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The Groundhog Problem

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The Groundhog Problem is stated as follows: "A groundhog has made an infinite number of holes roughly a metre apart in a straight line in both directions on an infinite plane. Every day it travels a fixed number of holes in one direction. A farmer would like to catch the groundhog by shining a torch, but only once a night, into one of the holes at midnight when it is asleep. What strategy can the farmer use to ensure that he catches the groundhog eventually?" It turns out that the solution of this problem relies on a set-theoretic concept usually taught at the tertiary level. One key purpose of this paper is to explicitly articulate the problem solving trajectory of a professional mathematician who is cognizant of his/her own problem solving disposition and thinking.

Keywords: tertiary mathematics • problem solving

Introducing the Groundhog

Groundhog Day is a traditional holiday, which originated from Pennsylvania German custom in eastern and central Pennsylvania, that is celebrated on February 2 in USA and Canada. Folklore has it that if it is cloudy when a groundhog emerges from its burrow on that day, then early arrival of spring is expected; if it is sunny, the groundhog supposedly sees its own shadow and retreats into its underground den, and winter will persist for another six weeks.

Figure 1. A groundhog

The *groundhog* (Marmota monax), also known as a *woodchuck*, or *whistlepig*, is a rodent of the family Sciuridae, belonging to the group of large ground squirrels known as marmots (see Figure 1). Monax, the American Indian name for woodchuck, means “the digger” – an attribution to its expertise in digging burrows. Groundhog burrows usually have two to five entrances called burrow holes, providing groundhogs their primary means of escape from predators. Burrows are particularly large, with up to 14 metres of tunnels buried up to 1.5 metres underground, and can pose a serious threat to agricultural and residential development by damaging farm machinery and even undermining building foundations (Wikipedia, 2020). Because of these threats, groundhogs' nemesis is none other than human beings.

The groundhog problem

Consider the hypothetical situation of a lazy, but mathematically inclined, farmer wanting to hunt down a lone groundhog which resides in an infinitely long underground tunnel extending indefinitely to the east and the west across his (infinitely long) farm. Rising up from this tunnel are a series of burrow holes, spaced about one metre apart. We assume that groundhogs sleep directly under a burrow hole at any point of time, and if the farmer shines a flashlight into the right burrow hole he will see the groundhog in his tunnel.

Groundhogs are known to stay very alert in the day and move very quickly so that it is hard to hunt them down. The farmer's only chance is to nab the groundhog at night when it lies sleeping in its tunnel under a hole. Being a lazy farmer, he shines light into *only one* burrow hole each night; if the groundhog is there, he catches it and goes to bed. If he doesn't catch it he tries again the next night.

There is a caveat, though, on the part of the groundhog: the motion of the groundhog is regular in the sense that as long as it lives the groundhog moves at a constant velocity, i.e., on any given day it moves *a fixed number of holes* in *one fixed direction* away from the resident hole of the previous night.

Problem statement. You are the farmer. Devise a strategy for inspecting the burrow holes, which guarantees that you will nab the groundhog in finite time. The problem is, however, that you don't know which hole it was in yesterday, how far it moved per day, or which direction it is heading in.

At this juncture, we invite the reader to *stop* reading and *try* solving the problem yourself.

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Thanks for resuming your perusal. Clearly, you found it unwise to randomly pick a hole each night to inspect and hope that the groundhog is there.

In what follows, we shall document our problem solving journey following the Problem Solving strategies put forth by George Pólya (Pólya, 1957): (UP) Understanding the Problem, (DP) Devising a Plan, (CP) Carrying out the Plan, and (CE) Checking & Expanding.

We begin by *acting it out*. For bookkeeping purposes, let us label the burrow holes with integers

$$\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots,$$

adopting the convention that integers run in an increasing order eastwards.
The set of integers is denoted by $\mathbb{Z}$ as usual.
The frustration of not being able to locate the groundhog stems from three unknowns:

(i) the initial hole number, p, where the groundhog was the day prior to the first day the farmer decides to take action;
(ii) the (constant) number of holes, q, that the groundhog moves per day; and
(iii) the direction in which the groundhog is heading, i.e., positive or negative.

Understanding the problem a little more, we proceed to create some useful definitions.

Definition. (Hole No. and q-groundhog)
We refer to the integral label of a burrow hole as a *Hole No*. Let the Hole No. inspected on the nth night be denoted by H_n, where $n \geq 1$. By (i), we define $H_0 = p$.
A groundhog that moves with velocity, q holes per day (where $q \in \mathbb{Z} - \{0\}$), is called a groundhog of type q or simply, a *q-groundhog*.

