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**THE NUMBER OF REPRESENTATIONS OF
A POSITIVE INTEGER BY TRIANGULAR, SQUARE AND
DECAGONAL NUMBERS**

UHA ISNAINI, RAY MELHAM, AND PEE CHOON TOH

ABSTRACT. Let $T_a D_b(n)$ and $T_a D'_b(n)$ denote respectively the number of representations of a positive integer n by $a(x^2 - x)/2 + b(4y^2 - 3y)$ and $a(x^2 - x)/2 + b(4y^2 - y)$. Similarly, let $S_a D_b(n)$ and $S_a D'_b(n)$ denote respectively the number of representations of n by $ax^2 + b(4y^2 - 3y)$ and $ax^2 + b(4y^2 - y)$. In this paper, we prove 162 formulas for these functions.

1. Introduction

Consider a positive definite binary quadratic form $ax^2 + bxy + cy^2$ where a, b, c are integers with $a > 0$ and discriminant $d = b^2 - 4ac < 0$. We let $R_{(a,b,c)}(n)$ denote the number of representations of an integer n by this quadratic form as x and y range over all integers. In other words, we have

$$(1) \quad R_{(a,b,c)}(n) = |\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : n = ax^2 + bxy + cy^2\}|.$$

Jacobi's celebrated two squares theorem [9] can be stated as

$$(2) \quad R_{(1,0,1)}(n) = 4 \sum_{d|n} \left(\frac{-4}{d} \right) \text{ for } n \geq 1,$$

where (\cdot) is the Jacobi symbol.

The problem of finding the number of representations of an integer by sums of squares has been studied by many mathematicians throughout history. Formulas for $R_{(1,0,2)}(n)$ and $R_{(1,0,3)}(n)$ are usually attributed respectively to Dirichlet and Lorenz.

$$(3) \quad R_{(1,0,2)}(n) = 2 \sum_{d|n} \left(\frac{-8}{d} \right),$$

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$$(4) \quad R_{(1,0,3)}(n) = 2 \sum_{d|n} \left(\frac{-3}{d} \right) + 4 \sum_{4d|n} \left(\frac{-3}{d} \right).$$

More details on the above formulas can be found in the accompanying notes to [3, Chpt. 3]. More recently, mathematicians have been interested in finding the number of representations in terms of polygonal numbers. The formula for the k -th polygonal number is given by

$$(5) \quad F_k(n) = \frac{n^2(k-2) - n(k-4)}{2} \quad \text{for } k \geq 3.$$

In particular,

$$(6) \quad F_3(n) = \frac{n^2 + n}{2}, \quad F_4(n) = n^2, \quad F_5(n) = \frac{3n^2 - n}{2} \quad \text{and} \quad F_7(n) = \frac{5n^2 - 3n}{2},$$

giving us respectively, the triangular, square, pentagonal and heptagonal numbers.

The results that we briefly survey below all concern formulas for the number of representations of n as a sum of a copies of a polygonal number and b copies of another polygonal number. In other words, they are all closely related to representations by binary quadratic forms.

In [5], Hirschhorn used elementary methods to prove 14 formulas for representations in terms of various combinations of triangular and square numbers. Subsequently, he [6] proved another 27 formulas, where each formula contained at least a pentagonal or an octagonal number. Baruah and Sarmah [1] used one of Ramanujan's theta function identity, namely [2, p. 48, Entry 31], to prove another 25 formulas. Each of their formulas contained at least a heptagonal, decagonal, hendecagonal, dodecagonal or octadecagonal number.

Meanwhile, in a series of papers [14] to [17], Sun used results on binary quadratic forms that he obtained together with Williams [18] to prove 191 formulas involving triangular, square and pentagonal numbers.

Independently, Melham released an unpublished manuscript [10] in 2007 that contained a total of 298 conjectured formulas involving polygonal numbers from triangular to dodecagonal numbers. It is interesting to note that none of the 298 conjectured formulas coincides with the 66 formulas proved in [1, 5, 6]. Melham [11] subsequently extracted and published 21 of these conjectures which involved triangular, pentagonal and heptagonal numbers. He noted in his paper that three of these conjectures involving triangular numbers were equivalent to formulas proved by Sun [15]. In fact, another of these conjectures involving the sum of one pentagonal number and five copies of another pentagonal number was already proved in [14]. A further eight can be found in [16] which appeared in print in 2011. So 12 of the 21 conjectures in [11], in addition to another 21 conjectures in [10] had in fact been proved by Sun. Toh [20] subsequently proved all 21 conjectures in [11]. He also described a uniform approach to proving the remaining 277 conjectures in [10], and in doing

so provided proofs for 13 of these 277 as examples. In Toh’s paper, he mentioned that Hirschhorn had already proved three of the 21 conjectures in [11]. Hirschhorn’s proofs had remained unpublished until recently, and they may now be found in [7, Chpt. 29]. Using Ramanujan’s ${}_1\psi_1$ summation formula, Sarmah [12] also proved three of the 21 conjectures in [11]. He concluded his paper by remarking that the rest of Melham’s conjectures may be formulated in terms of the ${}_1\psi_1$ summation formula but these “might be too complicated to actually have a proof.” We note that two of the three conjectures proved by Sarmah coincided with the three by Hirschhorn and these four conjectures were a subset of those proved by Sun. Finally Humby, in his unpublished Masters thesis [8], also proved all of Melham’s 21 conjectures in [11] by using the theory of modular forms.

