Properties of π-skew graphs with applications*

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Abstract

The skewness of a graph G is the minimum number of edges in G whose removal results in a planar graph. It is a parameter that measures how non-planar a graph is, and it also has important applications to VLSI design, but the results for skewness are quite limited. For any connected graph G on p vertices and q edges with girth g, one can easily verify that $sk(G) \geq \pi(G)$, where $\pi(G) = \left\lceil q - \frac{g}{g-2}(p-2) \right\rceil$, and a graph G is said to be π-skew if the equality holds. The purpose of this note is to characterize the structures of the π-skew graphs. Some families of π-skew graphs are obtained by applying these properties, including Cartesian product of two graphs and join product of two graphs, as well as some complete multipartite graphs etc. Moreover, we also discuss the threshold for the existence of a spanning triangulation. Among other results some sufficient conditions in terms of the minimum degree and size for a graph, which ensure a spanning triangulation, are given.

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1 Introduction

All graphs considered here are simple, finite and undirected. All graph theory terminology not defined here are referred to [6]. A *drawing* of a graph $G = (V, E)$ is a mapping D that assigns to each vertex in V a distinct node in the plane and to each edge uv in E a continuous arc (i.e., a homeomorphic image of a closed interval) connecting $D(u)$ and $D(v)$, not passing through the image of any other node. The nodes and arcs are also called vertices and edges of D respectively. A graph G is said to be *planar* if there exists a drawing of G so that its edges intersect only at their endpoints. Such a drawing of a planar graph G is called a *planar drawing* or *plane graph* of G. A *triangulation* (also called maximal planar graph) is a planar drawing in which each face has degree three. Clearly, adding an edge between any two nonadjacent vertices in a triangulation would destroy its planarity. A *spanning triangulation* of graph G is a triangulation of some spanning subgraph of G.

The *skewness* of a graph G, denoted by $sk(G)$, is defined as the minimum number of edges in G whose removal results in a planar graph. This is complementary, and computationally equivalent, to the *Maximum Planar Subgraph Problem* that asks for a planar subgraph of G with the maximum number of edges (see [19] for more details).

Probably the first reference on the skewness is Kotzig [17], where he gave a formula for the skewness of the complete graph and the complete bipartite graph. As a topological invariant of a graph, it is an important research object in topological graph theory, and it also plays important roles in automatic graph drawing and VLSI design [14, 18, 21]. But the results about skewness are quite limited, compared with other topological invariants, e.g., genus, thickness and crossing number.

The problem of determining the skewness of a given graph is known to be NP-complete [20, 26]. For certain classes of graphs, i.e., complete graph K_m [17], complete bipartite graph $K_{m,n}$ [17], n-cube Q_n [11], complete k-partite graphs for $k \leq 4$ [8], some generalized Petersen graphs [9] and Cartesian product of two cycles [7], the skewness is known. For more about skewness of graph, see [9, 19, 13] and the references therein.

The girth of a connected graph is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles (i.e. it's an acyclic graph), in general its girth is defined to be infinity. In this paper, the girth of an acyclic graph is defined to be twice the number of edges and all acyclic graphs considered have at least two edges without specially indication. The girth of a disconnected graph is equal to the minimum value of the girths of its all connected components. Let G be a connected graph on p vertices and q edges with girth g. Then it is not difficult to obtain from Euler's polyhedron formula that $sk(G) \geq \pi(G)$, where

$$\pi(G) = \lceil q - \frac{g}{g-2}(p-2) \rceil, \tag{1.1}$$

and a graph G is said to be *π-skew* if $sk(G) = \pi(G)$. The concept of π-skew was first proposed by Chia et al. [5, 6], and they gave some families of π-skew graphs, such as complete graph, complete bipartite graph, n-cube and so on. It is therefore natural to ask when does $sk(G) = \pi(G)$ hold? We are interested in the structures for the π-skew graphs, particularly, the graphs with girth 3 which actually contain a triangulation as a spanning subgraph, i.e. a spanning maximal planar subgraph. This is a very interesting extremal question and some recent results can been seen in [15, 16, 1].

This paper is organized as follows. We first give some structural characterizations for the π-skew graphs in Section 3. Applying these properties, we then obtain some new families of π-skew graphs, including join products of two graphs as well as some complete multipartite graphs in Section 4, and Cartesian products of complete bipartite graph and n-cube with trees in Section 5. In the final section we focus on the threshold for the existence of a spanning triangulation. Among other results some sufficient conditions in terms of the minimum degree and size for a graph, which ensure a spanning triangulation, are given.

2 Preliminaries

The Cartesian product $G_1 \Box G_2$ of two vertex-disjoint graphs G_1 and G_2 is defined to be the graph with vertex set $V(G_1) \times V(G_2) = \{(u, v) : u \in V(G_1), v \in V(G_2)\}$ and edge set $E(G_1 \Box G_2) = \{\{(x_1, y_1), (x_2, y_2)\} : x_1 = x_2$ and $y_1y_2 \in E(G_2)$ or $y_1 = y_2$ and $x_1x_2 \in E(G_1)\}$.

For vertex-disjoint graphs G_1 and G_2, the *union* $G_1 \cup G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, and the *join product* of G_1 and G_2, denoted by $G_1 + G_2$, is obtained from $G_1 \cup G_2$ by adding edges joining u and v for all $u \in V(G_1)$ and $v \in V(G_2)$.

Throughout this paper, with $\mathcal{PSK}(G)$ we denote the family of planar drawings of spanning subgraphs of G which are planar and of size $|E(G)| - sk(G)$. Clearly, any planar subgraph of G contains at most $|E(G)| - sk(G)$ edges. The degree of a vertex v of a graph G is denoted by $deg_G(v)$. The maximum degree of a graph G is denoted by $\Delta(G)$, and the minimum degree of a graph is denoted by $\delta(G)$. Let $\overline{G}$ denote the complement of the graph G.

The following basic and mostly well-known properties of skewness can be easily obtained by the definition of skewness of a graph.

Property 1 *If two graphs G_1 and G_2 are homeomorphic, then $sk(G_1) = sk(G_2)$.*

Property 2 *If graph G contains H as a subgraph, then $sk(H) \leq sk(G)$.*

Property 3 *Let G_1 and G_2 be two edge-disjoint subgraph of G, then*

$$sk(G_1 \cup G_2) \geq sk(G_1) + sk(G_2).$$

Property 4 *For any planar subgraph H of G, we have*

$$sk(G) \leq |E(G)| - |E(H)|.$$

Property 5 *For any connected graph G, D is connected for each $D \in \mathcal{PSK}(G)$.*

The *n-cube*, denoted by Q_n, is the complete graph K_2 if $n = 1$, while for $n \geq 2$, Q_n is defined inductively as $Q_{n-1}\Box K_2$. The following lemmas can be proved by Euler's polyhedron formula and can be seen in [8, 9, 11, 17].

Lemma 1 ([17]) $sk(K_m) = \binom{m-3}{2}$ *and* $sk(K_{m,n}) = (m-2)(n-2)$.

Lemma 2 ([11]) $sk(Q_n) = 2^n(n-2) - n2^{n-1} + 4$.