I saw it yesterday: Solving a simpler problem

Let us now get to the drawing board *devising a plan*. Why not *solve a simpler problem* first? I saw the groundhog was at Hole No. 0 last night but I was too sleepy to catch it then! Now, where will it be tonight?

It moved eastwards, I swear. I am sure I heard it scurrying to the east! So in this case, we have $p = 0$ and the groundhog moves in the positive direction. We just have to guess what the positive integer q is. On second thought, perhaps we can *guess and check* in a systematic way. *Making suppositions*. Suppose this underground fellow is a 1-groundhog. Then I will go to Hole No. 1 tonight and find it there. If I don't find it there, it must have been *at least* a 2-groundhog. So, on the second night, I will inspect Hole No. 4 for suppose it were a 2-groundhog it would have moved $2 \times 2 = 4$ holes eastwards in 2 days. The inspection schedule for the first 6 days appears in Table 1, with its first 3 days enactment in Figure 2.

Table 1.
Inspection schedule for p = 0 and q > 0

n	q	H_n	= ?
1	$+1$	$1 \times (+1)$	1
2	$+2$	$2 \times (+2)$	4
3	$+3$	$3 \times (+3)$	9
4	$+4$	$4 \times (+4)$	16
5	$+5$	$5 \times (+5)$	25
6	$+6$	$6 \times (+6)$	36

Inspection enacted for p = 0 and q > 0

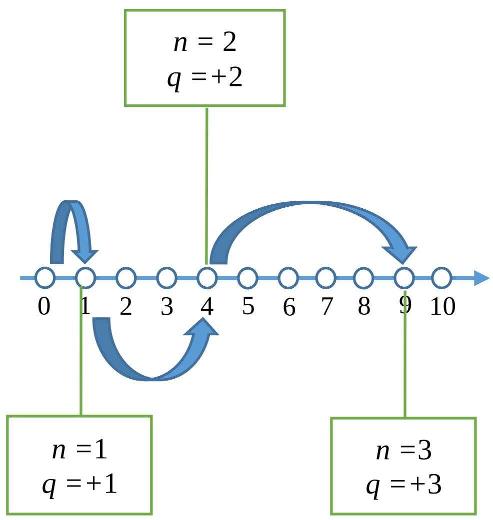

Figure 2.

Going on this way, the supposition that the groundhog hiding in my farm were an *n*-groundhog will be made on the *n*th night. Thus, on an *n*th night I will head to Hole No. $n \times n = n^2$. Since any groundhog out in the field is some *q*-groundhog, I will certainly catch it one of these nights – a *q*th-groundhog's doomsday, according to my strategy, is the *q*th night.

God knows where it has headed. If I stick to the eastward strategy described in the preceding section, I might miss the groundhog altogether because it could have been a negative groundhog, i.e., a *q*-groundhog with $q < 0$. But by symmetry, it wouldn't work either by just relying on a westward version of the above strategy. This naturally leads us to the idea of *interleaving* the two strategies: on the odd nights, we go eastwards searching for an eastward groundhog and on the even nights the westward one. Let's *carry out this plan*!

Making a systematic list. We plan our inspection schedule in the form of Table 2 below, and the corresponding enactment for the first 3 days is given in Figure 3.

Table 2.
Inspection schedule for p = 0 and q ≠ 0

n	q	H_n	= ?
1	$+1$	$1 \times (+1)$	1
2	-1	$2 \times (-1)$	-2
3	$+2$	$3 \times (+2)$	6
4	-2	$4 \times (-2)$	-8
5	$+3$	$5 \times (+3)$	15
6	-3	$6 \times (-3)$	-18

Inspection enacted for p = 0 and q ≠ 0

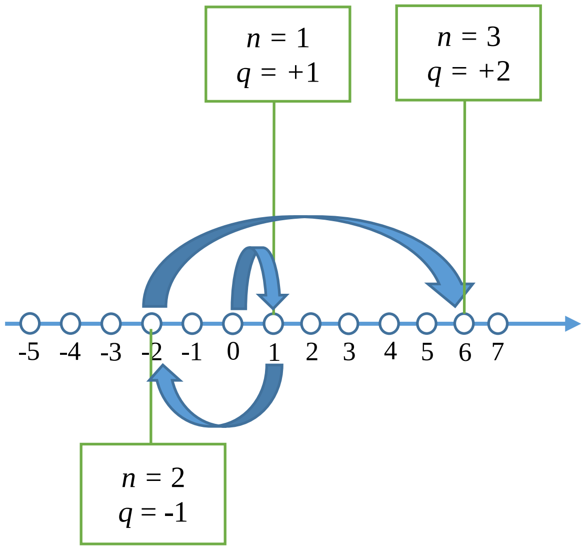

Figure 3.