In this paper, we focus on representations of integers by a combination of triangular or square numbers, and decagonal numbers. We shall denote these numbers respectively as

$$(7) \quad T(n) = F_3(-n) = \frac{n^2 - n}{2}, \quad S(n) = F_4(n), \quad D(n) = F_{10}(n) = 4n^2 - 3n,$$

and adopt the following notation

$$(8) \quad T_a D_b(n) = \left| \left\{ (x, y) \in \mathbb{Z}^+ \times \mathbb{Z} : n = a \left(\frac{x^2 - x}{2} \right) + b(4y^2 - 3y) \right\} \right|,$$

$$(9) \quad S_a D_b(n) = \left| \left\{ (x, y) \in \mathbb{Z} \times \mathbb{Z} : n = ax^2 + b(4y^2 - 3y) \right\} \right|,$$

where a and b are positive integers. It turns out that every formula for $T_a D_b(n)$ that we found has a companion formula for $T_a D'_b(n)$ where

$$(10) \quad T_a D'_b(n) = \left| \left\{ (x, y) \in \mathbb{Z}^+ \times \mathbb{Z} : n = a \left(\frac{x^2 - x}{2} \right) + b(4y^2 - y) \right\} \right|.$$

The same holds for every $S_a D_b(n)$ and the corresponding $S_a D'_b(n)$ defined by

$$(11) \quad S_a D'_b(n) = \left| \left\{ (x, y) \in \mathbb{Z} \times \mathbb{Z} : n = ax^2 + b(4y^2 - y) \right\} \right|.$$

We summarise our main findings below. First of all, we relate $T_a D_b(n)$ and $S_a D_b(n)$ to representations by binary quadratic forms.

Theorem 1.1. *Let $a, b, n \in \mathbb{Z}^+$. If b is odd, then*

$$4T_a D_b(n) = R_{(8a, 8a, 2a+b)}(16n + 2a + 9b).$$

The following is a consequence of Theorem 1.1.

Corollary 1.2. *If $4 \nmid a$ and b is odd, then*

$$4T_a D_b(n) = R_{(2a, 0, b)}(16n + 2a + 9b).$$

Likewise for $S_a D_b(n)$, we have:

Theorem 1.3. *Let $a, b, n \in \mathbb{Z}^+$. If both a and b are odd, and $4 \nmid (a - b)$, then*

$$2S_a D_b(n) = R_{(a, 0, b)}(16n + 9b).$$

We also have the following corresponding results for the companion functions $T_a D'_b(n)$ and $S_a D'_b(n)$.

Theorem 1.4. *Let $a, b, n \in \mathbb{Z}^+$. If b is odd, then*

$$4T_a D'_b(n) = R_{(8a, 8a, 2a+b)}(16n + 2a + b).$$

Corollary 1.5. *If b is odd, and $4 \nmid a$, then*

$$4T_a D'_b(n) = R_{(2a, 0, b)}(16n + 2a + b).$$

Theorem 1.6. *Let $a, b, n \in \mathbb{Z}^+$. If a and b are odd, and $4 \nmid (a - b)$, then*

$$2S_a D'_b(n) = R_{(a, 0, b)}(16n + b).$$

Corollary 1.2 and Theorem 1.3 allow us to find the explicit formulas for $T_a D_b(n)$ and $S_a D_b(n)$ through utilizing existing formulas for $R_{(a,b,c)}(n)$ from [19] and [4]. In total, we proved formulas for $T_a D_b(n)$ for 72 values of (a, b) and $S_a D_b(n)$ for 9 values of (a, b) . These include the seven cases already proved by Baruah and Sarmah [1] and all the eight conjectures for $T_a D_b(n)$ in [10, Chpt. 11]. The remaining 66 formulas are new. Likewise, all the 72 formulas for $T_a D'_b(n)$ and 9 formulas for $S_a D'_b(n)$ are also new. The complete list is given in the following table.

