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PAPER

Efficient fusion of photonic W -states with nonunitary partial-swap gates

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Hai-Rui Wei^{1,5}, Wen-Qiang Liu¹ and Leong-Chuan Kwek^{2,3,4}¹ School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, People's Republic of China² Centre for Quantum Technologies, National University of Singapore, Singapore 117543, Singapore³ MajuLab, CNRS-UNS-NUS-NTU International Joint Research Unit, Singapore UMI 3654, Singapore⁴ National Institute of Education and Institute of Advanced Studies, Nanyang Technological University, Singapore 637616, Singapore⁵ Author to whom any correspondence should be addressed.E-mail: hrwei@ustb.edu.cn**Keywords:** W -state fusion, multipartite entanglement, quantum gate**Abstract**

We introduce a nonunitary partial-swap gate for fusing arbitrary small-sized photonic W -states into a large-scale entangled network of W -state efficiently without ancillary photons. A partial-swap gate is designed in an optical architecture based on linear optics elements. By introducing auxiliary degree of freedom, this gate provides a higher success probability with less cost. Our implementation can create a larger target state with a simpler set-up than previous proposals for W -state fusion. Also all 'garbage' states are recyclable, i.e., there is no complete failure output in our scheme in principle.

1. Introduction

Quantum entanglement is a key resource in many quantum information processing tasks [1], such as quantum teleportation [2, 3], quantum superdense coding [4], quantum key distribution [5], quantum algorithms [6], and measurement-based quantum computation [7]. In particular, depending on the requirement of a specific task, the preparation and the manipulation of multiqubit entangled states with different multipartite features (for instance, the Greenberger *et al* [8], Dicke [9], W [10], and cluster states [11]) are needed, and these different types of entangled states cannot be converted into each other by using local operations and classical communications. However, there are still theoretical and experimental challenges in study of multipartite entanglement due to the more complex mathematical form and rapidly growing resource overhead with the number of particles increasing. In recent years, the weblike property of the W -state, due to its robustness against particle loss and decoherence effects, has rendered it to be a useful resource for quantum communication [12]. Indeed, the W -state has been shown to be relevant for many schemes and applications ranging from its use in the foundation of quantum mechanics [13–15], in anonymous quantum networks [16], in quantum telecloning and teleportation [17], in quantum computation [18, 19], in quantum memories [20] and as a probe for reading information [21].

So far, many theoretical proposals and realistic experiments for generating small-size W states have been proposed [22–24]. Currently, there are two efficient ways to generate large-scale photonic W states: expansion and fusion. Both schemes have now been achieved in a wide range of physical settings [25–28]. In 2008, Tashima *et al* [25] proposed a scheme for locally expanding any polarization-based n -photon W ($|W_n\rangle$) state to an $(n+2)$ -photon W ($|W_{n+2}\rangle$) state by accessing just one of the qubits with a success probability of $(n+2)/(16n)$. This scheme was subsequently demonstrated experimentally in 2010 [29]. Schemes for expanding $|W_n\rangle$ locally to $|W_{n+1}\rangle$ with a success probability of $(n+1)/(5n)$ were also proposed in 2009 [30], and even one for $|W_n\rangle$ to $|W_{n+k}\rangle$ was proposed in 2011 [31]. Notably, the success probability of the expansion from $|W_n\rangle$ to $|W_{n+k}\rangle$ decreases with an approximately exponential dependence with increasing n .

Fusion, on the other hand, was first proposed in 2011 by Özdemir *et al* [32]. The idea was to join $|W_n\rangle$ and $|W_m\rangle$ to give the $|W_{n+m-2}\rangle$ with a success probability of $(n+m-2)/(nm)$. In 2013, enhancement the

W -state fusion process was proposed through the use of a Fredkin gate, Bugu *et al* [33] then proposed a mechanism to fuse $|W_n\rangle$ and $|W_m\rangle$ with one ancillary single photon into $|W_{n+m-1}\rangle$ with a success probability of $(n+m-1)/(mn)$. In 2014, Ozaydin *et al* [34] generalized the setup for fusing three W states: $|W_n\rangle$, $|W_m\rangle$, and $|W_t\rangle$ states and one ancillary single photon were joined into $|W_{n+m+t-3}\rangle$ with a success probability of $(n+m+t-3)/(mnt)$ with a Fredkin gate. Using three CNOT gates and one Toffoli gate, Yesilyurt *et al* [35] further generalized the scheme for creating $|W_{n+m+t+z-4}\rangle$ from $|W_n\rangle$, $|W_m\rangle$, $|W_z\rangle$, and $|W_t\rangle$ states with a success probability of $(n+m+t+z-4)/(mntz)$. However, the success probabilities of the required CNOT [36–39], Toffoli [40–44], and Fredkin [45–49] gates with linear optical elements were generally not considered. Currently, nonlinear fusion schemes for fusing $|W_n\rangle$ and $|W_m\rangle$ into $|W_{n+m}\rangle$ without qubit loss have also been proposed [50, 51].