Lemma 3 ([8, 9]) *Complete graphs, complete bipartite graphs and n-cubes are π-skew.*

Lemma 4 ([9]) *For any connected graph G, we have $sk(G) \geq \pi(G)$.*

3 Properties for π-skew graphs

Let $F(D)$ denote the set of faces of a plane graph D. For each face $f \in F(D)$, its *degree* in D, denoted by $d_D(f)$, is defined to be the number of edges in D which are on the boundary of f, where each bridge of D on the boundary of f is counted twice. For any connected plane graph D, let $g'(D) = \min\{d_D(f) : f \in F(D)\}$ and

$$\epsilon(D) = \sum_{f \in F(D)} (d_D(f) - g'(D)).$$

Clearly, $\epsilon(D) \geq 0$ and $2|E(D)| = g'(D)|F(D)| + \epsilon(D)$. By Euler's polyhedron formula, we have

$$|F(D)| = \frac{2}{g'-2}\left(p - 2 - \frac{\epsilon}{2}\right) \tag{3.1}$$

and

$$|E(D)| = \frac{g'}{g'-2}\left(p - 2 - \frac{\epsilon}{g'}\right), \tag{3.2}$$

where $p = |V(D)|$, $g' = g'(D)$ and $\epsilon = \epsilon(D)$.

Lemma 5 *For any disconnected graph G, $sk(G) \geq \pi(G) + 2$.*

Proof. Suppose that G has p vertices, q edges and girth g. Let $G_1, G_2, \cdots, G_k$ be components of G, where $k \geq 2$. By Property 3 and Lemma 4,

$$\begin{aligned} sk(G) &\geq \sum_{i=1}^{k} sk(G_i) \geq \sum_{i=1}^{k} \pi(G_i) \\ &= \sum_{1 \leq i \leq k} \left\lceil q_i - \frac{g_i(p_i - 2)}{g_i - 2} \right\rceil \\ &= \sum_{1 \leq i \leq k} \left(q_i - (p_i - 2) - \frac{2(p_i - 2)}{g_i - 2} \right) \\ &\geq q - 2p + 2k - \sum_{1 \leq i \leq k} \frac{2(p_i - 2)}{g - 2} \\ &\geq q - \frac{g}{g-2}(p-2) + \frac{2g(k-1)}{g-2}. \end{aligned} \tag{3.3}$$

Note that $k \geq 2$, it follows that

$$\frac{2g}{g-2}(k-1) = (2 + \frac{4}{g-2})(k-1) \geq 2 + r,$$

where $r > 0$, which implies that

$$\begin{aligned} sk(G) &\geq q - \frac{g}{g-2}(p-2) + \frac{2g}{g-2}(k-1) \\ &\geq \pi(G) + 1 + r. \end{aligned}$$

This implies that the claim follows, since $sk(G)$ is an integer. $\square$

Remark. Lemma 5 implies that all π-skew graphs are connected. Combining with Lemma 4, we conclude that $sk(G) \geq \pi(G)$ holds for any graph G.

Lemma 6 *Let G be a π-skew graph on p vertices with girth g, then for any $D \in \mathcal{PSK}(G)$,*

$$|E(D)| = \lfloor \frac{g}{g-2}(p-2) \rfloor = \frac{g'}{g'-2}(p-2-\frac{\epsilon}{g'}),$$

where $g' = g'(D)$ and $\epsilon = \epsilon(D)$.

Proof. The definitions of π-skew and $\mathcal{PSK}(G)$ imply that the first equality follows, and the second equality holds by equality (3.2). $\square$

We now present the following two simple, but useful, sufficient conditions.

Lemma 7 *A graph G with girth g is π-skew if G contains a spanning subgraph H with girth g such that H is π-skew.*

Proof. Set $p = |V(G)|$ and $q = |E(G)|$. Let $D \in \mathcal{PSK}(H)$. It follows from Lemma 6 that

$$|E(D)| = \lfloor \frac{g}{g-2}(p-2) \rfloor.$$

Note that D is also a plane subgraph of G, thus

$$sk(G) \leq q - |E(D)| = q - \lfloor \frac{g}{g-2}(p-2) \rfloor = \pi(G).$$

Consider that $sk(G) \geq \pi(G)$, thus G is π-skew. $\square$

Lemma 8 *Let G be a connected graph with girth g. Assume that D is a planar drawing of a spanning and connected subgraph of G such that $g'(D) = g$ and $\epsilon(D) \leq g-3$. Then G is π-skew.*

Proof. Set $p = |V(G)|$, $q = |E(G)|$ and $\frac{g}{g-2}(p-2) = a + r$, where a is an integer and $0 \le r < 1$. Suppose that D is a planar drawing of satisfying given conditions. Then, equality (3.2) and the fact that $\epsilon(D) \le g-3$ imply that

$$|E(D)| = \frac{g}{g-2}(p-2-\frac{\epsilon(D)}{g}) \ge \frac{g}{g-2}(p-2) - \frac{g-3}{g-2} > a + r - 1,$$

Note also that

$$\pi(G) = q - \lfloor \frac{g}{g-2}(p-2) \rfloor = q - a,$$

thus,

$$sk(G) \le q - |E(D)| < q - a - r + 1 = \pi(G) - r + 1.$$

As $0 \le r < 1$ and $sk(G)$ is an integer, we further have $sk(G) \le \pi(G)$. Hence G is π-skew, since $sk(G) \ge \pi(G)$ holds for any graph G. $\square$

Now we give the following characterizations on π-skew graphs in two cases.

Theorem 1 *Let G be a connected graph on p vertices with girth g. If $2(p-2)$ is divisible by $g-2$, then the following three statements are equivalent:*

(i) *G is π-skew;*

(ii) *for each $D \in \mathcal{PSK}(G)$, $g'(D) = g$ and $d_D(f) = g$ for each $f \in F(D)$;*

(iii) *there exists a planar drawing D of a spanning and connected subgraph of G such that $g'(D) = g$ and $\epsilon(D) \le g-3$.*

Proof. Note that (ii) $\Rightarrow$ (iii) is obvious and (iii) $\Rightarrow$ (i) follows from Lemma 8. Thus it suffices to prove (i) $\Rightarrow$ (ii). Assume now that G is π-skew. Let $D \in \mathcal{PSK}(G)$, and then by Lemma 6,

$$\frac{g'}{g'-2}(p-2-\frac{\epsilon}{g'}) = \lfloor \frac{g}{g-2}(p-2) \rfloor, \tag{3.4}$$

where $g' = g'(D)$ and $\epsilon = \epsilon(D)$. As $2(p-2)$ is divisible by $g-2$, $g(p-2)$ is also divisible by $g-2$ and so equality (3.4) can be written as

$$\frac{g'}{g'-2}(p-2-\frac{\epsilon}{g'}) = \frac{g}{g-2}(p-2).$$

Since $g' \ge g$ and $\epsilon \ge 0$, thus

$$0 \le \epsilon = \frac{2(p-2)(g-g')}{(g-2)} \le 0.$$

The above equality implies that $g' = g$ and $\epsilon = 0$. Clearly, $\epsilon = 0$ also implies that $d_D(f) = g$ for all $f \in F(D)$. Hence (ii) follows. $\square$

Theorem 2 *Let G be a connected graph on p vertices with girth g. If $p \geq \binom{g-1}{2}+2$, then the following three statements are equivalent:*

(i) *G is π-skew;*

(ii) *for each $D \in \mathcal{PSK}(G)$, $g'(D) = g$ and $\epsilon(D) \leq g-3$;*

(iii) *there exists a planar drawing D of a spanning and connected subgraph of G such that $g'(D) = g$ and $\epsilon(D) \leq g-3$.*