Formula	(a, b)	Location of formula
$T_a D_b(n)$ or $T_a D'_b(n)$	(1, 1)	Theorem 4.1 and [1, (37)]
	(1, p), (p , 1), $p = 3, 5, 11, 29$	Theorem 4.2 and [10, Chpt. 11]
	(2, 1)	Theorem 4.3 and [1, (30)]
	(1, 9), (9, 1)	Theorem 4.4
	(6, 1), (2, 3)	Theorem 4.5 and [1, (32), (29)]
	(14, 1), (2, 7)	Theorem 4.5
	(30, 1), (2, 15), (10, 3), (6, 5)	Theorem 4.6
	(1, 15), (3, 5), (5, 3), (15, 1)	Theorem 4.7
	(1, 21), (3, 7), (7, 3), (21, 1)	Theorem 4.8
	(1, 35), (5, 7), (7, 5), (35, 1)	Theorem 4.9
	(1, 39), (3, 13), (13, 3), (39, 1)	Theorem 4.10
	(1, 51), (3, 17), (17, 3), (51, 1)	Theorem 4.11
	(1, 65), (5, 13), (13, 5), (65, 1)	Theorem 4.12
	(1, 95), (5, 19), (19, 5), (95, 1)	Theorem 4.13
	(1, 105), (3, 35), (5, 21), (7, 15) (15, 7), (21, 5), (35, 3), (105, 1)	Theorem 4.14
	(1, 165), (3, 55), (5, 33), (11, 15) (15, 11), (33, 5), (55, 3), (165, 1)	Theorem 4.15
(1, 231), (3, 77), (7, 33), (11, 21) (21, 11), (33, 7), (77, 3), (231, 1)	Theorem 4.16	
$S_a D_b(n)$ or $S_a D'_b(n)$	(1, 1)	Theorem 4.17 and [1, (31)]
	(1, 3), (3, 1)	Theorem 4.18 and [1, (28), (33)]
	(1, 7), (7, 1)	Theorem 4.18
	(1, 15), (15, 1), (3, 5), (5, 3)	Theorem 4.19

In the next section, we recall known results required for our proofs. In Section 3, we prove Theorems 1.1 to 1.6. The 162 explicit formulas are presented in Section 4.

2. Preliminary results

In this section, we first recall some results from the literature. It is known that the associated L -series of a genus character of an imaginary quadratic field with discriminant d can be decomposed into a product of two Dirichlet L -series [13, p. 62, Th. 4]. Consequently, for certain quadratic forms $ax^2 + bxy + cy^2$ with discriminant d , it is possible to write $R_{(a,b,c)}(n)$ as a convolution of two divisor sums with characters [19]. Toh used this property to deduce $R_{(a,0,c)}(n)$ for 11 pairs of (a, c) which are associated with imaginary quadratic fields with class number 2 [19, p. 232].

Theorem 2.1 (Toh [19]). *If $p = 5, 13$ or 37 , then*

$$(12) \quad R_{(1,0,p)}(n) = \sum_{d|n} \left(\frac{-4p}{d}\right) + \sum_{d|n} \left(\frac{p}{d}\right) \left(\frac{-4}{n/d}\right).$$

If $p = 3, 5, 11$ or 29 , then

$$(13) \quad R_{(1,0,2p)}(n) = \sum_{d|n} \left(\frac{-8p}{d}\right) + \sum_{d|n} \left(\frac{d}{p}\right) \left(\frac{-\left(\frac{-1}{p}\right)2}{n/d}\right)$$

and

$$(14) \quad R_{(2,0,p)}(n) = \sum_{d|n} \left(\frac{-8p}{d}\right) - \sum_{d|n} \left(\frac{d}{p}\right) \left(\frac{-\left(\frac{-1}{p}\right)2}{n/d}\right).$$

We remark that formulas equivalent to Theorem 2.1 also can be found in [18]. In the following, we recall several theorems for $R_{(a,0,c)}(n)$ that were proved by Chan and Toh [4].

Theorem 2.2 (Theorem 2.1 from [4]). *The following identities hold.*

$$\begin{aligned} R_{(1,0,15)}(n) &= \sum_{d|n} \left(\frac{-15}{d}\right) + \sum_{d|n} \left(\frac{-3}{d}\right) \left(\frac{5}{n/d}\right) - 2 \sum_{2d|n} \left(\frac{-60}{d}\right) \\ &\quad + 2 \sum_{2d|n} \left(\frac{-3}{d}\right) \left(\frac{20}{n/(2d)}\right), \\ R_{(3,0,5)}(n) &= \sum_{d|n} \left(\frac{-15}{d}\right) - \sum_{d|n} \left(\frac{-3}{d}\right) \left(\frac{5}{n/d}\right) - 2 \sum_{2d|n} \left(\frac{-60}{d}\right) \\ &\quad - 2 \sum_{2d|n} \left(\frac{-3}{d}\right) \left(\frac{20}{n/(2d)}\right). \end{aligned}$$

Theorem 2.3 (Theorem 3.2 from [4]). *The following identity holds.*

$$R_{(1,0,4)}(n) = 2 \sum_{d|n} \left(\frac{4}{d}\right) \left(\frac{-4}{n/d}\right) + 4 \sum_{4d|n} \left(\frac{-4}{d}\right).$$