In this paper, we propose a protocol for fusing W states of arbitrary size into a larger one via nonunitary partial-swap gates. By introducing auxiliary spatial degrees of freedom and using $(n-1)$ partial-swap gates, a $|W_{nm-n+1}\rangle$ state can be created from n arbitrary-sized $|W_m\rangle$ states. All the ‘garbage’ states are recyclable, and our scheme avoids failed outcomes. Moreover, additional ancillary photon is not required for our scheme. The length (cost or complexity) of our scheme (measured by the number of the two-qubit entangling gates needed to construct the scheme) is much less than the Fredkin- and CNOT-Toffoli-based schemes [33, 35].

2. Simplifying a fusion-based W state with nonunitary partial-swap gate

2.1. Fusion of $|W_n\rangle$ and $|W_m\rangle$ to give $|W_{n+m-1}\rangle$

Suppose Alice and Bob possess n - and m -partite polarization encoded W states, $|W_n\rangle_A$ and $|W_m\rangle_B$, respectively, and they wish to fuse their states together. A schematic example for the fusion process of two three-partite W -states is depicted in figure 1. The entangled W -states of Alice ($|W_n\rangle_A$) and Bob ($|W_m\rangle_B$) can be written as

$$|W_n\rangle_A = (|(n-1)_H\rangle_a |1_V\rangle_1 + \sqrt{n-1} |W_{n-1}\rangle_a |1_H\rangle_1) / \sqrt{n}, \quad (1)$$

$$|W_m\rangle_B = (|(m-1)_H\rangle_b |1_V\rangle_2 + \sqrt{m-1} |W_{m-1}\rangle_b |1_H\rangle_2) / \sqrt{m}, \quad (2)$$

where $|(N-k)_H\rangle_i |k_V\rangle_j$ represents the superposition of all possible permutations of $N-k$ photons with a horizontal polarization (H) in mode i and k photons with a vertical polarization (V) in mode j . Capital letters A and B label the W states are held by Alice and Bob, respectively. Therefore, the initial state of the system composed of Alice and Bob can be written as

$$\begin{aligned} |W_n\rangle_A \otimes |W_m\rangle_B &= \frac{1}{\sqrt{nm}} |(n-1)_H\rangle_a |(m-1)_H\rangle_b |1_V\rangle_1 |1_V\rangle_2 + \frac{\sqrt{m-1}}{\sqrt{nm}} |(n-1)_H\rangle_a |W_{m-1}\rangle_b |1_V\rangle_1 |1_H\rangle_2 \\ &+ \frac{\sqrt{n-1}}{\sqrt{nm}} |W_{n-1}\rangle_a |(m-1)_H\rangle_b |1_H\rangle_1 |1_V\rangle_2 + \frac{\sqrt{(n-1)(m-1)}}{\sqrt{nm}} |W_{n-1}\rangle_a |W_{m-1}\rangle_b |1_H\rangle_1 |1_H\rangle_2. \end{aligned} \quad (3)$$

As shown in figure 2, the fusion of $|W_m\rangle_A$ and $|W_n\rangle_B$ states into a larger W state is achieved by sending photons in mode 1 (2), i.e., $|1_H\rangle_1$ and $|1_V\rangle_1$ ($|1_H\rangle_2$ and $|1_V\rangle_2$), into the partial-swap gate and those in mode a (b) are kept intact at Alice’s (Bob’s) site. Note that the partial-swap gate swaps the states of the two photons if the first photon is H polarization, and has no effect otherwise (see table 1). That is, the action of this gate on the four input states yields

$$\begin{aligned} |1_H\rangle_1 |1_H\rangle_2 &\xrightarrow{\text{p-swap}} |1_H\rangle_1 |1_H\rangle_2, & |1_H\rangle_1 |1_V\rangle_2 &\xrightarrow{\text{p-swap}} |1_V\rangle_1 |1_H\rangle_2, \\ |1_V\rangle_1 |1_H\rangle_2 &\xrightarrow{\text{p-swap}} |1_V\rangle_1 |1_H\rangle_2, & |1_V\rangle_1 |1_V\rangle_2 &\xrightarrow{\text{p-swap}} |1_V\rangle_1 |1_V\rangle_2. \end{aligned} \quad (4)$$

Therefore, such partial-swap gate completes the transformation

$$\begin{aligned} |W_n\rangle_A \otimes |W_m\rangle_B &\rightarrow \frac{1}{\sqrt{nm}} |(n-1)_H\rangle_a |(m-1)_H\rangle_b |1_V\rangle_1 |1_V\rangle_2 + \frac{\sqrt{m-1}}{\sqrt{nm}} |(n-1)_H\rangle_a |W_{m-1}\rangle_b |1_V\rangle_1 |1_H\rangle_2 \\ &+ \frac{\sqrt{n-1}}{\sqrt{nm}} |W_{n-1}\rangle_a |(m-1)_H\rangle_b |1_V\rangle_1 |1_H\rangle_2 + \frac{\sqrt{(n-1)(m-1)}}{\sqrt{nm}} |W_{n-1}\rangle_a |W_{m-1}\rangle_b |1_H\rangle_1 |1_H\rangle_2 \\ &= \frac{\sqrt{n+m-1}}{\sqrt{nm}} |W_{n+m-1}\rangle_{a,b,2} |1_V\rangle_1 + \frac{\sqrt{(n-1)(m-1)}}{\sqrt{nm}} |W_{n-1}\rangle_a |W_{m-1}\rangle_b |1_H\rangle_1 |1_H\rangle_2. \end{aligned} \quad (5)$$