Proof. Note that (ii) $\Rightarrow$ (iii) is obvious and (iii) $\Rightarrow$ (i) follows from Lemma 8. Thus it suffices to prove (i) $\Rightarrow$ (ii). Suppose now that G is π-skew, and let $D \in \mathcal{PSK}(G)$. By Lemma 6,

$$\frac{g'}{g'-2}(p-2-\frac{\epsilon}{g'}) = \lfloor \frac{g}{g-2}(p-2) \rfloor, \tag{3.5}$$

where $g' = g'(D)$ and $\epsilon = \epsilon(D)$. Equality (3.5) implies that

$$\frac{g+r}{g+r-2}(p-2-\frac{\epsilon}{g+r}) > \frac{g}{g-2}(p-2)-1,$$

where $r = g'-g$. Solving this inequality yields that

$$0 \leq (g-2)\epsilon < (g^2+gr-2rp-4g+2r+4), \tag{3.6}$$

implying that $g^2+gr-2rp-4g+2r+4 > 0$, which further implies that

$$r < \frac{(g-2)^2}{2p-g-2}.$$

Since $p \geq \binom{g-1}{2}+2$, we have $r < 1$. Further, $r = 0$ and $g' = g$ because r is an integer. Hence, inequality (3.6) implies that $\epsilon < g-2$. Note also that ϵ is an integer, it has $\epsilon \leq g-3$. Therefore (ii) follows. $\square$

4 π-skew join graphs

Let P_n be the path on n vertices. A graph that contains a Hamiltonian path is called a *traceable graph*.

Theorem 3 *Let G_1 and G_2 be two traceable graphs. Then $G_1 + G_2$ is π-skew.*

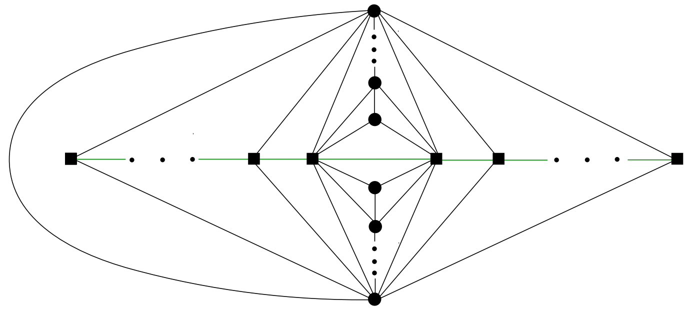

Figure 1: A spanning triangulation of $P_m + P_n$

Proof. Set $|V(G_1)| = m$ and $|V(G_2)| = n$. By Lemma 7, it is enough to prove that $P_m + P_n$ is π-skew, since $G_1 + G_2$ contains $P_m + P_n$ as a spanning subgraph and they both have girth 3. Figure 1 illustrates a spanning triangulation of $P_m + P_n$. The result hence follows from Theorem 1. □

Now we define two operations introduced by Chia et al in [8], which are called vertex-triangulation of a face and building faces on a given edge.

Definition. Let D denote a planar drawing.

(i) Let f be a face of D. By a vertex-triangulation of f, we mean inserting a new vertex x inside f and joining x to each vertex on the boundary of f;

(ii) Let uv be an edge of D. By building a fetch on uv with a new vertex x, we mean adding edges joining x to both u and v.

Theorem 4 *Let G be a π-skew graph on p vertices with girth g. Assume that $2(p-2) = k(g-2)$ holds for some integer k, then $G + H$ is π-skew for any graph H on s vertices, where $s \leq k$ and $s = k$ whenever $g \geq 4$.*

Proof. By Lemma 7, it suffices to show that the result holds when $H = \overline{K}_s$, since any graph on s vertices contains $\overline{K}_s$ as a spanning subgraph.

Let $D \in \mathcal{PSK}(G)$. It follows that $|F(D)| = k$ from Lemma 6 and Euler's polyhedron formula. Moreover, $d_D(f) = g$ for each $f \in F(D)$ by Theorem 1.

If $g = 3$, then, each face in D is a triangle. As $s \leq k$, we can get a spanning triangulation of $G + \overline{K}_s$ by vertex-triangulating s different faces of D. Thus, $G + \overline{K}_s$ is π-skew by Theorem 1 in this case.

If $g \geq 4$, then, by vertex-triangulating $s = k$ faces of D, we obtain a spanning triangulation of $G + \overline{K}_s$. Theorem 1 hence implies that $G + \overline{K}_s$ is π-skew again. □

Remark. The result of Theorem 4 for $g = 3$ and $H = \overline{K}_s$ can been also found in [8].

Theorem 5 *Let G_1 be a π-skew graph on p vertices with girth g, and let G_2 be a traceable graph on s vertices. Assume $2(p-2) = k(g-2)$ holds for some integer k, then $G_1 + G_2$ is π-skew, where $s \geq 1$ and $s \geq k$ whenever $g \geq 4$.*

Proof. From Lemma 7 we only need to prove that $G_1 + P_s$ is π-skew, since $G_1 + G_2$ contains $G_1 + P_s$ as a spanning subgraph with girth 3.

We denote the vertices of P_s with $t_1, t_2, \cdots, t_s$ and let $t_i t_{i+1} \in E(P_s)$ for $i = 1, 2, \cdots, s-1$. Let $D \in \mathcal{PSK}(G)$. By Lemma 6 and Euler's polyhedron formula, we have $|F(D)| = k$. Furthermore, $d_D(f) = g$ for each $f \in F(D)$ by Theorem 1.

Since $G + P_s$ contains $G + \overline{K}_s$ as a spanning subgraph and $G + \overline{K}_s$ is π-skew when $s \leq k$ for $g = 3$ and $s = k$ for $g \geq 4$ by Theorem 4, thus $G + P_s$ is π-skew in this case by Lemma 7.

Assume now that $s > k$. We first vertex-triangulate k faces of D with vertices $t_1, t_2, \cdots, t_k$. Without loss of generality, assume that t_k lie in the face of D with v_1, v_2, v_3 on its boundary. And then we place the vertices $t_{k+1}, \cdots, t_s$ as well as the edges between them inside the face with t_k, v_1, v_2 on its boundary and join t_i, $i = k+1, \cdots, s$, to v_1 and v_2, as well as t_{k+1} to t_k. The local drawing shown in Figure 2 may help us understand the operation. One can easily observe that the resulting drawing is a spanning triangulation of $G + P_s$. Theorem 1 hence implies that $G + P_s$ is π-skew. $\square$

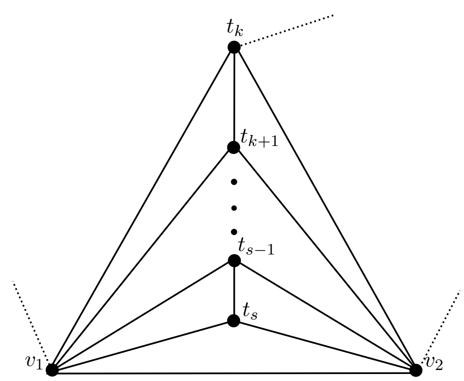

Figure 2: A local drawing of spanning triangulation of $G + P_s$

Though the following lemma is intuitively obvious, the proof is tedious and hence it is omitted.