In closed form, we have, for any positive integer *n*,

$$H_n = \begin{cases} -\dfrac{n^2}{2} & \text{if } n \equiv 0 \pmod 2 \\ \dfrac{n(n+1)}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

What have we so got so far?

All the hard work put into our first two days (yes, two days!) of problem solving has started to pay off. The simplified problem is substantially easier since we have the extra information about the initial position of the groundhog, i.e., $p = 0$. Actually, there is nothing special about $p = 0$; we could have fixed your favourite p' value, and the corresponding strategy is obtained by applying the previous strategy for $p = 0$, followed by a constant shift of p' holes. In that case, the closed formula would look like:

$$H_n = \begin{cases} p' - \dfrac{n^2}{2} & \text{if } n \equiv 0 \pmod 2 \\ p' + \dfrac{n(n+1)}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases} \quad (1)$$

The original Groundhog Problem is much more vexing, or at least the way it is being phrased, because we have to grapple with two unknown integers *p* and *q*, each of which can take on infinitely many values. We are really clueless how to go on. Should we abandon mission?

Metacognitive stance. Wait! But in the first place didn't we already have an infinitude of *q*'s to grapple with in the simplified problem? It looks like having an infinite set of possible values of *q* (be it the first case where *q* takes only positive integral values or in the second case where *q* takes only all possible integral values) did not deny us of a solution – we have a way of handling it. What we need to see is exactly which handle we used to crank the machine!

A closer look at the Tables 1 and 2 may shed some insights. For the first case, *q* is a positive integer. In Table 1, for each day *n*, we are checking out the *n*th possible value of *q*, making sure not to miss out any positive integer *q*. Indeed you will never miss out because $q = n$, for some *n*. As you move on from one day to the other, what you are checking is whether the next *n* value for *q* is the type of the groundhog out there in the field, knowing that the previous one isn't.

As for the second case, we wish to run through all the possible integral values of *q* for each *n*th day. Again, for each natural number *n*, we must match up a unique integral value of *q* in such a way that no integers are missed out. Eye-balling the elements along the first two columns headed by "*n*" and "*q*" in Table 2 makes us realize that we were actually numbering off the possible integral values of *q*, without missing out any single one of them in this counting process.

Table 3 shows how we can number off the set $\mathbb{Z}^+$ of all positive integers (respectively, the set $\mathbb{Z} - \{0\}$ of all non-zero integers) without missing any element.

Table 3.
Numbering off elements in $\mathbb{Z}^+$ and $\mathbb{Z} - \{0\}$ without missing out any elements

Case 1 $p = 0, q > 0$	$q \in \mathbb{Z}^+$	+1	+2	+3	+4	+5	+6
		↕	↕	↕	↕	↕	↕
	$n \in \mathbb{N}$	1	2	3	4	5	6
		↕	↕	↕	↕	↕	↕
Case 2 $p = 0, q \neq 0$	$q \in \mathbb{Z}$-{0}	+1	-1	+2	-2	+3	-3

The 'reason' for not missing out on any elements in $\mathbb{Z}^+$ or $\mathbb{Z} - \{0\}$ is that there is a one-to-one correspondence between the elements in each of these sets with those in $\mathbb{N}$, denoted by the symbol ↕ which we use in Table 3. This two-headed arrow is chosen with intention. Suppose one wishes to number off elements in a non-empty set *S*. Then this one-to-one correspondence will enable us to assign for each element *s* of *S* one and only one natural number *n*, and also allow us to go backwards, i.e., for this assigned natural *n* one can trace back exactly to the unique element *s* in *S* that was assigned to *n*.

