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DUAL CONCEPTION OF FUNCTIONS? WHAT EXACTLY DO YOU MEAN?

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ABSTRACT

The study reported in this paper explored three secondary mathematics teachers' conceptions of functions and their related pedagogical content knowledge. Do teachers view functions as structural objects or do they perceive functions purely from a procedural orientation? How do these teachers' orientations influence their teaching?

The interpretative paradigm underlying this study is complemented by uncovering the nature of the teachers' conceptions about functions through an open-ended questionnaire and a classification task. The teachers were interviewed to understand their pedagogical practices. The study found that the experienced teachers were able to recognize the dual nature of functions and even regarded the dual conceptions as complementary. On the other hand, the novice teacher held a predominantly operational conception of functions and his process-

oriented approach was not helpful in moving students towards a broader view of functions as a dynamic object. The ability to recognize the dual nature of mathematics concepts is crucial for the learning of mathematics and these preliminary findings have significant implications on professional teacher development. The paper concludes with suggestions how to sensitize teachers to the dual nature of mathematical concepts - knowledge vital to help improve pedagogical practice. The improved pedagogical practice may help more students advance to higher mathematics with a broader view of mathematical concepts.

INTRODUCTION

At certain stages of knowledge acquisition, the amount of information may grow so expansively that the old schema becomes saturated and practically impervious to any enrichment. Sfard (1991) described structural thinking as a powerful weapon against the limitations of our working memory. At a more philosophical level, she argued that structural conception is what probably underlies the relational understanding, defined by Skemp (1976) as "knowing both what and why to do".

In this paper, we shall examine the nature of three teachers' conceptions of mathematical notions, particularly functions. Their responses were analyzed by comparing the extent of their structural conception and their pedagogical practices.

THEORETICAL PERSPECTIVES

Sfard's Model of Learning Mathematics

A mathematical concept can be developed in two ways – structurally as an object, and operationally as a process. The structural conception means seeing a mathematical notion as an entity with a static structure, integrative and recognizable "at a glance" without going into details. Sfard (1991) argued that this visual approach to abstract ideas makes it more tangible

and therefore more readily manipulated like real objects. In contrast, the operational way of thinking about mathematical notions emphasizes on the sequential and dynamic process of performing algorithms and actions, rather than about objects.

The distinction between the procedures that a learner needs to acquire in order to manipulate mathematical objects, and the concepts on which the learner operate with the skills suggests a dichotomy between the two conceptions. Sfard (1989) argued that although processes and objects are ostensibly incompatible, they are in fact complementary. The different conceptions arose from the different facets of the same mathematical notion. She further presented evidences to support the conjecture that in the process of concept formation, operational conceptions precede the structural. This transition from computational operations to abstract objects was accomplished in three stages: interiorization, condensation and reification. The following is an illustration of how a mathematical idea such as *gradient* can be evolved from an operational to a structural conception.

According to the spiralling Mathematics curriculum in Singapore, in the lower secondary level, particularly Year 2, the idea of the gradient of a line is developed operationally first. At the stage of interiorization, a learner is familiarized with the operations performed on lower-level mathematical objects. Hence a diagram (see Figure 1) is often used to help them visualize and calculate the ratio of rise (*BC*) and run (*AC*) on a grid.

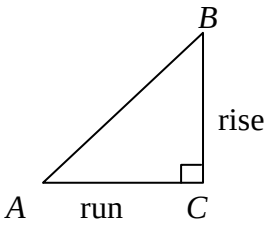

Figure 1

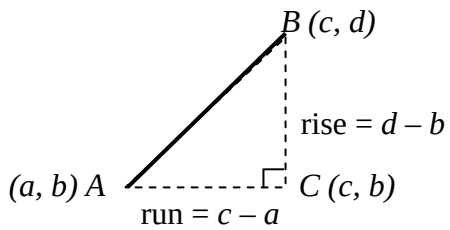

Figure 2

This figure forms a mental image to concretize the concept of steepness. From the lengths of a triangle, the students then progress to the use of coordinates of *A B* and *C* to form a triangle *ABC*, as shown in Figure 2.