Lemma 9 *The complete k-partite graph $K_{p_1,p_2,\cdots,p_k}$ is Hamiltonian if and only if $p_k \leq \sum_{j=1}^{k-1} p_j$, where $3 \leq k$ and $p_i \leq p_{i+1}$ for $1 \leq i \leq k-1$.*

Corollary 1 *The complete k-partite graph $K_{p_1,p_2,\cdots,p_k}$ is π-skew if $p_k \leq \sum_{j=1}^{k-1} p_j - 2$, where $3 \leq k$ and $2 \leq p_i \leq p_{i+1}$ for $1 \leq i \leq k-1$.*

Proof. The partite sets of $K_{p_1,p_2,\cdots,p_k}$ are denoted by $V_1, V_2, \cdots, V_k$ with cardinalities $p_1, p_2, \cdots, p_k$, respectively. The claim follows for the case $k = 3, 4$ from [8]. So from now on we may assume that $k \geq 5$. Let $s = p_k - \sum_{j=3}^{k-1} p_j$. Two cases now arise, depending on whether $s > 0$ or not.

Case 1. $s \leq 0$. By Lemma 9, $K_{p_3,p_4,\cdots,p_k}$ is a traceable graph with $\sum_{j=3}^{k} p_j$ vertices. Thus, the results follows from Theorem 5, since $K_{p_1,p_2,\cdots,p_k} \cong K_{p_1,p_2} + K_{p_3\cdots,p_k}$ and $\sum_{j=3}^{k} p_j > p_1 + p_2 - 2$.

Case 2. $s > 0$. Let $p'_k = p_k - (p_1 + p_2 - 2)$. Clearly, $0 < p'_k \leq \sum_{j=3}^{k-1} p_j$. Let $D \in \mathcal{PSK}(K_{p_1,p_2})$, then $|F(D)| = p_1 + p_2 - 2$ and $d_D(f) = 4$ for each $f \in F(D)$. Let D' be the planar drawing obtained from D by vertex-triangulating each face of D with $p_1 + p_2 - 2$ vertices from V_k. By Lemma 9, $K_{p_3,p_4,\cdots,p'_k}$ contains a Hamiltonian cycle. We place the vertices of $K_{p_3,p_4,\cdots,p'_k}$ inside some face of D'. And then the similar operation in D' as in the proof of Theorem 5 can obtain a triangulation which is indeed a spanning triangulation of $K_{p_1,p_2,\cdots,p_k}$, since a careful construction can guarantee the vertices from V_k are nonadjacent to each other in the triangulation.

Thereby the above two cases finish the proof of our result. □

Theorem 6 *Let G be a π-skew graph on p vertices with girth g. Assume that $2(p-2) = k(g-2)$ holds for some integer k, then for any $s \geq k$,*

$$sk(G + \overline{K}_s) = \pi(G) + (p-2)(s-2).$$

Proof. Let $D \in \mathcal{PSK}(G)$. Lemma 6 and Euler's polyhedron formula imply that $|F(D)| = k$. Furthermore, $d_D(f) = g$ for each $f \in F(D)$ by Theorem 1.

We will obtain a spanning triangulation of $G + \overline{K_s}$ from D. For this purpose, we first vertex-triangulate all k faces in D, and then build $s-k$ fetches on a fixed edge of D. It is not hard to see that the resulting drawing D' has $p+s$ vertices and $|E(D)| + gk + 2(s-k)$ edges. This implies that

$$\begin{aligned} sk(G + \overline{K_s}) &\leq |E(G + \overline{K_s})| - |E(D')| \\ &= |E(G)| + ps - |E(D)| - gk - 2(s-k) \\ &= \pi(G) + (p-2)(s-2). \end{aligned}$$

Note that $G + \overline{K}_s = G \cup K_{p,s}$ and $sk(K_{p,s}) = (p-2)(s-2)$. Then it follows that $sk(G+\overline{K}_s) \geq sk(G) + sk(K_{p,s}) = \pi(G) + (p-2)(s-2)$ from Property 3. Therefore, the proof is completed. $\square$

Corollary 2 ([8]) *Let G be a π-skew graph on p vertices with girth 3, Then, for any positive integer s,*

$$sk(G+\overline{K_s}) = \begin{cases} \pi(G+\overline{K_s}) & if\ s \leq 2p-4 \\ \pi(G) + (p-2)(s-2) & otherwise. \end{cases}$$

Proof. The result follows directly from Theorem 4 and Theorem 6. $\square$

5 π-skew Cartesian product graphs

Given any graph $G = (V, E)$ and any set I of positive integers, let $\Omega(G, I)$ be the set of mappings $\psi : E(G) \to I$ such that $\psi(e_1) \neq \psi(e_2)$ whenever e_1 and e_2 are adjacent edges in G. Thus, for any $i \in I$, the set $\{e \in E : \psi(e) = i\}$ is a matching of G. Let $W = \{d_i : i \in I\}$ be a multi-set of real numbers corresponding to I, where it is possible that $d_i = d_j$. In the case that $\Omega(G, I) \neq \emptyset$, define

$$\omega(G, I, W) = \max_{\psi \in \Omega(G,I)} \sum_{e \in E(G)} d_{\psi(e)}.$$

It is an interesting problem to determine $\omega(G, I, W)$ for any given G, I and W. If $I = \{1, 2, \cdots, k\}$ and $d_1 \geq d_2 \geq \cdots \geq d_k$, then

$$\omega(G, I, W) = \max_{(E_1, E_2, \cdots, E_r) \in \mathcal{P}} \sum_{i=1}^{r} d_i |E_i|,$$

where $\mathcal{P}$ is the set of all $(E_1, E_2, \cdots, E_r)$'s such that $\{E_1, E_2, \cdots, E_r\}$ is a partition of E, each E_i is a matching of G and $|E_1| \geq |E_2| \geq \cdots \geq |E_r|$.

For a planar drawing D and any face f in D, let $v_D(f)$ denote the number of vertices on the boundary of f in D. Observe that $v_D(f) \leq d_D(f)$, where the equality holds whenever D is bridgeless.

Lemma 10 *Let D be a connected planar drawing of planar graph G with faces $f_1, f_2, \cdots, f_r$, and let T be a tree with n vertices and $\Delta(T) \leq r$. Then $G \square T$ has a planar subgraph H with*

$$|E(H)| \geq n|E(G)| + \omega(T, I, W),$$

where W is the multi-set $\{v_D(f_i) : i \in I\}$ and $I = \{1, 2, \cdots, r\}$.

Proof. As T is a tree with $\Delta(T) \le r = |I|$, it is trivial to show that $\Omega(T,I) \ne \emptyset$. It suffices to prove that for any $\psi \in \Omega(T,I)$, $G\Box T$ always has a planar subgraph H with

$$|E(H)| = n|E(G)| + \sum_{e\in E(T)} v_D(f_{\psi(e)}). \tag{5.1}$$

Let $t_1, t_2, \cdots, t_n$ be the vertices of T such that for all $q = 1, 2, \cdots, n$, the subgraph T_q of T induced by $\{t_i : 1 \le i \le q\}$ is connected. Thus t_q is a leaf of T_q and T_q is a tree with $\Delta(T_q) \le r$ for all $q = 1, 2, \cdots, n$. For $q = 2, 3, \cdots, n$, let e_q be the edge in T_q incident with t_q and $t_{b(e_q)}$, where $1 \le b(e_q) < q$. Clearly e_1 does not exists here.

Note that $\{(u, t_i) : u \in V(G), i \in I\}$ is the vertex set of $G\Box T$, and (u_1, t_i) and (u_2, t_j) are adjacent in $G\Box T$ if and only if either $u_1 = u_2$ and $t_it_j \in E(T)$ or $i = j$ and $u_1u_2 \in E(G)$.