Theorem 2.4 (Theorem 4.2 from [4]). *The following identities hold.*

$$R_{(1,0,18)}(n) = \sum_{d|n} \left(\frac{9}{d}\right) \left(\frac{-72}{n/d}\right) + \sum_{d|n} \left(\frac{-3}{d}\right) \left(\frac{24}{n/d}\right) + 2 \sum_{9d|n} \left(\frac{-8}{d}\right),$$

$$R_{(2,0,9)}(n) = \sum_{d|n} \left(\frac{9}{d}\right) \left(\frac{-72}{n/d}\right) - \sum_{d|n} \left(\frac{-3}{d}\right) \left(\frac{24}{n/d}\right) + 2 \sum_{9d|n} \left(\frac{-8}{d}\right).$$

Theorem 2.5 (Theorem 5.1 from [4]). *If $p = 3$ or 7 , set $D = -p$, $N_3 = 6$ and $N_7 = 2$, then*

$$R_{(1,0,4p)}(n) = \sum_{d|n} \left(\frac{4}{d}\right) \left(\frac{4D}{n/d}\right) + 2 \sum_{4d|n} \left(\frac{4}{d}\right) \left(\frac{4D}{n/(4d)}\right) + N_p \sum_{16d|n} \left(\frac{D}{d}\right)$$

$$+ \sum_{d|n} \left(\frac{-4}{d}\right) \left(\frac{-4D}{n/d}\right),$$

$$R_{(4,0,p)}(n) = \sum_{d|n} \left(\frac{4}{d}\right) \left(\frac{4D}{n/d}\right) + 2 \sum_{4d|n} \left(\frac{4}{d}\right) \left(\frac{4D}{n/(4d)}\right) + N_p \sum_{16d|n} \left(\frac{D}{d}\right)$$

$$- \sum_{d|n} \left(\frac{-4}{d}\right) \left(\frac{-4D}{n/d}\right).$$

Theorem 2.6 (Theorem 5.3 from [4]). *The following identities hold.*

$$R_{(1,0,60)}(n) = \frac{1}{2} \left(1 + \left(\frac{-1}{n}\right)\right) \sum_{d|n} \left(\frac{4}{d}\right) \left(\frac{-60}{n/d}\right) + \sum_{4d|n} \left(\frac{4}{d}\right) \left(\frac{-60}{n/(4d)}\right)$$

$$+ \sum_{16d|n} \left(\frac{-15}{d}\right) + \frac{1}{2} \left(1 + \left(\frac{-1}{n}\right)\right) \sum_{d|n} \left(\frac{-12}{d}\right) \left(\frac{20}{n/d}\right)$$

$$+ \sum_{4d|n} \left(\frac{-12}{d}\right) \left(\frac{20}{n/(4d)}\right) + \sum_{16d|n} \left(\frac{-3}{d}\right) \left(\frac{5}{n/(16d)}\right),$$

$$R_{(3,0,20)}(n) = \frac{1}{2} \left(1 - \left(\frac{-1}{n}\right)\right) \sum_{d|n} \left(\frac{4}{d}\right) \left(\frac{-60}{n/d}\right) + \sum_{4d|n} \left(\frac{4}{d}\right) \left(\frac{-60}{n/(4d)}\right)$$

$$+ \sum_{16d|n} \left(\frac{-15}{d}\right) - \frac{1}{2} \left(1 - \left(\frac{-1}{n}\right)\right) \sum_{d|n} \left(\frac{-12}{d}\right) \left(\frac{20}{n/d}\right)$$

$$- \sum_{4d|n} \left(\frac{-12}{d}\right) \left(\frac{20}{n/(4d)}\right) - \sum_{16d|n} \left(\frac{-3}{d}\right) \left(\frac{5}{n/(16d)}\right),$$

$$\begin{aligned}
 R_{(4,0,15)}(n) &= \frac{1}{2} \left(1 - \left(\frac{-1}{n} \right) \right) \sum_{d|n} \left(\frac{4}{d} \right) \left(\frac{-60}{n/d} \right) + \sum_{4d|n} \left(\frac{4}{d} \right) \left(\frac{-60}{n/(4d)} \right) \\
 &\quad + \sum_{16d|n} \left(\frac{-15}{d} \right) + \frac{1}{2} \left(1 - \left(\frac{-1}{n} \right) \right) \sum_{d|n} \left(\frac{-12}{d} \right) \left(\frac{20}{n/d} \right) \\
 &\quad + \sum_{4d|n} \left(\frac{-12}{d} \right) \left(\frac{20}{n/(4d)} \right) + \sum_{16d|n} \left(\frac{-3}{d} \right) \left(\frac{5}{n/(16d)} \right), \\
 R_{(5,0,12)}(n) &= \frac{1}{2} \left(1 + \left(\frac{-1}{n} \right) \right) \sum_{d|n} \left(\frac{4}{d} \right) \left(\frac{-60}{n/d} \right) + \sum_{4d|n} \left(\frac{4}{d} \right) \left(\frac{-60}{n/(4d)} \right) \\
 &\quad + \sum_{16d|n} \left(\frac{-15}{d} \right) - \frac{1}{2} \left(1 + \left(\frac{-1}{n} \right) \right) \sum_{d|n} \left(\frac{-12}{d} \right) \left(\frac{20}{n/d} \right) \\
 &\quad - \sum_{4d|n} \left(\frac{-12}{d} \right) \left(\frac{20}{n/(4d)} \right) - \sum_{16d|n} \left(\frac{-3}{d} \right) \left(\frac{5}{n/(16d)} \right).
 \end{aligned}$$