Definition. (Bijection, Countably infinite)
A function $f: A \longrightarrow B$ between two sets is a *one-to-one correspondence* (also known as *bijection*) if there exists a (necessarily unique) function $f^{-1}: B \longrightarrow A$ with the property that for each $a \in A$ and $b \in B$, $f^{-1}\big(f(a)\big) = a$ and $f\big(f^{-1}(b)\big) = b$.
A set *S* is *countably infinite* if there exists a bijection e_S (i.e., one-to-one correspondence) between *S* and the set of natural numbers, $\mathbb{N} = \{1,2,3,\cdots\}$. In that case, we call e_S an *enumeration* of *S*.

In the context of the simplified Groundhog Problem, where $p = 0$ and $q \in \mathbb{Z} - \{0\}$, the map

$$e_{\mathbb{Z}-\{0\}}: \mathbb{Z} - \{0\} \longrightarrow \mathbb{N}, q \mapsto \begin{cases} -2q & \text{if } q < 0; \\ 2q - 1 & \text{if } q > 0 \end{cases}$$

that counts off the numbers appearing in the "*q*" column, i.e., $e_{\mathbb{Z}-\{0\}}(q) = n$, is indeed a bijection between $\mathbb{Z} - \{0\}$ and $\mathbb{N}$. In fact, the inverse of $e_{\mathbb{Z}-\{0\}}$ is given by

$$e_{\mathbb{Z}-\{0\}}^{-1}: \mathbb{N} \longrightarrow \mathbb{Z} - \{0\}, n \mapsto \begin{cases} -\frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}.$$

We encourage the enthusiastic reader to check that these maps do indeed satisfy the equations

$$e_{\mathbb{Z}-\{0\}}\left(e_{\mathbb{Z}-\{0\}}^{-1}(m)\right) = m \text{ and } e_{\mathbb{Z}-\{0\}}^{-1}\left(e_{\mathbb{Z}-\{0\}}(n)\right) = n$$

hold for all $m \in \mathbb{N}$ and $n \in \mathbb{Z} - \{0\}$. So, the bijection $e_{\mathbb{Z}-\{0\}}$ gives the precise mathematical definition for the one-to-one correspondence, ↕, between the 3rd and 5th rows of Table 3.

On the *n*th night we are targeting the $e_{\mathbb{Z}-\{0\}}^{-1}(n)$-groundhog, and if the groundhog out there in my farm is indeed such, then I ought to be able to find it in Hole No. H_n, where

$$H_n = p + n \times e_{\mathbb{Z}-\{0\}}^{-1}(n) = \left(p + n \times e_{\mathbb{Z}-\{0\}}^{-1}(\cdot)\right)(n).$$

Note that the preceding equation is equivalent to Equation (1). The second representation "$\left(p + n \times e_{\mathbb{Z}-\{0\}}^{-1}(\cdot)\right)(n)$" is intentionally chosen because the groundhog's movement when encoded as

$$\lceil n \rfloor := p + e_{\mathbb{Z}-\{0\}}^{-1}(n)$$

may be regarded as an instruction for the farmer to carry out his inspection, i.e., multiplication by $e_{\mathbb{Z}-\{0\}}^{-1}(n)$ followed by addition of p is indeed a *function*! In summary, the simplified Groundhog Problem is solved in two stages:
(1) enumerating the set of possible *q*'s using the number, *n*, of nights that passed as the counting index allows us to target the corresponding *q*-groundhog (where $q = e_{\mathbb{Z}-\{0\}}^{-1}(n)$), and
(2) evaluating the movement function $\lceil n \rfloor$ at the input *n* enables us to determine the exact location of the groundhog once the target is locked on.

Building my resource for catching groundhogs

As described earlier, instead of the one-dimensional parameterization of a groundhog via the notion of a *q*-groundhog we now have a two-dimensional analogue:

Definition. ((*p*, *q*)-groundhog)
Let $p, q \in \mathbb{Z}$ $(q \neq 0)$ be given. A groundhog whose starting position is p and velocity q is called a (*p*, *q*)-*groundhog*.

Set theoretically speaking, instead of dealing with $\mathbb{Z} - \{0\}$ in the simplified Groundhog Problem, we now have to deal with the set $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ to which the possible pairs of numbers (p, q) belongs. This is to say that $\mathbb{Z} \times (\mathbb{Z} - \{0\}) \coloneqq \{(p, q): p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}\}$.
Drawing a diagram. To stomach this set better, let us visualize that each element (p, q) in the above set can be represented as a point in the two dimensional plane: p reads as the first component in the horizontal axis; and q reads as the second component in the vertical axis. So the set $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ can be represented by a set of dots in Figure 3. As you can see there are many types of groundhogs, such as the $(0,1)$-groundhog which is represented by the point $(0,1)$, and the $(17, -2)$-groundhog which is represented by the point $(17, -2)$. So, Figure 3 can be thought of as a chart that records all types of groundhogs.