For $1 \le k \le n$, let H_k be the spanning subgraph of $G\Box T_k$ with vertex set $\{(u, t_i) : u \in V(G), 1 \le i \le k\}$ and edge set as follows:

$$E_k = \bigcup_{i=1}^{k} \{\{(u_1, t_i), (u_2, t_i)\} : u_1u_2 \in E(G)\} \cup \bigcup_{q=2}^{k} \{\{(u, t_{b(e_q)}), (u, t_q)\} : u \in V(f_{\psi(e_q)})\},$$

where $V(f_{\psi(e_q)})$ is the set of vertices on the boundary of face $f_{\psi(e_q)}$ of D. Thus $|V(f_{\psi(e_q)})| = v_D(f_{\psi(e_q)})$.

Clearly H_k is a spanning subgraph of $G\Box T_k$ with size given in (5.1) when n is changed k and T is changed T_k. It remains to show that H_k is planar for all $k = 1, 2, \cdots, n$.

For $k = 1, 2, \cdots, n$, let D'_k be a planar drawing obtained from D by changing each vertex u in D to (u, t_k) in D'_k and, whenever $k \ge 2$, setting face f_s as the external face of D'_k, where $s = \psi(e_k)$. The face in D'_k corresponding to face f_s of D is also called the s'th face in D'_k.

For $k = 1, 2, \cdots, n$, we construct a planar drawing H'_k of H_k from the planar drawings $D'_1, D'_2, \cdots, D'_k$, by the following steps:

(i) let H'_1 be exactly the planar drawing D'_1;

(ii) let H'_2 be the planar drawing obtained from H'_1 by moving D'_2 to face f_s of H'_1 and adding an edge joining (u, t_1) and (u, t_2) for each vertex u in the set $V(f_s)$, where $s = \psi(e_2)$;

(iii) for $1 \le k < n$, suppose H'_k is obtained with the property that for $j = 1, 2, \cdots, k$ and $i = 1, 2, \cdots, r$, the i'th face of D'_j is also a face of H'_k, unless $i = \psi(e_s)$ for some s with $j \le s \le k$ and either $s = j > 1$ or $b(e_s) = j$;

(iv) by the definition of $\Omega(T, I)$ and the above property on H'_k, then the s'th face of the planar drawing D'_c is also a face in H'_k, where $s = \psi(e_{k+1})$ and $c = b(e_{k+1})$;

(v) let H'_{k+1} be the planar drawing obtained from H'_k by moving D'_{k+1} into the face of H'_k which is originally the s'th face of the planar drawing D'_c and adding an edge joining (u, t_c) and (u, t_{k+1}) for each u in the set $V(f_s)$.

Clearly, H'_n is a planar drawing of H_n, thus equality (5.1) holds. $\square$

Applying Lemma 10, we get an upper bound for $sk(G\Box T)$, where T is a tree.

Theorem 7 *Let G be a connected graph on p vertices and q edges, and let T be a tree on n vertices with $\Delta(G) \leq q - p - sk(G) + 2$. Then, for any $D \in \mathcal{PSK}(G)$, we have*

$$sk(G\Box T) \leq n \times sk(G) + (n-1)p - \omega(T, I, W),$$

where $I = \{1, 2, \cdots, r\}$, r is the number of faces of D, W is the multi-set $\{v_D(f_i) : f_i \in F(D), i \in I\}$ and $F(D)$ is the set of faces of D.

Proof. Let $D \in \mathcal{PSK}(G)$. Then $|E(D)| = q - sk(G)$. By Euler's polyhedron formula, the number of faces in D is

$$|F(D)| = q - sk(G) - p + 2.$$

Clearly, D is a planar drawing of some planar graph, say G_1. By Lemma 10, $G_1\Box T$ has a planar subgraph with size at least $n|E(D)|+\omega(T, I, W)$. This planar subgraph is also a subgraph of $G\Box T$. Thus

$$\begin{aligned} sk(G\Box T) &\leq |E(G\Box T)| - n|E(D)| - \omega(T, I, W) \\ &= nq + (n-1)p - n(q - sk(G)) - \omega(T, I, W) \\ &= n \times sk(G) + (n-1)p - \omega(T, I, W), \end{aligned}$$

as stated and the proof is completed. $\square$

Corollary 3 *Let G be a connected cyclic graph on p vertices and q edges, and let T be a tree with n vertices and $\Delta(T) \leq q - p - sk(G) + 2$. Then*

$$sk(G\Box T) \leq n \times sk(G) + (n-1)(p - g).$$

Proof. By Theorem 7, it suffices to show that $v_D(f) \geq g$ holds for each face f of D, where $D \in \mathcal{PSK}(G)$. As G is connected cyclic graph (and also D), the vertices on the boundary of each face f of D induce cycles, implying that $v_D(f) \geq g$. Thus the result follows. $\square$

Theorem 8 *Let G be a π-skew graph on p vertices and q edges with girth 4, and let T be a tree on n vertices with $\Delta(T) \leq p-2$. Then*

$$sk(G\Box T) = n \times sk(G) + (n-1)(p-4),$$

and hence $G\Box T$ is π-skew.

Proof. As the girth of G is 4 and G is π-skew, $sk(G) = q - 2p + 4$. The girth of $G\Box T$ is also 4. Thus,

$$\pi(G\Box T) = nq + (n-1)p - 2(np-2) = n \times sk(G) + (n-1)(p-4).$$

Since $\Delta \leq p-2 = q-p-sk(G)+2$, Corollary 3 implies that $sk(G\Box T) \leq \pi(G\Box T)$. Hence, the claim follows from the fact that $sk(G\Box T) \geq \pi(G\Box T)$. $\Box$

The following result, due to Chia et al. [9], is a special case of Theorem 8.

Corollary 4 ([9]) *Let G be a π-skew graph on p vertices with girth 4. Then $sk(G\Box K_2) = 2sk(G) + p - 4$ and hence $G\Box K_2$ is π-skew.*

However, if $g = 3$ or 5, Theorem 8 does not hold.

Theorem 9 *Let G be a connected graph with $p \geq 2$ vertices and girth g and T be a tree with $n \geq 2$ vertices. If $g \neq 4$ and $np \geq \binom{g-1}{2} + 2$, then $G\Box T$ is not π-skew.*

Proof. Assume that $g = 3$. Assume that $G\Box T$ is π-skew. As the girth of $G\Box T$ is also 3, by Theorem 2, there exists $D \in \mathcal{PSK}(G\Box T)$ such that $g'(D) = 3$ and $\epsilon(D) = 0$, implying that each edge of D is contained in a triangle, which further implies that D does not contain any edge joining (u, t_i) and (u, t_j) for some $u \in V(G)$ and $t_i, t_j \in V(T)$ and so D is disconnected, contradicting Property 5.
Now consider the case $g \geq 5$. Assume that $G\Box T$ is π-skew. The girth of $G\Box T$ is also 4. Since the number of vertices in $G\Box T$ is $np \geq \binom{g-1}{2} + 2$, by Theorem 2, there exists $D \in \mathcal{PSK}(G\Box T)$ such that $g'(D) = 4$ and $\epsilon(D) \leq 1$. Thus $\epsilon(D) = 0$; otherwise, $d_D(f) = 4$ for all faces f except one face f_0 with $d_D(f_0) = 5$, implying the dual graph D^* of D contains exactly one odd vertex, a contradiction.
Taking any edge $e = uv$ in one copy of G. If e is also an edge in D, then e must be on the boundaries of exactly two faces, say f_1 and f_2. Clearly $d_D(f_i) = 4$ for $i = 1, 2$. At least one of the two faces, say f_1, is bounded by edges in one copy of G, contradicting the condition that the girth g of G is at least 5.