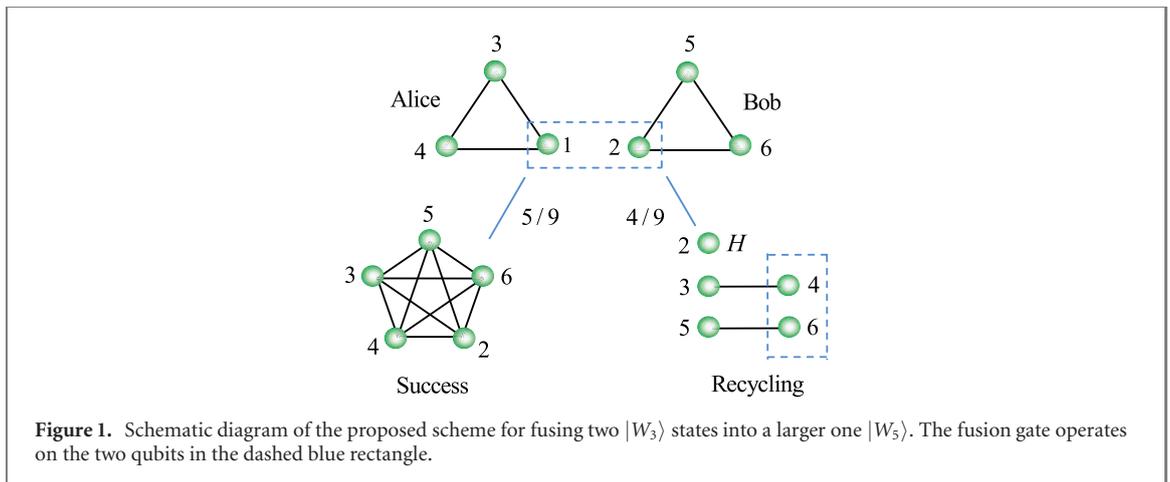


Figure 1. Schematic diagram of the proposed scheme for fusing two $|W_3\rangle$ states into a larger one $|W_5\rangle$. The fusion gate operates on the two qubits in the dashed blue rectangle.

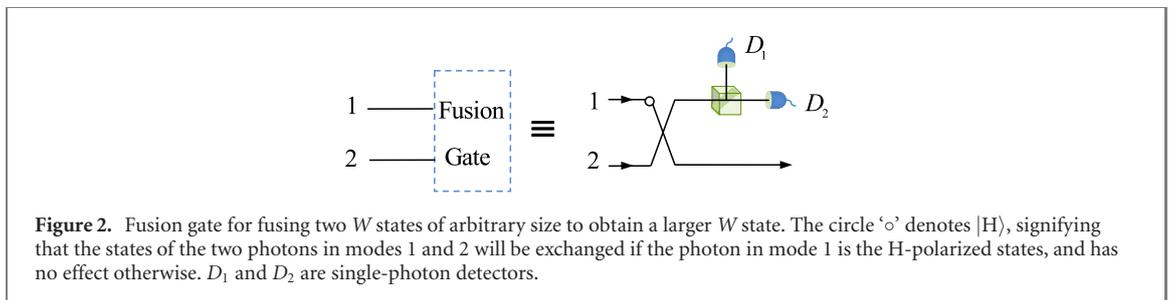


Figure 2. Fusion gate for fusing two W states of arbitrary size to obtain a larger W state. The circle ‘o’ denotes $|H\rangle$, signifying that the states of the two photons in modes 1 and 2 will be exchanged if the photon in mode 1 is the H-polarized states, and has no effect otherwise. D_1 and D_2 are single-photon detectors.

Table 1. Truth table of the polarization partial-swap gate.

x_1x_2	y_1y_2
H_1H_2	H_1H_2
H_1V_2	V_1H_2
V_1H_2	V_1H_2
V_1V_2	V_1V_2

The photon in mode 1 is then measured in the $\{|H\rangle, |V\rangle\}$ basis. From equation (5), one sees that (i) When the photon in mode 1 is V-polarized and detector D_1 clicks, the scheme is successful with probability (success probability) of $(n + m - 1)/(nm)$, and the system collapses into the desired state