We also require a formula for $R_{(1,0,7)}(n)$ which was known to Ramanujan [2, p. 302].

$$(15) \quad R_{(1,0,7)}(n) = 2 \sum_{d|n} \left(\frac{-7}{d} \right) - 4 \sum_{2d|n} \left(\frac{-28}{d} \right).$$

We end this section with the following lemma from Sun [16, Lemma 2.2].

Lemma 2.7. *Let $a, b, n \in \mathbb{Z}^+$, with $2 \nmid n$.*

i) *If $2 \nmid a$ and $4 \nmid (a - b)b$, then*

$$(16) \quad R_{(a,0,4b)}(n) = \begin{cases} R_{(a,0,b)}(n), & \text{if } n \equiv a \pmod{4}; \\ 0, & \text{otherwise.} \end{cases}$$

ii) *If $2 \nmid a$, $2 \mid b$ and $8 \nmid b$, then*

$$(17) \quad R_{(a,0,4b)}(n) = \begin{cases} R_{(a,0,b)}(n), & \text{if } n \equiv a \pmod{8}; \\ 0, & \text{otherwise.} \end{cases}$$

iii) *If $2 \nmid (a + b)$ and $8 \nmid ab$, then*

$$(18) \quad R_{(4a,4a,a+b)}(n) = \begin{cases} R_{(a,0,b)}(n), & \text{if } n \equiv a + b \pmod{8}; \\ 0, & \text{otherwise.} \end{cases}$$

3. Relating $T_a D_b(n)$ and $S_a D_b(n)$ to $R_{(a,b,c)}(n)$

In this section, we prove our main results by relating $T_a D_b(n)$ and $S_a D_b(n)$ to $R_{(a,b,c)}(n)$.

Proof of Theorem 1.1. Since $T(n + 1) = T(-n)$, we have

$$\begin{aligned}
 4T_a D_b(n) &= 2 \left| \left\{ (x, y) \in \mathbb{Z}^2 : n = a \left(\frac{x^2 - x}{2} \right) + b(4y^2 - 3y) \right\} \right| \\
 &= 2 \left| \left\{ (x, y) \in \mathbb{Z}^2 : 16n + 2a + 9b = 2a(2x - 1)^2 + b(8y - 3)^2 \right\} \right| \\
 (19) \quad &= \left| \left\{ (x, y) \in \mathbb{Z}^2 : 16n + 2a + 9b = 2ax^2 + by^2, \right. \right. \\
 &\quad \left. \left. 2 \nmid x, y \equiv \pm 3 \pmod{8} \right\} \right|.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 (20) \quad 4T_a D'_b(n) &= \left| \left\{ (x, y) \in \mathbb{Z}^2 : 16n + 2a + b = 2ax^2 + by^2, \right. \right. \\
 &\quad \left. \left. 2 \nmid x, y \equiv \pm 1 \pmod{8} \right\} \right|.
 \end{aligned}$$

Combining (19) and (20) gives us

$$\begin{aligned}
 &4T_a D_b(n) + 4T_a D'_b \left(n + \frac{b}{2} \right) \\
 &= \left| \left\{ (x, y) \in \mathbb{Z}^2 : 16n + 2a + 9b = 2ax^2 + by^2, 2 \nmid xy \right\} \right| \\
 &= \left| \left\{ (x, y) \in \mathbb{Z}^2 : 16n + 2a + 9b = 2ax^2 + by^2, 2 \mid (x - y) \right\} \right| \\
 &\quad - \left| \left\{ (x, y) \in \mathbb{Z}^2 : 16n + 2a + 9b = 2ax^2 + by^2, 2 \mid x, 2 \mid y \right\} \right| \\
 &= \left| \left\{ (x, y) \in \mathbb{Z}^2 : 16n + 2a + 9b = 2a(2x + y)^2 + by^2 \right\} \right| \\
 &\quad - \left| \left\{ (x, y) \in \mathbb{Z}^2 : 16n + 2a + 9b = 2a(2x)^2 + b(2y)^2 \right\} \right| \\
 &= \left| \left\{ (x, y) \in \mathbb{Z}^2 : 16n + 2a + 9b = 8ax^2 + 8axy + (2a + b)y^2 \right\} \right| \\
 &\quad - \left| \left\{ (x, y) \in \mathbb{Z}^2 : 16n + 2a + 9b = 8ax^2 + 4by^2 \right\} \right| \\
 (21) \quad &= R_{(8a, 8a, 2a+b)}(16n + 2a + 9b) - R_{(8a, 0, 4b)}(16n + 2a + 9b).
 \end{aligned}$$