As the students become more skilled at computing the gradient of a straight line, both positive and negative values, without using a diagram or grid, their conception of gradient gradually moves towards the next stage of condensation. By their year 3, the students begin

to think about the computational process as a whole, without feeling the urge to go into details. In the topic of coordinate geometry, the students are ready to apply the concept of gradient to prove the collinearity of points and perpendicularity of lines. There are also occasions when the students need to alternate between different representations of the concept. For example, the students may geometrically prove the collinearity of three points and algebraically determine the equation of the line formed by the three points. For many year 4 students, their conceptual development of gradient was 'expected' to enter the last stage of reification. When learning about increasing and decreasing functions, it is not uncommon to see many students struggling to make the instantaneous quantum leap; from seeing gradient as a property of lines to a property existing in non-linear functions as well. At this stage, the idea of gradient has progressed to that of gradient function. It can be seen as an object used to describe the behaviour of a function. For example, the range of values of a gradient function is an essential condition for determining whether its root function is increasing or decreasing in a given domain. In other words,

f is an increasing function $\Leftrightarrow$ $f'(x) > 0$

f is a decreasing function $\Leftrightarrow$ $f'(x) < 0$

f is a constant function $\Leftrightarrow$ $f'(x) = 0$

Sfard (1991) described this transition as an ontological shift in which the learner acquired a sudden ability to see a familiar mathematical entity in a totally new light. It is at this stage that the learner conceived a notion as a static structure, a full fledged object. From her three-phase schema, Sfard (1991) also proposed that the operation at one level become reified as objects to become basic units of higher level concepts.

Even's Analytical Framework of Subject Matter Knowledge for Teaching

The insights gained from examining the possible causes of children's difficulties in learning algebra serve to expand teachers' subject matter knowledge about the specific mathematical topic. This knowledge is in turn influenced by what they know across different demands of knowledge. Therefore, analysis of teachers' subject matter knowledge about a specific piece of mathematics should integrate several bodies of knowledge. In Even's (1990) study, he identified seven main facets of teachers' subject matter knowledge for teaching about a

specific mathematical topic, of which three are elaborated below.

One aspect of the framework deals with the *different representations* of the concept. Even (1990) commented that teachers need to understand concepts in different representations, and be able to translate and form linkages among and between them. The "different representations give different insights which allow for a better, deeper, more powerful and complete understanding of the concept". Another aspect concerns the teachers' *alternative ways of approaching* a concept. The appearance of a complex concept in various forms, representations, labels and notations are enhanced by the different uses of the concept. Hence, teachers should develop utility in using alternative ways of approaching the same concept. This will enable them make good decisions about different available choices in various situations. Even (1990) also examined the *strength of a concept* which is rooted in the new opportunities it opens. Teachers should have a good understanding of the unique characteristics of the concept in order to capture the essence of the definition as well as a more sophisticated formal mathematical knowledge.

Rationale for the Study

The developmental scheme proposed by Sfard (1989) has important implications on teachers' instructional actions. The teachers' instructional practices are related to their own conceptions of algebra. Together, they contribute to the effective learning in the subject matter. Hence it is essential to ask questions regarding teachers' conceptions about algebra and their pedagogical approaches towards the structural outlook.

A critical junction of secondary school algebra is the transition from a purely operational conception of a symbolic formula to the dual process-product interpretation (Linchecski,

Sfard, 1994). At the upper secondary school level, this transition can be identified when the concept of functions is formally introduced. In learning this topic, students often face difficulties in interpreting symbols and using letters as variables. This may be due to the change in focus from their numerical interpretation to the study of relationships.

Research suggests that knowledge of functions is problematic for prospective teachers who often hold simplified conceptions of functions that lead to misconceptions about ideas as basic as what constitutes a function (Even, 1993). In consideration of the above, we shall examine teachers' conceptions and their pedagogical practices in the context of teaching functions.

RESEARCH QUESTIONS

From the responses of three teachers to open-ended tasks and questions, we hope to gain insight into the nature of their conceptions about functions and their teaching approaches towards reification. This study seeks to answer the following questions:

1. What is the nature of the teachers' conceptions about functions?
2. How do the teachers teach for reification?

METHODS USED IN THE STUDY

Participants

Three teachers from the upper secondary level participated in this study. They are

- Mr Ang, a novice teacher with less than two years of teaching experience,
- Mr Bong with about five years of teaching experience, and
- Mrs Chan, a senior teacher with more than ten years of teaching experience.