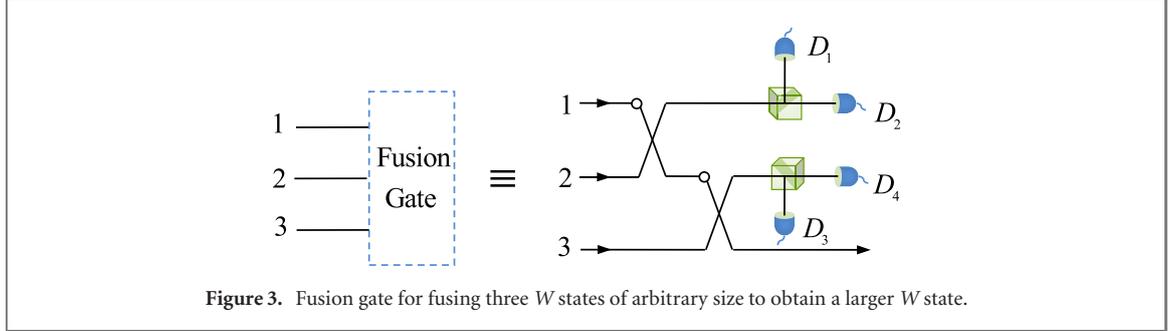
$$\frac{\sqrt{n + m - 1}}{\sqrt{nm}} |W_{n+m-1}\rangle_{a,b,2}. \tag{6}$$

(ii) When the photon in mode 1 is H-polarized state and detector D_2 clicks, then the remaining photon collapses into state $|W_{n-1}\rangle_a \otimes |W_{m-1}\rangle_b \otimes |1_H\rangle_2$ with probability (recycle probability) of $(n - 1)(m - 1)/(nm)$. It is interesting to see that the ‘garbage’ state $|W_{n-1}\rangle_a \otimes |W_{m-1}\rangle_b$ of Alice and Bob remains a W state but with a reduced number of qubits, and therefore this state can be recycled, much like a repeat-until-success scheme [52]. Remarkably, the fail probability of the designed scheme is zero in principle as the system can not collapse into the failure states, such as $|(n - 1)_H\rangle_a |(m - 1)_H\rangle_b |1_V\rangle_2$.

In table 2, we compare our scheme with previous protocols. Here the success probabilities of the Fredkin, Toffoli, CNOT, and partial-swap gates are disregarded. The linear optical entangling gates are inherently probabilistic. The optimal cost of a Fredkin or Toffoli gate is five two-qubit gates [53, 54], therefore, the complexity of our partial-swap-based scheme is much lower than the Fredkin-based one [33] and the Toffoli-CNOT-based one [55]. Moreover, extra ancillary photon is necessary for the schemes presented in references [33, 55], and is not required in our scheme. Remarkably, our protocol is less complex with a higher success probability than any of the other protocols for the generation of a larger W state with the same size [32, 33, 55].

Table 2. Quantum resource required and success probability of various protocols for creating a larger W state. H is an extra ancillary H-polarized photon required for the fusion program.

Proposed protocol	Initial state	Success result	Success probability	Recycle probability	Fail probability
With I [32]	W_m, W_n	W_{m+n-2}	$\frac{m+n-2}{mn}$	$\frac{(m-1)(n-1)}{mn}$	$\frac{1}{mn}$
With 1 Fredkin [33]	W_m, W_n, H	W_{m+n-1}	$\frac{m+n-1}{mn}$	$\frac{(m-1)(n-1)}{mn}$	0
With 1 Toffoli, 1 CNOT [55]	W_m, W_n, H	W_{m+n-1}	$\frac{m+n-1}{mn}$	$\frac{(m-1)(n-1)}{mn}$	0
Ours with 1 partial-swap	W_m, W_n	W_{m+n-1}	$\frac{m+n-1}{mn}$	$\frac{(m-1)(n-1)}{mn}$	0

**Figure 3.** Fusion gate for fusing three W states of arbitrary size to obtain a larger W state.

2.2. Fusing n arbitrary-sized $|W_m\rangle$ states into a large-scalable $|W_{nm-n+1}\rangle$ state

Figure 3 displays a scheme for fusing $|W_n\rangle_A$, $|W_m\rangle_B$, and $|W_t\rangle_C$ states into $|W_{n+m+t-2}\rangle$ by using two partial-swap gates. We denote polarization-based entangled W states of Alice *et al* as

$$|W_n\rangle_A = (|(n-1)_H\rangle_a |1_V\rangle_1 + \sqrt{n-1} |W_{n-1}\rangle_a |1_H\rangle_1) / \sqrt{n}, \quad (7)$$

$$|W_m\rangle_B = (|(m-1)_H\rangle_b |1_V\rangle_2 + \sqrt{m-1} |W_{m-1}\rangle_b |1_H\rangle_2) / \sqrt{m}, \quad (8)$$

$$|W_t\rangle_C = (|(t-1)_H\rangle_c |1_V\rangle_3 + \sqrt{t-1} |W_{t-1}\rangle_c |1_H\rangle_3) / \sqrt{t}. \quad (9)$$