Thus D does not contain any edge which is in some copy of G, implying that D is a subgraph of $N_p \Box T$ and so D is disconnected, contradicting Property 5. $\Box$

Since both $K_{r,m}$ and Q_m are π-skew graphs with girth 4, applying Theorem 8 immediately get the following conclusions.

Corollary 5 *Let T be a tree on n vertices. If $\Delta(T) \leq r + m - 2$, then $K_{r,m} \Box T$ is π-skew and*

$$sk(K_{r,m} \Box T) = rmn - (r + m)(n + 1) + 4.$$

where $2 \leq r \leq m$.

Corollary 6 *Let T be a tree on n vertices. If $\Delta(T) \leq 2^m - 2$, then $Q_m \Box T$ is π-skew for $m, n \geq 1$, namely,*

$$sk(Q_m \Box T) = 2^{m-1}(nm - 2n - 2) + 4.$$

6 π-skew graphs with girth 3

In this section, the general π-skew graphs with girth 3 are considered. For convenience, Let Π_3 be the family of π-skew graphs with girth 3. By Theorem 1, a graph $G \in \Pi_3$ if and only if G has a spanning triangulation. The thresholds for the existence of spanning triangulation in terms of the minimum degree for a graph are given by Kühn, et al [16]. Specifically, among other results they proved the following theorem.

Theorem 10 ([16]) *There exists an integer n_0 such that every graph G of order $n \geq n_0$ and minimum degree at least $2n/3$ contains a triangulation as a spanning subgraph (i.e., $G \in \Pi_3$).*

Unfortunately, it seems that we could neither omit the condition that $n \geq n_0$ nor know the exact value of n_0 in [16]. Some sufficient conditions in terms of the minimum degree and size for a graph, which ensure a spanning triangulation, will be given in the rest of this section.

First we set up some notation. For any graph $G = (V, E)$, let $\blacktriangle(G)$ be the set of triangles in G, i.e.,

$$\blacktriangle(G) = \{\{u_1, u_2, u_3\} : G[\{u_1, u_2, u_3\}] \cong C_3\}.$$

Let $\blacktriangle_G$ be the graph with vertex set $\blacktriangle(G)$ such that $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ is adjacent in $\blacktriangle_G$ whenever $|\{u_1, u_2, u_3\} \cap \{v_1, v_2, v_3\}| = 2$.

A graph G is said to be locally Hamiltonian if the subgraph induced by the neighborhood of each vertex $x \in V(G)$, $G[N_G(x)]$, is Hamiltonian. We denote by $G \cdot e$ the graph obtained from G by contracting $e \in E(G)$, i.e. by deleting e, identifying its endvertices, and merging each resulting multiple edge, if any, into a single edge. A edge $e \in E(G)$ is said to be a contractible edge of G if $G \in \Pi_3$ and $G \cdot e \in \Pi_3$.

Next we start with the following properties for the triangulation.

Proposition 1 ([23]) *Every triangulation with at least four vertices is 3-connected.*

Proposition 2 ([22]) *Every triangulation with at least four vertices is locally Hamiltonian.*

Proposition 3 ([3]) *Every minimal cut-set of a triangulation with at least four vertices induces a cycle*

Note that a plane graph is 3-connected if and only if its dual graph is 3-connected [24], the following assertion is intuitively obvious.

Proposition 4 *The dual graph of every triangulation with at least four vertices is a 3-regular and 3-connected plane graph.*

It turns out in [12] that every vertex in a triangulation with at least four vertices is incident with at least two contractible edges. Therefore, the following claim is clear.

Proposition 5 *Every triangulation with at least four vertices has at least p contractible edges.*

From Propositions 1-5 and Lemma 7, we have following results. The proofs are almost intuitively obvious and hence they are omitted.

Lemma 11 *Let $G = (V, E)$ be a connected graph on $p \geq 4$ vertices with girth 3.*

(i) *If $G \in \Pi_3$, then G is 3-connected;*

(ii) *If $G \in \Pi_3$, then the subgraph induced by $N_G(x)$ contains cycles for each $x \in V$;*

(iii) *If $e \in E(G)$ is contained in at most one triangle of G, then $G \in \Pi_3$ if and only if $G - e \in \Pi_3$;*

(iv) *For any edge $e = v_1v_2$, if there are no cycle C_i in $G[N_G(v_i)]$ containing a path $u_1v_{3-i}u_2$ for both $i = 1, 2$, where $u_1, u_2 \in N_G(v_1) \cap N_G(v_2)$, then $G \in \Pi_3$ if and only if $G - e \in \Pi_3$;*

(v) *If $G \in \Pi_3$, then $\blacktriangle_G$ has an induced subgraph of order $2p-4$ which is 3-regular, 3-connected and planar;*

(vi) *If $G \in \Pi_3$, then for any vertex-cut set S, $|S| \geq 3$ and $G[S]$ contains cycles;*

(vii) *If $G \in \Pi_3$, then G has at least p contractible edges.*

Before stating our main results, we give the following lemmas which will be used.

Lemma 12 *Let C_n be a cycle on n vertices, then $\overline{C}_n \in \Pi_3$ for $n \geq 8$.*

Proof. Let $V(C_n) = \{v_0, v_1, \cdots v_{n-1}\}$ and,

$$E(C_n) = \begin{cases} \{v_0v_1\} \cup (\bigcup_{i=0}^{\frac{n-4}{2}} v_{2i+1}v_{2i+3}) \cup \{v_{n-1}v_{n-2}\}(\bigcup_{i=1}^{\frac{n-2}{2}} v_{n-2i}v_{n-2i-2}) & if\ n\ is\ even, \\ \\ \{v_0v_1\} \cup (\bigcup_{i=0}^{\frac{n-5}{2}} v_{2i+1}v_{2i+3}) \cup \{v_{n-1}v_{n-2}\}(\bigcup_{i=0}^{\frac{n-3}{2}} v_{n-2i-1}v_{n-2i-3}) & otherwise. \end{cases}$$

Let F_n be a spanning subgraph of $\overline{C}_n$ with edge set

$$E(F_n) = (\bigcup_{i=1}^{n-3} v_iv_{i+1}) \cup (\bigcup_{i=4}^{n-2} v_1v_i) \cup (\bigcup_{i=2}^{n-5} v_{n-2}v_i) \cup (\bigcup_{i=1}^{4} v_{n-1}v_i) \cup (\bigcup_{i=1}^{4} v_0v_{n-i-1})$$

for $n \equiv 0\ (mod\ 2)$, and for $n \equiv 1\ (mod\ 2)$

$$\begin{aligned} E(F_n) &= (\bigcup_{i=1}^{n-4} v_iv_{i+1}) \cup (\bigcup_{i=4}^{n-3} v_1v_i) \cup (\bigcup_{i=2}^{n-6} v_{n-3}v_i) \cup (\bigcup_{i=1}^{4} v_{n-2}v_i) \cup (\bigcup_{i=4}^{5} v_{n-1}v_i) \\ &\quad \cup \{v_{n-1}v_1\} \cup (\bigcup_{i=1}^{4} v_0v_{n-i-2}). \end{aligned}$$

Figure 3 and 4 illustrate a triangulation of the graph F_8 and F_9, respectively. The natural extension of these two drawings implies that there is a triangulation of F_n for any $n \geq 8$. This implies that $\overline{C}_n \in \Pi_3$ for $n \geq 8$. □