Apart from their varied length of teaching experience, the three teachers were teaching algebra and functions in the regular (Special stream) and advanced classes (for mathematically promising students from Special and GE stream) in the same school.

Procedures

The procedure to gather information about the nature of the teachers' conceptions about functions and their teaching approaches comprised of three stages. In the first stage, the teachers were posed a series of open-ended questions about how they would define, identify and interpret functions. They were given unlimited time to think and organize their thoughts about this mathematical notion of functions. Their written responses were then examined in the light of functions as processes and as objects.

In the second stage, it was through a sorting activity that the interpretation of their responses was further verified by the way they organized their knowledge about functions and their representations. The teachers were asked to sort the cards, or subset of cards, in different ways and to describe and justify their thinking while sorting. This triangulation of information was necessary for the interpretation of the nature of their conception to be as precise as possible.

Lastly, the interviews with the teachers provided opportunities for accessing

- qualitative information about their knowledge of students' difficulties in learning functions, and
- insights into their pedagogical practices that helped students make the connection between algebra of an unknown value and functional algebra with variables.

During the interview, they were each asked the following questions:

1. What is your main emphasis in teaching functions?
2. What are the major difficulties that your students encounter in learning function?
3. What do you think are the sources of these difficulties?
4. In your opinion, what is the role of functions in the study of algebra?

Instruments

In the first stage, open-ended questions were used to examine the nature of participants' conceptions of functions. To check their understanding of functions as relationships or computational operations, the participants were asked about their definition(s) of functions in the context of teaching algebra. To obtain a fuller picture of their definition(s), they were asked to interpret a general function in the form $ax^2 + bx + c$ in as many ways as they possibly can. Their ability to see the connectedness between representations, both structural and operational, provided details about their mental picture of the notion of functions. Lastly, the participants were asked to compare and contrast two expressions $y = 2x - 3$ and $g(x - 1) = 2x - 5$. The activity served to provide concrete evidence for examining the extent of their operational and structural conception of functions. In the second stage, the card-sorting activity was adapted from Wilson's (1994) study on a teacher's understanding of functions. The activity consisted of 25 cards, each describing a different function. These functions were represented graphically, algebraically, in tables (as sets of ordered pairs) and as real world situations.

DISCUSSIONS OF THE FINDINGS

Dual Nature of Mathematical Conceptions

Sfard (1991) stated that the dual nature of mathematical constructs can be observed through verbal descriptions and symbolic representations. Although such property as structurality lies

in the eyes of the beholder rather than in the symbols themselves, some representations appear to be more susceptible of structural interpretation than others. Hence the different approaches to the concept of functions can be expected from the three participants.

From the teachers' definition and interpretation of functions, we shall examine the extent of their structural and operational outlook of this notion. Table 1 shows the salient points from the teachers' responses.

	Mr Ang	Mr Bong	Mrs Chan
Q1 : Definition of functions	• Functions represents ***relationships*** between variables • The function gives the ***rule*** for determining a ***unique y*** based on *x*.	• Functions is a ***relation*** that shows how a ***dependent variable*** is related to other ***independent variables*** • …functions give meaning to ***algebraic expression***	• Functions consists of three ***essential components***: rule, domain and range • Every input ***must*** have an output • Output must be ***unique*** • Domain of a rule must be given or assumed for the function to be ***well defined***
Q2 : Interpretation of functions	• The value of $f(x)$ is obtained ***by substituting*** specific values of x • If $x = k$, then $\boldsymbol{f(k) = ak^2 + bk + c}$ under certain boundary conditions	• Expression is an example of a quadratic ***function*** • The variable x is a ***parameter***. In some contexts, it can represent a displacement function with parameter t • … the general form of a quadratic function can ***have other equivalent forms***	• An input ***produces an output*** through the given ***rule*** • The given function can be observed in other forms of ***representations*** such as a mapping, a (quadratic) graph or an operator

Q 3 : Compare and Contrast	• Both expressions have x as a variable representing an ***input value***	• Both expressions represent the same ***function***	• Both expressions represent injective and ***functions*** • Both expressions are ***equivalen***t and satisfy the univalence property of functions
	• An expression of the form $y = 2x - 3$ shows a ***relation*** while $g(x - 1) = 2x - 5$ shows a ***function*** with an unique output	• An expression of the form $y = 2x - 3$ shows an ***explicit*** relationship between x and y • An expression of the form $g(x - 1) = 2x - 5$ shows an ***implicit*** relationship between input and output	• The expression of the form $y = 2x - 3$ shows a ***direct relationship*** between two variables, $g(x - 1) = 2x - 5$ represents a ***transformation*** of the first expression as can be seen from a mapping using ordered pairs