When b is odd, we get $T_a D'_b(n + \frac{b}{2}) = 0$ and there are no solutions for

$$16n + 2a + 9b = 8ax^2 + 4by^2,$$

which implies $R_{(8a, 0, 4b)}(16n + 2a + 9b)$ equals zero. This completes the proof. \square

We now deduce Corollary 1.2.

Proof. From Theorem 1.1, when b is odd we have

$$4T_a D_b(n) = R_{(8a, 8a, 2a+b)}(16n + 2a + 9b).$$

If we further assume $4 \nmid a$, then $2 \nmid (2a + b)$, $8 \nmid (2a)b$ and

$$16n + 2a + 9b \equiv 2a + b \pmod{8}.$$

By (18),

$$R_{(4(2a), 4(2a), 2a+b)}(16n + 2a + 9b) = R_{(2a, 0, b)}(16n + 2a + 9b)$$

and the proof follows. \square

The proofs of Theorem 1.4 and Corollary 1.5 for $T_a D'_b(n)$ follow in an analogous manner. We now proceed to the proof of Theorem 1.3.

Proof. We have

$$\begin{aligned}
 2S_a D_b(n) &= 2 \left| \{ (x, y) \in \mathbb{Z}^2 : n = ax^2 + b(4y^2 - 3y) \} \right| \\
 &= 2 \left| \{ (x, y) \in \mathbb{Z}^2 : 16n + 9b = 16ax^2 + b(8y - 3)^2 \} \right| \\
 (22) \qquad &= \left| \{ (x, y) \in \mathbb{Z}^2 : 16n + 9b = 16ax^2 + by^2, y \equiv \pm 3 \pmod{8} \} \right|.
 \end{aligned}$$

Similarly,

$$(23) \quad 2S_a D'_b(n) = \left| \{ (x, y) \in \mathbb{Z}^2 : 16n + b = 16ax^2 + by^2, y \equiv \pm 1 \pmod{8} \} \right|.$$

Combining (22) and (23) gives us

$$\begin{aligned}
 &2S_a D_b(n) + 2S_a D'_b \left(n + \frac{b}{2} \right) \\
 &= \left| \{ (x, y) \in \mathbb{Z}^2 : 16n + 9b = 16ax^2 + by^2, 2 \nmid y \} \right| \\
 &= \left| \{ (x, y) \in \mathbb{Z}^2 : 16n + 9b = 16ax^2 + by^2 \} \right| \\
 &\quad - \left| \{ (x, y) \in \mathbb{Z}^2 : 16n + 2a + 9b = 2ax^2 + b(2y)^2 \} \right| \\
 (24) \qquad &= R_{(16a,0,b)}(16n + 9b) - R_{(16a,0,4b)}(16n + 9b).
 \end{aligned}$$

When b is odd, there are no solutions for

$$16n + 9b = 16ax^2 + 4by^2,$$

which implies $R_{(16a,0,4b)}(16n + 9b)$ equals zero. In other words,

$$(25) \qquad 2S_a D_b(n) = R_{(16a,0,b)}(16n + 9b).$$

If we further assume that a is also odd, then all the conditions of (17) are satisfied and we get

$$R_{(b,0,4(4a))}(16n + 9b) = R_{(b,0,4a)}(16n + 9b).$$

We can simplify further by adding another assumption that $4 \nmid (b - a)$. Then by (16) we get

$$R_{(b,0,4a)}(16n + 9b) = R_{(b,0,a)}(16n + 9b),$$

which completes the proof. □

4. Explicit formulas

In this section, we present the explicit formulas for $T_a D_b(n)$ and $S_a D_b(n)$. As mentioned in the introduction, each formula has a companion formula. The formulas for $T_a D_b(n)$ are stated as a divisor sum for $m = 16n + 2a + 9b$. To obtain the companion formula for $T_a D'_b(n)$, one simply replaces m by $m' = 16n + 2a + b$.