As shown figure 3, after Alice *et al* send one of their photons ($|1_H\rangle_1$ and $|1_V\rangle_1$, $|1_H\rangle_2$ and $|1_V\rangle_2$, $|1_H\rangle_3$ and $|1_V\rangle_3$) to the two partial-swap gates through modes 1, 2, and 3, respectively, the two partial-swap gates lead to the following transformations:

$$\begin{aligned}
& |W_n\rangle \otimes |W_m\rangle \otimes |W_t\rangle \\
& \rightarrow \frac{1}{\sqrt{nm t}} |(n-1)_H\rangle_a |(m-1)_H\rangle_b |(t-1)_H\rangle_c |1_V\rangle_1 |1_V\rangle_2 |1_V\rangle_3 \\
& + \frac{\sqrt{n-1}}{\sqrt{nm t}} |W_{n-1}\rangle_a |(m-1)_H\rangle_b |(t-1)_H\rangle_c |1_V\rangle_1 |1_V\rangle_2 |1_H\rangle_3 \\
& + \frac{\sqrt{m-1}}{\sqrt{nm t}} |(n-1)_H\rangle_a |W_{m-1}\rangle_b |(t-1)_H\rangle_c |1_V\rangle_1 |1_V\rangle_2 |1_H\rangle_3 \\
& + \frac{\sqrt{t-1}}{\sqrt{nm t}} |(n-1)_H\rangle_a |(m-1)_H\rangle_b |W_{t-1}\rangle_c |1_V\rangle_1 |1_V\rangle_2 |1_H\rangle_3 \\
& + \frac{\sqrt{(n-1)(m-1)}}{\sqrt{nm t}} |W_{n-1}\rangle_a |W_{m-1}\rangle_b |(t-1)_H\rangle_c |1_H\rangle_1 |1_V\rangle_2 |1_H\rangle_3 \\
& + \frac{\sqrt{(n-1)(t-1)}}{\sqrt{nm t}} |W_{n-1}\rangle_a |(m-1)_H\rangle_b |W_{t-1}\rangle_c |1_V\rangle_1 |1_H\rangle_2 |1_H\rangle_3 \\
& + \frac{\sqrt{(m-1)(t-1)}}{\sqrt{nm t}} |(n-1)_H\rangle_a |W_{m-1}\rangle_b |W_{t-1}\rangle_c |1_V\rangle_1 |1_H\rangle_2 |1_H\rangle_3 \\
& + \frac{\sqrt{(n-1)(m-1)(t-1)}}{\sqrt{nm t}} |W_{n-1}\rangle_a |W_{m-1}\rangle_b |W_{t-1}\rangle_c |1_H\rangle_1 |1_H\rangle_2 |1_H\rangle_3.
\end{aligned} \quad (10)$$

Equation (10) implies the following four possible outcomes:

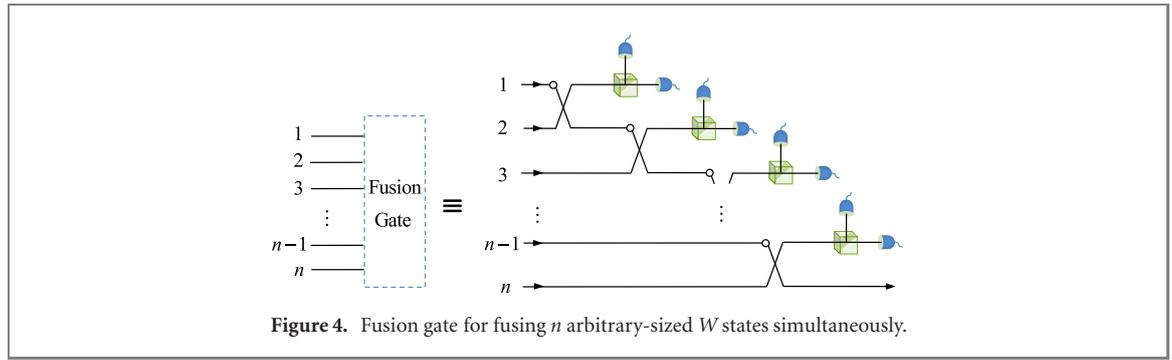


Figure 4. Fusion gate for fusing n arbitrary-sized W states simultaneously.

Table 3. Quantum resources required and success probabilities of various protocols for fusing multiple W states into a larger one simultaneously. $F = ((5 - m - n - z - t) - (n - 1)(m - 1)(t - 1)(z - 1))/(mntz)$.

Proposed protocol	Initial state	Achieved state	Success probability	Fail probability
With 1 Fredkin [33]	W_m, W_n, W_t, H	$W_{m+n+t-3}$	$\frac{m+n+t-3}{mnt}$	$\frac{(t-1)(m+n-2)+1}{mnt}$
Ours with 2 partial-swaps	W_m, W_n, W_t	$W_{m+n+t-2}$	$\frac{m+n+t-2}{mnt}$	0
With 1 Toffoli, 3 CNOT's [35]	W_m, W_n, W_t, W_z	$W_{m+n+t+z-4}$	$\frac{m+n+t+z-4}{mntz}$	F
Ours with 3 partial-swaps	W_m, W_n, W_t, W_z	$W_{m+n+t+z-3}$	$\frac{m+n+t+z-3}{mntz}$	0