Table 1

From Mr Ang's definition of functions, he indicated his understanding of functions as a relationship that satisfied the property of univalence. He explained this property by using the input-output approach. Despite the openness of question 2, his substitution method showed that Mr Ang interpreted only the surface structure of the given expression $f(x) = ax^2 + bx + c$. The computational operation of $f(k)$ further provided evidence for the way he saw functions (represented in symbolic form) largely as an operational process. In comparing the expressions $y = 2x - 3$ and $g(x - 1) = 2x - 5$ in question 3, Mr Ang based his criteria for comparison on the surface structure of the functions without examining their differences at the systemic level. After verbally clarifying the ambiguities in his responses, at this stage, it is apparent that Mr Ang's conception of functions was predominantly operational in nature.

Mr Bong defined functions as relations that provide meaning to algebraic expressions. He clarified that through the idea of functions, an algebraic expression becomes more dynamic in its use as a mathematical model. His propensity to see an algebraic expression as a model was

suggestive of his seeing it as an object. In addition, the use of examples in question 2 to illustrate that x is a parameter and the expression has other equivalent forms explained why he thought this object is dynamic in the case of mathematical modelling. Without the use of numbers as a referent, and simply representing the variable as an arbitrary symbol, Usiskin (1988) identified this use of variable in the conception of algebra as the study of structures. In comparing the two given functions, Mr Bong noted their difference in the explicit and implicit relationship of input and output. Later in the interview, he added that the implicit relationship can lead to an explicit relationship by algebraic manipulations. Although Mr Bong did not elaborate further about the alternatives of showing $y = 2x - 3$ and $g(x - 1) = 2x - 5$ as the same object, it was clear that at this stage, his conception of functions was predominantly structural in nature.

Mrs Chan's definition of functions was similar to that of Mr Ang. Not only did she emphasized on the univalence property of functions, she also provided greater details for the functions to be well defined. However, in question 2, Mrs Chan moved away from the input-output approach towards functions, and demonstrated her versatility in interpretation – a particular trait of algebraic thinking. Sfard and Linchevski (1994) attributed this flexibility to a jump from operational to structural mode of thinking, from the detailed and diffuse to the general and concise. Mrs Chan's mode of structural thinking was made even more apparent in her explanation of how the two given functions were related in question 3. Like Mr Bong, she adeptly identified the explicit and implicit relationships of input and output. Besides using algebraic manipulations to show that y is equivalent to $g(x - 1)$, she also explained the equivalence through graphical transformation. In this approach, she treated $y = 2x - 3$ as a dynamic object that can be operated on to obtain another object. This adaptability in manipulating functions is a strong indication of structural thinking.

Different Representations of Functions

Even (1990) stressed the importance of teachers' understanding of mathematical concepts in different representations. From the *object perspective*, a function or relation and any of its representations are thought of as entities—for example, algebraically as members of parameterized classes, or in the plane as graphs that are thought of as being 'picked up whole' and rotated or translated (Moschkovich et al., 1993). Hence the second stage of this study was an attempt to engage the participants in making connections between different representations. Their responses were observed from the structural perspective.

The following shows the different criteria with which the participants used to group the cards. The number of different ways of grouping the cards is used as an indication of the level of flexibility in moving between representations. The generalized criteria were adapted from Wilson's (1994) study on a teacher's understanding of functions.

Mr Ang sorted the cards based on:

- type of representation (equations, graphs, tables, problem sums),
- graphical properties (shape),
- existence of contexts (problem and non problem).

Mr Bong sorted the cards based on:

- type of representation (equations, graphs, tables),
- algebraic properties (polynomials, exponential, logarithms).

Mrs Chan sorted the cards based on:

- type of representation (equations, graphs, tables, problem sums),
- algebraic properties (polynomials, exponential, logarithms),
- graphical properties (shape, continuity, increasing or decreasing),
- existence of contexts (problem and non problem).

On the whole, all the participants were observed to be able to switch representations of functions. In comparison, Mrs Chan demonstrated greater facility in moving from one representation to another because she was able to give more groupings under a generalized criterion. For example, she identified various behaviours of a graph by examining whether it is increasing or decreasing and whether it has turning points or not.