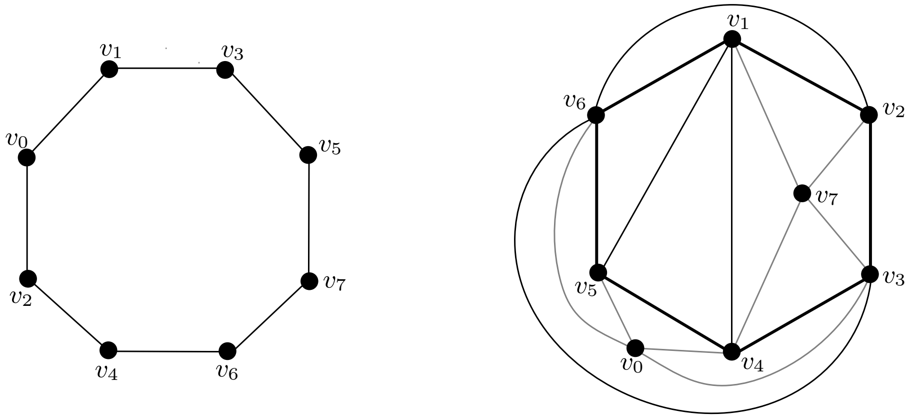

Figure 3: The cycle C_8 and a triangulation of F_8

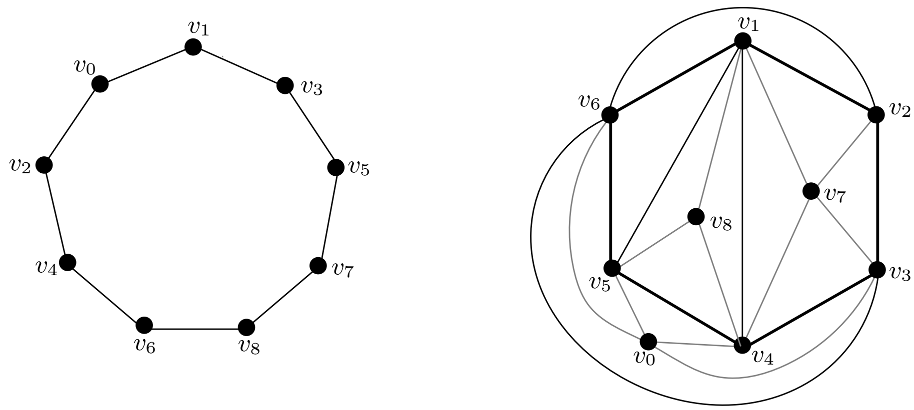

Figure 4: The cycle C_9 and a triangulation of F_9

For brevity, it is convenient to denote

$$K_{p_1,p_2,\cdots,p_k}(n_1,n_2,\cdots,n_k) = K_{\underbrace{p_1,p_1,\cdots,p_1}_{n_1},\underbrace{p_2,p_2,\cdots,p_2}_{n_2},\cdots,\underbrace{p_k,p_k,\cdots,p_k}_{n_k}}$$

Lemma 13 *Let C_n be a cycle on n vertices, then $\overline{C}_n$ contains $\lfloor (n-3)/2 \rfloor$ edge-disjoint Hamiltonian cycles for $n \geq 3$.*

Proof. From [2], we know that K_n contains $\lfloor (n-1)/2 \rfloor$ edge-disjoint Hamiltonian cycles for $n \geq 1$, which implies that our claim follows. □

Lemma 14 *Let H_1 and H_2 be vertex-disjoint Hamiltonian graphs, then $H_1 + H_2$ is Hamiltonian.*

Proof. The proof is almost intuitively obvious. □

Lemma 15 *$K_{3,4}(n_1, n_2)$ is a traceable graph when $n_1+n_2 \geq 2$, and further $K_{3,4}(n_1, n_2) \in \Pi_3$ when $n_1 + n_2 \geq 3$.*

Proof. The second assertion follows directly from Corollary 1. Thus, we only need to show that the first claim holds.

Obviously, both $K_3(n)$ and $K_4(n)$ are Hamiltonian. Therefore, $K_{3,4}(n_1, n_2)$ is Hamiltonian for $n_1, n_2 \geq 2$ by Lemma 14, since $K_{3,4}(n_1, n_2) \cong K_3(n_1) + K_4(n_2)$. Additionally, it is a routine task to prove that $K_{3,4}(1, 1)$ is a traceable graph. Hence, it suffices to show that both $K_{3,4}(1, n_2)$ and $K_{3,4}(n_1, 1)$ are traceable graphs for $n_1, n_2 \geq 2$. Note that $K_{3,4}(1, n_2) \cong \overline{K}_3 + K_4(n_2)$ and $K_{3,4}(n_1, 1) \cong K_3(n_1) + \overline{K}_4$, together with the fact that both $K_3(n_1)$ and $K_4(n_2)$ are Hamiltonian, we also conclude that the first assertion follows. The proof of this observation is merely technical. □

Lemma 16 *Let H be a Hamiltonian graph, then $(K_2 \cup K_2) + H \in \Pi_3$.*

Proof. Let $|V(H)| = n$. By Lemma 7, it suffices to show that the result holds when $H = C_n$. Figure 5 illustrates a spanning triangulation of $(K_2 \cup K_2) + C_n$. Hence, the claim follows. □

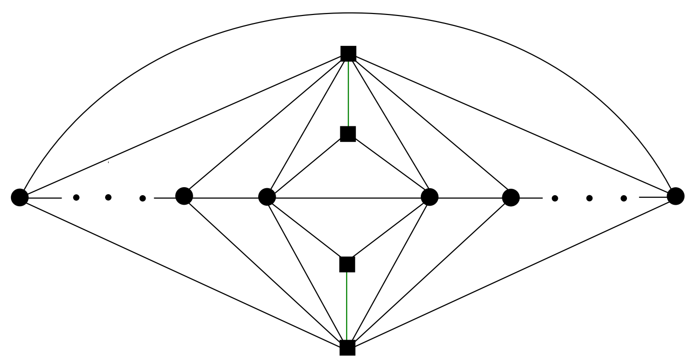

Figure 5: A spanning triangulation of $(K_2 \cup K_2) + C_n$

Lemma 17 *Let H be a Hamiltonian graph but not a cycle, then $\overline{K}_3 + H \in \Pi_3$.*

Proof. It is not hard to observe that H contains a spanning subgraph H', which consists of a Hamiltonian cycle of H along with a chord of the cycle. Let D be a planar drawing of H'. Then $|F(D)| = 3$. Inserting three new vertices inside three faces of D, respectively, and joining each new vertex v to all vertices on the boundary of face which contains v. The resulting drawing is a spanning triangulation of $\overline{K}_3 + H \in \Pi_3$. The result thus follows. □

Now we give the following two thresholds for the existence of a spanning triangulation, depending on the minimum degree and size of graph, respectively.

Theorem 11 *Let G be a $(p-3)$-regular graph on p vertices. Then $G \in \Pi_3$ when $p \geq 9$.*

Proof. Since G is a $(p-3)$-regular graph on p vertices, the complement graph $\overline{G}$ of G is 2-regular. With $\mathcal{C}$ we denote the family of connected components of $\overline{G}$. Obviously, each member of $\mathcal{C}$ is homeomorphic to a cycle. Let

$$\mathcal{C}_i = \{F : F \in \mathcal{C} \text{ and } |V(F)| = i+2\},$$

and

$$\mathcal{C}_3 = \{F : F \in \mathcal{C} \text{ and } |V(F)| \geq 5\}.$$

where $i = 1, 2$. Set $|\mathcal{C}| = c$ and $|\mathcal{C}_i| = c_i$ for $i = 1, 2, 3$. Clearly,

$$\mathcal{C} = \bigcup_{i=1}^{3} \mathcal{C}_i \text{ and } c = \sum_{i=1}^{3} c_i.$$

One can easily see that $\overline{F} \cong \overline{K}_3$ if $F \in \mathcal{C}_1$, and $\overline{F} \cong K_2 \cup K_2$ if $F \in \mathcal{C}_2$. By Lemma 13, we know that $\overline{F}$ is Hamiltonian if $F \in \mathcal{C}_3$ and $\overline{F}$ is not a cycle further if $|V(F)| \geq 6$. We distinguish the following four cases depending on the value of $c_1 + c_2$.