- (a) When the photon in mode 1 is V-polarized state, the photon in mode 2 is also V-polarized state and detectors D_1 and D_3 click, the system collapses into the successful state $|W_{n+m+t-2}\rangle$

$$\begin{aligned}
 |W_{n+m+t-2}\rangle = & \frac{1}{\sqrt{mnt}} |(n-1)_H\rangle_a |(m-1)_H\rangle_b |(t-1)_H\rangle_c |1_V\rangle_3 \\
 & + \frac{\sqrt{n-1}}{\sqrt{mnt}} |W_{n-1}\rangle_a |(m-1)_H\rangle_b |(t-1)_H\rangle_c |1_H\rangle_3 \\
 & + \frac{\sqrt{m-1}}{\sqrt{mnt}} |(n-1)_H\rangle_a |W_{m-1}\rangle_b |(t-1)_H\rangle_c |1_H\rangle_3 \\
 & + \frac{\sqrt{t-1}}{\sqrt{mnt}} |(n-1)_H\rangle_a |(m-1)_H\rangle_b |W_{t-1}\rangle_c |1_H\rangle_3.
 \end{aligned} \tag{11}$$

- (b) When the photon in mode 1 is H-polarized state, the photon in mode 2 is also H-polarized state and detectors D_2 and D_4 click, the system collapses into the recyclable state

$$\frac{\sqrt{(n-1)(m-1)(t-1)}}{\sqrt{mnt}} |W_{n-1}\rangle_a |W_{m-1}\rangle_b |W_{t-1}\rangle_c |1_H\rangle_3. \tag{12}$$

- (c) When the photon in mode 1 is H-polarized state, the photon in mode 2 is V-polarized state and detectors D_2 and D_3 click, the system collapses into the ‘garbage’ state

$$\frac{\sqrt{(n-1)(m-1)}}{\sqrt{mnt}} |W_{n-1}\rangle_a |W_{m-1}\rangle_b |(t-1)_H\rangle_c |1_H\rangle_3. \tag{13}$$

We call this case ‘a partial recyclable’ because the states between Alice and Bob remain a W -state but Charlie needs to prepare a new W state for the subsequent round of the fusion process.

- (d) When the photon in mode 1 is V-polarized state, the photon in mode 2 is H-polarized state and detectors D_1 and D_4 click, the system collapses into the ‘garbage’ state

$$\frac{\sqrt{(t-1)(m+n-2)}}{\sqrt{mnt}} |W_{n+m-2}\rangle_{a,b} |W_{t-1}\rangle_c |1_H\rangle_3. \tag{14}$$

We call this case ‘a partial success’ because the state between Alice and Bob has been fused but not Charlie.

Figure 4 shows a scheme for fusing multiple W states of arbitrary size simultaneously. Table 3 compares the success and failure probabilities and an estimation of the required quantum resources for our proposal against previous schemes. Compared with other proposals for generating a W state of given size, our proposal scores a higher success probability and a lower failure probability, with a simpler network.

3. Linear-optics fusion-based W state using auxiliary spatial degrees of freedom

Based on section 2, one can see that the key component of our fusion gates is the partial-swap gate described by equation (4). The matrix form of this partial-swap gate in the basis $\{|1_H\rangle|1_H\rangle, |1_H\rangle|1_V\rangle, |1_V\rangle|1_H\rangle, |1_V\rangle|1_V\rangle\}$ can be written as

$$N_{p\text{-swap}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (15)$$

Obviously, this operation is not a unitary one due to $NN^\dagger \neq N^\dagger N \neq I$, with I being an identity matrix.

The nonunitary gate can be implemented by utilizing the framework of quantum measurement, or by expanding the state space to a larger one, and then performing a proper unitary operation and an orthogonal measurement in the enlarged space in succession. Here, we employ auxiliary spatial degrees of freedom introduced by polarizing beam splitters PBS_1 and PBS_2 (see figure 5) to implement the nonunitary polarization partial-swap gate. Next, we provide a step-by-step description of our protocol for implementing this partial-swap gate.

We consider photons 1 and 2 as being initially prepared in an arbitrary two-qubit polarization-encoded state

$$|\psi_0\rangle = \alpha_1|1_H\rangle_{in}|1_H\rangle_{in'} + \alpha_2|1_H\rangle_{in}|1_V\rangle_{in'} + \alpha_3|1_V\rangle_{in}|1_H\rangle_{in'} + \alpha_4|1_V\rangle_{in}|1_V\rangle_{in'}. \quad (16)$$

In the first step, as shown in figure 5, photons 1 and 2 pass through PBS_1 and PBS_2 , respectively. Next, photons in modes 1, 3, and 4 interact with HWP oriented at 45° (HWP^{45°), 67.5° ($HWP^{67.5^\circ}$), and 22.5° ($HWP^{22.5^\circ}$), respectively. The PBSs and HWPs cause the state to evolve from $|\psi\rangle_0$ to

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} (\alpha_1|1_V\rangle_1(|1_H\rangle_4 + |1_V\rangle_4) + \alpha_2|1_V\rangle_1(|1_H\rangle_3 + |1_V\rangle_3) \\ + \alpha_3|1_V\rangle_2(|1_H\rangle_4 + |1_V\rangle_4) + \alpha_4|1_V\rangle_2(|1_H\rangle_3 + |1_V\rangle_3)). \quad (17)$$