Theorem 4.1. *Let $m = 16n + 11$. Then*

$$2T_1D_1(n) = \sum_{d|m} \left(\frac{-8}{d}\right).$$

Proof. From Corollary 1.2, we have

$$4T_1D_1(n) = R_{(2,0,1)}(16n + 11),$$

and the result follows from Dirichlet’s formula (3). □

Theorem 4.2. *If $p = 3, 5, 11$ or 29 , then*

$$2T_1D_p(n) = \sum_{d|m} \left(\frac{-8p}{d}\right), \text{ where } m = 16n + 2 + 9p;$$

$$2T_pD_1(n) = \sum_{d|m} \left(\frac{-8p}{d}\right), \text{ where } m = 16n + 2p + 9.$$

Proof. We consider the case $p = 3$ or 11 . Corollary 1.2 and (14) give,

$$\begin{aligned} 4T_1D_p(n) &= R_{(2,0,p)}(16n + 9p + 2) \\ &= \sum_{d|16n+9p+2} \left(\frac{-8p}{d}\right) - \sum_{d|16n+9p+2} \left(\frac{d}{p}\right) \left(\frac{2}{d}\right)^2 \left(\frac{2}{\frac{16n+9p+2}{d}}\right) \\ &= \sum_{d|16n+9p+2} \left(\frac{-8p}{d}\right) - \sum_{d|16n+9p+2} \left(\frac{-2p}{d}\right) \left(\frac{2}{16n + 9p + 2}\right) \\ &= \sum_{d|16n+9p+2} \left(\frac{-8p}{d}\right) \left(1 - \left(\frac{2}{16n + 9p + 2}\right)\right) \\ &= 2 \sum_{d|16n+9p+2} \left(\frac{-8p}{d}\right). \end{aligned}$$

The other cases can be proved in a similar manner. □

The following theorem is obtained from Corollary 1.2 and Theorem 2.3.

Theorem 4.3. *Let $m = 16n + 13$. Then*

$$2T_2D_1(n) = \sum_{d|m} \left(\frac{-4}{d}\right).$$

The following theorem is an immediate consequence from Corollary 1.2 and Theorem 2.4.

Theorem 4.4. *If $(a, b, \alpha) = (1, 9, -1)$ or $(9, 1, 1)$, then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{9}{d}\right) \left(\frac{-72}{m/d}\right) + \alpha \sum_{d|m} \left(\frac{-3}{d}\right) \left(\frac{24}{m/d}\right) + 2 \sum_{9d|m} \left(\frac{-8}{d}\right),$$

where $m = 16n + 2a + 9b$.

By Theorem 2.5, we can employ computations similar to those in the proof of Theorem 4.2 to obtain the following.

Theorem 4.5. *If $p = 3$ or 7 , then*

$$2T_{2p}D_1(n) = \sum_{d|m} \left(\frac{-p}{d}\right), \text{ where } m = 16n + 4p + 9;$$

$$2T_2D_p(n) = \sum_{d|m} \left(\frac{-p}{d}\right), \text{ where } m = 16n + 4 + 9p.$$

Likewise, by Theorem 2.6 we can prove the following.

Theorem 4.6. *If $(a, b, \alpha) = (30, 1, 1), (2, 15, 1), (10, 3, -1)$ or $(6, 5, -1)$, then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{4}{d}\right) \left(\frac{-60}{m/d}\right) + \alpha \sum_{d|m} \left(\frac{-12}{d}\right) \left(\frac{20}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

The proofs of the remaining theorems follow in an analogous manner by using Corollary 1.2 and the corresponding $R_{(a,b,c)}(n)$ formulas in [19] or [4].

Theorem 4.7. *If $(a, b, \alpha) = (15, 1, 1), (5, 3, 1), (3, 5, -1)$ or $(1, 15, -1)$, then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{-120}{d}\right) + \alpha \sum_{d|m} \left(\frac{10}{d}\right) \left(\frac{-3}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.8. *If $(a, b, \alpha) = (21, 1, 1), (3, 7, 1), (7, 3, -1)$ or $(1, 21, -1)$, then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{-168}{d}\right) + \alpha \sum_{d|m} \left(\frac{14}{d}\right) \left(\frac{-3}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.9. *If $(a, b, \alpha) = (35, 1, 1), (7, 5, 1), (5, 7, -1)$ or $(1, 35, -1)$, then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{-280}{d}\right) + \alpha \sum_{d|m} \left(\frac{-14}{d}\right) \left(\frac{5}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.10. *If $(a, b, \alpha) = (39, 1, 1), (3, 13, 1), (13, 3, -1)$ or $(1, 39, -1)$, then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{-312}{d}\right) + \alpha \sum_{d|m} \left(\frac{26}{d}\right) \left(\frac{-3}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.11. *If $(a, b, \alpha) = (51, 1, 1), (17, 3, 1), (3, 17, -1)$ or $(1, 51, -1)$, then*

$$4T_a D_b(n) = \sum_{d|m} \left(\frac{-408}{d} \right) + \sum_{d|m} \left(\frac{34}{d} \right) \left(\frac{-3}{m/d} \right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.12. *If $(a, b, \alpha) = (65, 1, 1), (13, 5, 1), (5, 13, -1)$ or $(1, 65, -1)$, then*

