$\begin{array}{rcl} F\cap K_{i_0} & \subseteq & (\downarrow_X x_0\times\downarrow_Y y_0)\cap(p_X(K_{i_0}\cap F)\times p_Y(K_{i_0}\cap F)) \\ & = & (\downarrow_X x_0\cap p_X(K_{i_0}\cap F))\times(\downarrow_Y y_0\cap p_Y(K_{i_0}\cap F) \\ & \subseteq & U_1\times U_2 \\ & \subseteq & U. \end{array}$$

This contradicts $F\subseteq U^c$.

The proof is completed.

□

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