A PBS transmits the H-polarized and reflects the V-polarized components. Therefore, PBS_1 and PBS_2 impart the spatial degrees of freedom of the incident photon. The HWPs oriented at 45° (denoted HWP^{45°) induce the qubit-flip operation $|1_H\rangle \leftrightarrow |1_V\rangle$ while the $HWP^{67.5^\circ}$ results in

$$|1_H\rangle \leftrightarrow \frac{1}{\sqrt{2}}(-|1_H\rangle + |1_V\rangle), \quad |1_V\rangle \leftrightarrow \frac{1}{\sqrt{2}}(|1_H\rangle + |1_V\rangle). \quad (18)$$

Finally, the $HWP^{22.5^\circ}$ completes the transformation

$$|1_H\rangle \leftrightarrow \frac{1}{\sqrt{2}}(|1_H\rangle + |1_V\rangle), \quad |1_V\rangle \leftrightarrow \frac{1}{\sqrt{2}}(|1_H\rangle - |1_V\rangle). \quad (19)$$

In the second step, the photons in modes 2 and 3 are then mixed at PBS_3 before going through $HWP^{67.5^\circ}$ while the photons in modes 1 and 4 are mixed at PBS_4 before going through $HWP^{22.5^\circ}$. The completion of these operations leads to the joint state

$$|\psi_2\rangle = \frac{1}{2\sqrt{2}} (\alpha_1(|1_H\rangle_7 - |1_V\rangle_7)(|1_H\rangle_7 + |1_V\rangle_7 + |1_H\rangle_8 - |1_V\rangle_8) \\ + \alpha_2(|1_H\rangle_7 - |1_V\rangle_7)(-|1_H\rangle_5 + |1_V\rangle_5 + |1_H\rangle_6 + |1_V\rangle_6) \\ + \alpha_3(|1_H\rangle_5 + |1_V\rangle_5)(|1_H\rangle_7 + |1_V\rangle_7 + |1_H\rangle_8 - |1_V\rangle_8) \\ + \alpha_4(|1_H\rangle_5 + |1_V\rangle_5)(-|1_H\rangle_5 + |1_V\rangle_5 + |1_H\rangle_6 + |1_V\rangle_6)). \quad (20)$$

In the third step, the photons in modes 5 and 8 (6 and 7) are combined at PBS_5 (PBS_6), and the photon in mode 10 (12) passes through HWP^{45° (HWP^{45°). The operations $PBS_5 \rightarrow HWP^{45^\circ}$ and $PBS_6 \rightarrow HWP^{45^\circ}$ make $|\psi\rangle_2$ become

$$|\psi_3\rangle = \frac{1}{2\sqrt{2}} (\alpha_1(|1_H\rangle_{11} - |1_H\rangle_{12})(|1_H\rangle_{11} + |1_H\rangle_{12} + |1_H\rangle_9 - |1_H\rangle_{10}) \\ + \alpha_2(|1_H\rangle_{11} - |1_H\rangle_{12})(-|1_V\rangle_{10} + |1_V\rangle_9 + |1_V\rangle_{12} + |1_V\rangle_{11}) \\ + \alpha_3(|1_V\rangle_{10} + |1_V\rangle_9)(|1_H\rangle_{11} + |1_H\rangle_{12} + |1_H\rangle_9 - |1_H\rangle_{10}) \\ + \alpha_4(|1_V\rangle_{10} + |1_V\rangle_9)(-|1_V\rangle_{10} + |1_V\rangle_9 + |1_V\rangle_{12} + |1_V\rangle_{11})). \quad (21)$$