Pedagogical Approaches towards Functional Algebra

In functional algebra, Keiran (1997) observed that "the object of study is no longer number but relationships. The new objects of attention are the parameters – now represented by letters. The form of expression becomes a carrier of new information concerning the type of function or relation – linear, quadratic and so on. Translation from verbal or display representation to symbolic ones move to the background while translations between graphical and symbolic representations move to the foreground."

For teachers to effectively enable their students make this transition from the algebra of an unknown to the functional algebra of a variable, they should anticipate their students' difficulties in learning functions, understand the sources of these difficulties and take a more structural approach in their instructions. Hence one would question the correlation between the teachers' conceptions of functions and their instructional approaches.

The interviews with the three participants showed that their different views about functional algebra and varied instructional approaches. Their responses were highlighted in Table 2.

	(a) What is your main emphasis in teaching functions? *(b) What do you think is the role of functions in the study of algebra?*
Mr Ang	… the analogy of "the function machine helped to put across the idea of input-output very clearly"
Mr Bong	… the idea of a unique output for every input is "important for the students to appreciate the existence of an inverse process" later. … important for students to know that a function "can be contextualized and therefore made meaningful". Quite often, this function is seen to be "part of an equation" whose solution will lead to the answer to the given problem.
Mrs Chan	… time is taken to remind students "not to compute and manipulate functions at the surface level, but take a more systemic view". For example, using algebraic manipulation, we can obtain the inverse function of $f(x) = x^2$, $x \in R$ by simply reversing the process, but conceptually such a function does not even exist!" … the unique output for every input based on conditions and assumptions have a "domino effect on learning other operations" such as composing functions. … the "algebraic manipulation skills are taken to be a pre-requisite" at this point

Table 2

Mr Ang's responses were consistent with his understanding of functions primarily as computational activities or operations defined by algebraic expressions. Although the analogy of a function machine helps to concretize the way functions process inputs, this process-oriented approach does not help students move beyond the condensation stage and see a broader view of functions as a dynamic object. When asked about the role of functions in algebra, Mr Ang instinctively answered with a question "Shouldn't it be the role of algebra in functions?" Further probing revealed that Mr Ang's understanding of algebra was very much

limited to the study of algebraic manipulations. It was no wonder that his predominantly operational mode of thinking about functions and his perception of algebra as procedures shape his process- approach towards the teaching of functions.

Mr Bong emphasized that functions are important in modelling real-world situations that may have numerous parameters. These situations are modelled using functions, which in turn become "part of an equation". This response was consistent with his view of functions as an entity which can take other values. He also explained that students use their previous knowledge about algebraic manipulations to obtain equivalent expressions to solve the equation. Mr Bong also used Figure 1 to illustrate his perception of the relationship between algebra, functions and algebraic manipulations.

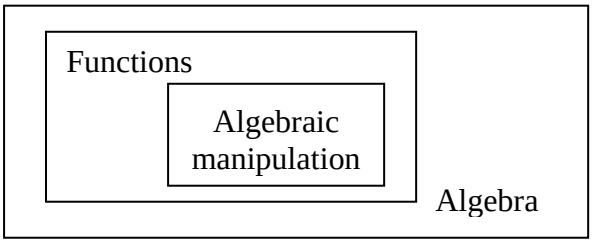

Figure 1

This visual representation reflected Mr Bong's thinking about algebraic manipulations being reified as functions to become basic units of higher level concepts in algebra at the upper secondary level.

Mrs Chan was keenly aware of the need for students to understand the process-object nature of functions. From the onset, she prepared the stage for structural learning by examining the macro-concept of change. In the interview, Mrs Chan skilfully used different representations to reveal the process nature of functions and suppressed it (using graphical representation) to explain how it can also be considered as an entity. When asked about the use of function machines, she thought it was an interesting way to show the "undoing" of processes but does not provide a deeper meaning to the answer. In her opinion, at the lower secondary level,

students study the surface structures of functions, an algebra of a fixed value. At the upper secondary level, she focused their learning on condensing the information and broadening their view about functions by oscillating between the operational and structural approaches.