$$4T_a D_b(n) = \sum_{d|m} \left(\frac{-520}{d} \right) + \sum_{d|m} \left(\frac{-26}{d} \right) \left(\frac{5}{m/d} \right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.13. *If $(a, b, \alpha) = (95, 1, 1), (5, 19, 1), (19, 5, -1)$ or $(1, 95, -1)$, then*

$$4T_a D_b(n) = \sum_{d|m} \left(\frac{-760}{d} \right) + \sum_{d|m} \left(\frac{-38}{d} \right) \left(\frac{5}{m/d} \right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.14. *Let $m = 16n + 2a + 9b$. Then*

$$\begin{aligned} 8T_a D_b(n) = & \sum_{d|m} \left(\frac{-840}{d} \right) + \alpha_2 \sum_{d|m} \left(\frac{280}{d} \right) \left(\frac{-3}{m/d} \right) \\ & + \alpha_3 \sum_{d|m} \left(\frac{-168}{d} \right) \left(\frac{5}{m/d} \right) + \alpha_4 \sum_{d|m} \left(\frac{120}{d} \right) \left(\frac{-7}{m/d} \right), \end{aligned}$$

where the values for a, b, α_i are listed in the following table.

a	b	α_2	α_3	α_4
1	105	-1	-1	1
3	35	-1	1	-1
5	21	1	1	-1
7	15	-1	1	1
15	7	1	-1	1
21	5	-1	-1	-1
35	3	1	-1	-1
105	1	1	1	1

Theorem 4.15. *Let $m = 16n + 2a + 9b$. Then*

$$\begin{aligned} 8T_a D_b(n) = & \sum_{d|m} \left(\frac{-1320}{d} \right) + \alpha_2 \sum_{d|m} \left(\frac{440}{d} \right) \left(\frac{-3}{m/d} \right) \\ & + \alpha_3 \sum_{d|m} \left(\frac{-264}{d} \right) \left(\frac{5}{m/d} \right) + \alpha_4 \sum_{d|m} \left(\frac{120}{d} \right) \left(\frac{-11}{m/d} \right), \end{aligned}$$

where the values for a, b, α_i are listed in the following table.

a	b	α_2	α_3	α_4
1	165	-1	-1	-1
3	55	1	1	-1
5	33	1	-1	-1
11	15	1	-1	1
15	11	-1	1	-1
33	5	-1	1	1
55	3	-1	-1	1
165	1	1	1	1

Theorem 4.16. *Let $m = 16n + 2a + 9b$. Then*

$$8T_a D_b(n) = \sum_{d|m} \left(\frac{-1848}{d}\right) + \alpha_2 \sum_{d|m} \left(\frac{616}{d}\right) \left(\frac{-3}{m/d}\right) + \alpha_3 \sum_{d|m} \left(\frac{264}{d}\right) \left(\frac{-7}{m/d}\right) + \alpha_4 \sum_{d|m} \left(\frac{168}{d}\right) \left(\frac{-11}{m/d}\right),$$

where the values for a, b, α_i are listed in the following table.

a	b	α_2	α_3	α_4
1	231	-1	1	-1
3	77	-1	-1	-1
7	33	-1	-1	1
11	21	1	1	-1
21	11	-1	1	1
33	7	1	-1	-1
77	3	1	-1	1
231	1	1	1	1

Formulas for $S_a D_b(n)$ and $S_a D'_b(n)$ are presented in the next three theorems. Although we only list the formulas for $S_a D_b(n)$ with divisors sum over $m = 16n + 9b$, each entry has a corresponding formula for $S_a D'_b(n)$ where m is replaced by $m' = 16n + b$.

Theorem 4.17. *Let $m = 16n + 9$. Then*

$$S_1 D_1(n) = \sum_{d|m} \left(\frac{-4}{d}\right).$$

Proof. From (25), we have

$$2S_1 D_1(n) = R_{(1,0,16)}(16n + 9).$$

From (17), we get

$$R_{(1,0,16)}(16n + 9) = R_{(1,0,4)}(16n + 9).$$

The proof follows from Theorem 2.3. \square

The next theorem follows from Theorem 1.3 and (4) and (15).

Theorem 4.18. *If $p = 3$ or 7 , then*

$$S_1 D_p(n) = \sum_{d|m} \left(\frac{-p}{d} \right), \text{ where } m = 16n + 9p;$$

$$S_p D_1(n) = \sum_{d|m} \left(\frac{-p}{d} \right), \text{ where } m = 16n + 9.$$

The following theorem is an immediate consequence of Theorem 1.3 and Theorem 2.2.

Theorem 4.19. *If $(a, b, \alpha) = (1, 15, 1), (15, 1, 1), (3, 5, -1)$ or $(5, 3, -1)$, then*

$$2S_a D_b(n) = \sum_{d|m} \left(\frac{-15}{d} \right) + \alpha \sum_{d|m} \left(\frac{-3}{d} \right) \left(\frac{5}{m/d} \right),$$

where $m = 16n + 9b$.

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