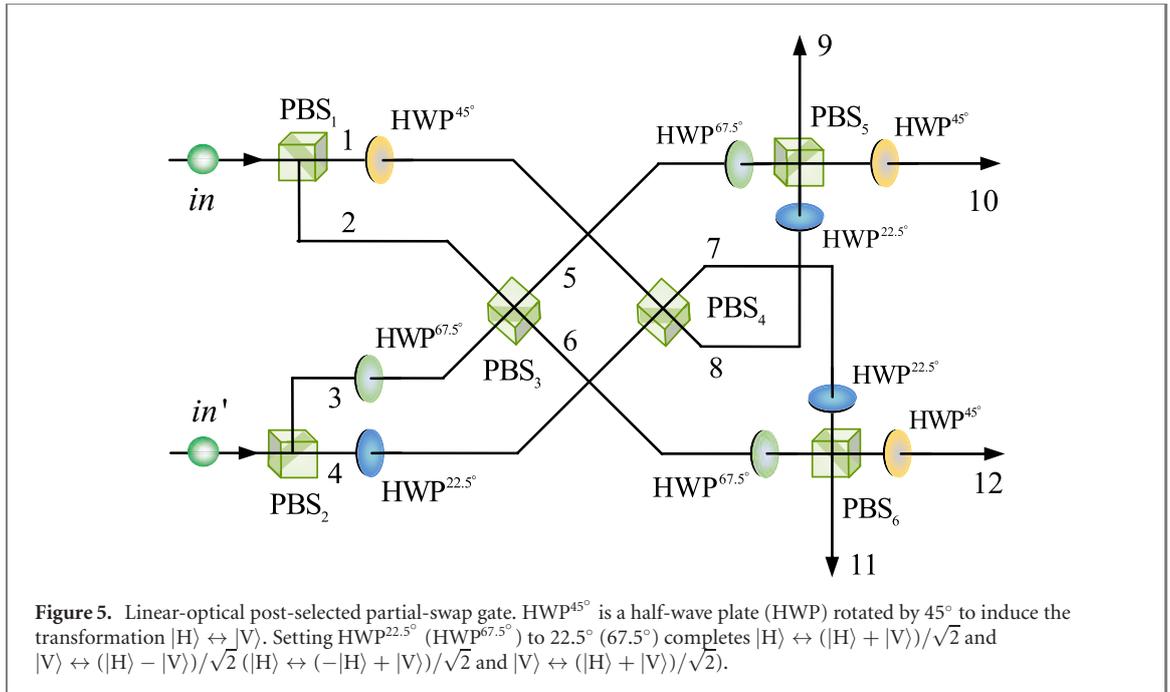


Table 4. Coincident expectation values calculated for the four logic basis inputs.

Input	$\langle n_{ H\rangle_9} n_{ H\rangle_{11}} \rangle$	$\langle n_{ H\rangle_9} n_{ V\rangle_{11}} \rangle$	$\langle n_{ V\rangle_9} n_{ H\rangle_{11}} \rangle$	$\langle n_{ V\rangle_9} n_{ V\rangle_{11}} \rangle$
	$\langle n_{ H\rangle_{10}} n_{ H\rangle_{12}} \rangle$	$\langle n_{ H\rangle_{10}} n_{ V\rangle_{12}} \rangle$	$\langle n_{ V\rangle_{10}} n_{ H\rangle_{12}} \rangle$	$\langle n_{ V\rangle_{10}} n_{ V\rangle_{12}} \rangle$
$ 1_H\rangle_{in} 1_H\rangle_{in'}$	1/8	0	0	0
$ 1_H\rangle_{in} 1_V\rangle_{in'}$	0	0	1/8	0
$ 1_V\rangle_{in} 1_H\rangle_{in'}$	0	0	1/8	0
$ 1_V\rangle_{in} 1_V\rangle_{in'}$	0	0	0	1/8

Equation (21) indicates that the two-qubit partial-swap operation (i.e., exchanges the information of the two photons, conditional on the first photon being H-polarized) is completed when a coincidence is observed between modes 9 and 11 (10 and 12). Table 4 lists the photon count rates in modes 9 and 11 (10 and 12) for computing basis inputs.

4. Discussion and conclusion

In this paper, we have proposed an effective scheme for fusing $|W_n\rangle$ and $|W_m\rangle$ states into a large-size $|W_{n+m-1}\rangle$ state by using a partial-swap gate. We have also designed a scheme for fusing multiple W states of arbitrary size simultaneously (see figure 4). By exploiting the spatial degrees of freedom of single-photons introduced by the PBSs, the partial-swap gate was implemented using an optically polarized architecture designed with linear-optical elements.

As shown in table 2, our scheme outperforms previous ones in fusing two W states of arbitrary size into a large-sized W state. An ancillary photon, which is necessary in the Fredkin- and Toffoli-based schemes [33, 55] to create a $|W_{n+m-1}\rangle$ state, is not required in our scheme. Moreover, our scheme minimizes failure outcomes. From table 3, one can see that, if the gate (Fredkin, Toffoli, CNOT, and partial-swap gates) operations are considered, the fail probability in the presented scheme is lower than that with the Fredkin- and Toffoli-based schemes [33, 35].

Our presented scheme has the further advantage of reducing cost in terms of the number of two-qubit gates. In previous studies [33, 55], the fusion of two W states required either one Fredkin gate, or one Toffoli and one CNOT gate. Our presented approach requires just one partial-swap gate. Notably, the optimal cost of an unconstructed Toffoli or Fredkin gate is five two-qubit gates [53, 54]. If we impose a further condition of using only CNOT gates, at least six CNOT gates are required to synthesize a Toffoli and at least eight for a Fredkin gate [56]. In contrast, the current proposals based on partial-swap gates surpass the Fredkin-gate scheme [33] and Toffoli-CNOT-scheme [55], and also surpass the scheme based on one Toffoli gate and three CNOT gates [35] (see tables 2 and 3).

Another important advantage of our scheme is its increased success probability. It is known that entangling quantum gates can be implemented only probabilistically using linear-optical elements. With a polynomial quantum resource, a linear-optical CNOT gate can be implemented with a maximal success probability of $3/4$ [36], and a post-selected CNOT gate with a success probability of $1/9$ [37]. The most efficient scheme for a CNOT gate with a success probability of $1/4$ is achieved with the help of a maximally entangled photon pair [38, 39]. Moreover, the ideal success probability of a Toffoli gate is $1/9$ under a linear-optics setup [42]. At present, the optimal success probability of a linear optical Fredkin gate is $1/64$ [46]. In contrast, the proposed partial-swap gate with a success probability of $1/4$ is achievable. To sum up, our partial-swap-based fusion schemes outperform the Fredkin-based [33] and Toffoli-CNOT-based schemes [35, 55] in term of the cost and success probability.

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