	(a) What are the major difficulties that students encounter in learning functions? *(b) What do you think are these sources of difficulties?*
Mr Ang	"No problems, no difficulties"
Mr Bong	… when the students do not clearly distinguish relations and functions, they will "have problem understanding the important implications of an input having a unique output" … the abstract idea of inverse functions causes them to "treat functions and their inverses exactly like addition is to subtraction and multiplication to division" … like in set theory, students must acquaint themselves with the new language (such as domain and range), new notations (such as f(x)) and different system of operating with and on the functions (such as composite functions)
Mrs Chan	… students often confuse arithmetic operations with those on functions, such as $f^{-1}(x) = \frac{1}{f(x)}$ and $f^{2}(x) = [f(x)]^2$ … when dealing with functions as an algebraic expression or part of an equation, the processes are obvious to them. However, they need more guidance on using the representation to help them "visualize the problem effectively to solve them" … many of the students are not aware of what functions really are. They are not merely function machines

Table 3

From Table 3, although Mr Ang's responses were positive and even somewhat comical, it was not surprising. Given his process-oriented approach to teaching functions to a group of able students in his school, it was not surprising that there were "no problems, no difficulties". That is possibly because computational processes can be learnt with or without meaning, especially for students with good retention power. On the other hand, Mr Bong and Mrs Chan were more able to pin-point specific problems related to their students' learning of functions and they often attribute these problems to the structural aspects of functions.

IMPLICATIONS OF THE STUDY

The Nature of the Teachers' Knowledge about Functions

Similar to the study conducted by Wilson (1994) to find out about a pre-service teacher's understanding of functions, Mr Ang's conception of functions as numerical operations was consistent with the way many beginning students think about functions (Sfard 1987, 1989). With richer teaching experiences and opportunities to observe students' work, teachers like Mr Bong and Mrs Chan were more able to recognize the duality of the concept. More importantly, instead of weighing the value of each conception separately, they were also able to perceive the dual conceptions as complementary. This coincided with Sfard's (1991) argument that this is because the dual conceptions "refer to different aspects of the same thing". In addition, the nature of the teachers' conceptions about functions is also reflective of their pedagogical approaches.

Teaching for Reification

"Mathematical objects are an outcome of reification – of our mind's eye's ability to envision the result of processes as permanent entities in their own right (Sfard, 1994)". With repeated reifications, a mathematical concept is strengthened. Freudenthal (1983) observed that the

ability to substitute functions into each other and invert them created new functions and helped with the study of differentials and integrals and this led to the explosive growth of the analysis.

To enable students appreciate the strength of the concept of functions, teachers like Mr Bong and Mrs Chan have shown to make attempts at helping them transit from processes to abstract objects to enhance their sense of understanding. The teachers continued to use the symbolic representation for evaluation and substitution, thereby focusing on the process nature of function. On the other hand, they also used graphical representations to project certain characteristics of the function, such as its shape; thereby focusing on the structural nature of functions. Schwartz and Yerushalmy (1992) would advocate this approach of expressing particular functions both symbolically and graphically from the outset because they thought this would help students appreciate the dual nature of the mathematical concept.

Changing Conceptions through Professional Discourse

Sfard (1991) argued that the ability to see "a function or number both as a process and as an object is indispensable for deep understanding of mathematics". In order to help novice teachers like Mr Ang establish a deep understanding of functions and successfully use a process-object approach to teach the subject matter, it is important for him to consistently engage in professional discourse with the other teachers. This will enable him to gain greater insight into the

- dual nature of mathematical notions such as functions,
- difficulties that most students face in developing the structural conception of these mathematical notions,

- teaching strategies that use the process- and object-oriented approaches complementarily.

These topics of discussion will benefit both teachers and their students. The teachers' operational and structural conception of these mathematics ideas can potentially "contribute to the improvement of the quality of subject matter preparation n of teachers and therefore the quality of teaching and learning (Even, 1990)" to in their teaching of algebra so that more students will advance to higher mathematics with a broader view of mathematical concepts.

CONCLUSION

This study on teachers' conceptions of mathematical notions serves as a good starting point for us to explore the relationships between the nature of their conceptions of algebra and their pedagogical content knowledge in this subject matter. The preliminary observations were suggestive of some correlation between the two constructs. To put it simply, this would mean that a teachers' instructional practices is related to the extent of their structural outlook of algebra, which would change with increased teaching experiences and greater engaged professional discourse.

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