Case 1. $c_1 + c_2 = 0$. In this case it means that $c_3 = c$. If $c_3 = 1$, then $\overline{G}$ is a cycle with at least 9 vertices, which implies that $G \in \Pi_3$ from Lemma 12 and the fact that $\overline{\overline{G}} \cong G$. If $c_3 \geq 2$, then G contains $C_{p_1} + C_{p_2} + \cdots, +C_{p_{c_3}}$ as a spanning subgraph, where $p = p_1 + p_2 + \cdots + p_{c_3}$. The result thus follows by applying Theorem 3 and 5 repeatedly.

Case 2. $c_1 + c_2 = 1$. It follows that $c_3 \geq 1$ from $p \geq 9$. If $c_1 = 1$, then it is not hard to check that G contains $\overline{K}_3 + H$ as a spanning subgraph by Lemma 14, where H is a Hamiltonian graph but not a cycle. Lemma 17 implies that the result holds. If $c_2 = 1$, then G contians $(K_2 \cup K_2) + C_{p-4}$ as a spanning subgraph, which means that $G \in \Pi_3$ by Lemma 16.

Case 3. $c_1 + c_2 = 2$. By $p \geq 9$, we know that $c_3 \geq 1$. The result follows directly from Theorem 3, since G contains $K_{3,4}(c_1, c_2) + C_{p-3c_1-4c_2}$ as a spanning subgraph and $K_{3,4}(c_1, c_2)$ is a traceable graph by Lemma 15.

Case 4. $c_1 + c_2 \geq 3$. If $c_3 = 0$, then G contains $K_{3,4}(c_1, c_2)$ as a spanning subgraph. The result therefore follows from Lemma 15. If $c_3 \geq 1$, then the same reason as the case 3 obtains an affirmative conclusion again.

We have exhausted all possible cases, so the result follows. $\square$

The following claim is an immediate consequence of Lemma 7 and Theorem 11.

Corollary 7 *Let G be a graph on p vertices with minimum degree at least $p-3$. Then $G \in \Pi_3$ when $p \geq 9$.*

Remark. The condition of minimum degree in Theorem 7 can not get weaker. Otherwise, there exist counterexamples. For instance, let $G = C_5 + \overline{K}_4$. Observe that G has minimum degree $|V(G)| - 4 = 5$. But it is impossible that $G \in \Pi_3$ by Lemma 11(ii), since the subgraph induced by $N_G(x)$ does not contain any cycle for any vertex $x \in V(G)$ of degree 6.

Theorem 12 *Let G be a graph on p vertices and at least $\binom{p-1}{2} + 3$ edges. Then $G \in \Pi_3$.*

Proof. Clearly, $p \geq 4$, as G is a simple graph. First we claim that G contains a vertex of degree greater than $p-3$. Otherwise,

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} deg_G(v) \leq \frac{p(p-3)}{2} < \binom{p-1}{2} + 3,$$

a contradiction.

We now prove the theorem by induction on p. Note that for $p = 4$ the theorem just says that $K_4 \in \Pi_3$. Suppose now the theorem is true for all graphs of order $p \leq k$ ($k \geq 4$), and we consider the case that $p = k+1$. The following two cases are distinguished according to $\Delta(G) = p-1$ or $p-2$, respectively.

Case 1. $\Delta(G) = p-1$. Let $deg_G(x) = p-1$ and set $G' = G-x$. If each pair of vertices in G' is connected by an edge of G'. Then, $G \cong K_p$ and the conclusion follows from Lemma 3. Consequently, without loss of generality, assume that $u, v \in V(G')$ are nonadjacent in G'. Let G'' be the graph obtained from G' by adding a new edge e_{uv} joining u to v. Then, $|V(G'')| = |V(G')| = p-1$ and

$$|E(G'')| = |E(G')| + 1 = |E(G)| - p + 2 \geq \binom{p-2}{2} + 3.$$

Therefore, $G'' \in \Pi_3$ by induction hypothesis. Let $D'' \in \mathcal{PSK}(G'')$. If $e_{uv} \notin E(D'')$, then $D'' \in \mathcal{PSK}(G')$. This means that $G' \in \Pi_3$. Moreover, $G = G' + x \in \Pi_3$ by Theorem 4. Assume now that $e_{uv} \in E(D'')$, and let $D' = D'' - e_{uv}$. Note that D''

is 3-connected, we know that D' is 2-connected. Since there are exactly two faces incident with e_{uv} in D'', all faces are triangles apart from one quadrangle face in D'. We will obtain a spanning triangulation of G from D'. For this purpose, we first insert vertex x inside the quadrangle face in D', and then join x to each vertex on the quadrangle. One can easily observe that the resulting drawing is a spanning triangulation of G. Therefore, the result follows in this case.

Case 2. $\Delta(G) = p-2$. Let $deg_G(x) = p-2$ and set $G' = G - x$. It follows that $|V(G')| = p-1$ and

$$|E(G')| = |E(G)| - p + 2 \geq \binom{p-2}{2} + 3.$$

From induction hypothesis it follows that $G' \in \Pi_3$. Let $D' \in \mathcal{PSK}(G')$, then $|F(D')| = 2p-6$. Assume that x and y are nonadjacent in G. We conclude that there exists at least one face $f \in F(D')$ which is not incident with y. Otherwise,

$$deg_{D'}(y) = |F(D')| = 2p - 6 \leq p - 2$$

which is absurd for $p \geq 5$. Let $f' \in F(D')$ be the face which is not incident with y. Then, placing x inside f' and joining x to each vertex on the boundary of f'. It is not difficult to see that the resulting drawing is a spanning triangulation of G. Thus, the claim holds again.

Thereby the above two cases finish the proof of our theorem. $\square$

Remark. The constraint condition of edges in Theorem 12 is best possible for all $p \geq 4$. Since, for any $p \geq 4$, there are graphs on p vertices and $\binom{p-1}{2} + 2$ edges without a spanning triangulation. For instance, let G denote the graph obtained from complete graph K_p by deleting $p-3$ edges which are incident with some vertex of K_p, say x. It is easily seen that G has p vertices and $\binom{p-1}{2}+2$ edges. However, x is a vertex of degree 2 of G, which means that $G \notin \Pi_3$ whenever $p \geq 4$ by Lemma 11(i).

Indeed, even if there are no vertices of degree 2, we can also not guarantee that the graph has a spanning triangulation. For example, let G be the graph obtained from K_p by deleting $p-4$ edges with which are incident some vertex of K_p, say x. With v_1, v_2, v_3 we denote the three vertices which are adjacent to x in G, and set $G' = G - v_1v_2$. Observe that G' has p vertices and $\binom{p-1}{2} + 2$ edges, and $\delta(G') = 3$. But it is impossible that $G \in \Pi_3$ by Lemma 11(ii), since the subgraph of G' induced by the neighborhood $\{v_1, v_2, v_3\}$ of x does not contain any cycle.

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