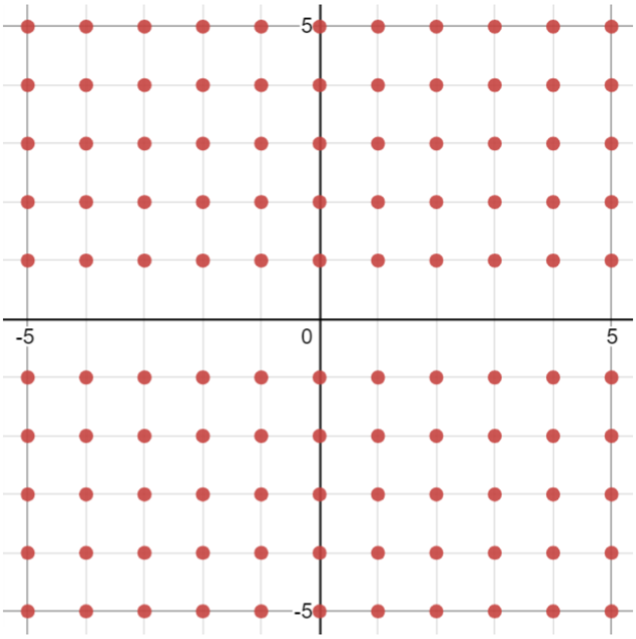

Figure 3. A chart of all types of groundhogs

Since the silver bullet for the simplified Groundhog Problem is the infinite countability of the set $\mathbb{Z}^+$(respectively, $\mathbb{Z} - \{0\}$), the most important question to ask is whether the set $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ is countable. Let us not be in a hurry to define the bijection between $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ and $\mathbb{N}$. Instead we look for an intuitive way to number off the elements of $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ without missing out a single element. We came up with two counting schemes, part of which are shown in Figure 4.

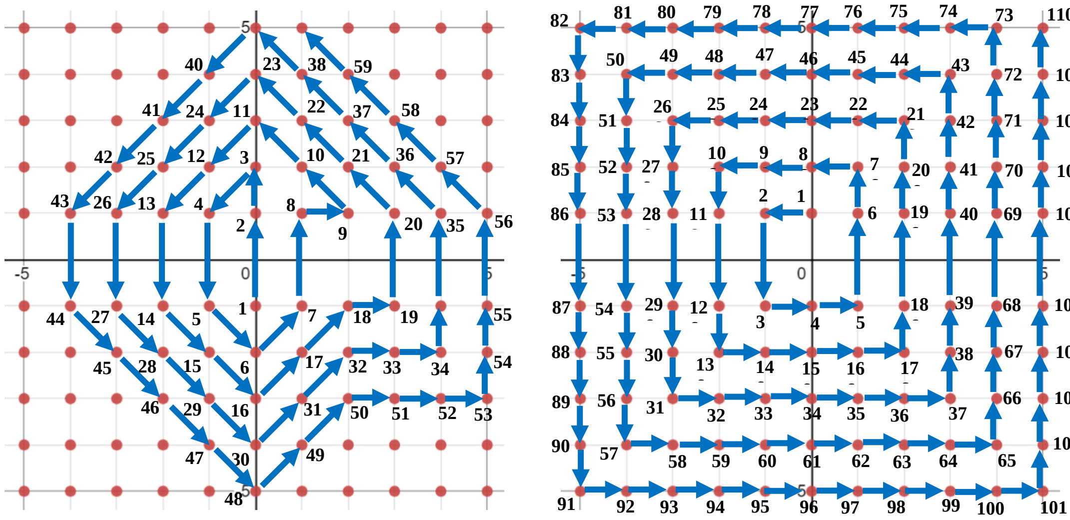

Figure 4. Two counting schemes for the set $\mathbb{Z} \times (\mathbb{Z} - \{0\})$

From the above two diagrammatical enumerations, it is clear that there is no unique enumeration. However, these two enumerations do not have simple formulae.

Here is another way to define an enumeration for $\mathbb{Z} \times (\mathbb{Z} - \{0\})$. For this, we rely on the standard enumeration of $\mathbb{N} \times \mathbb{N}$ using the bijection:

$$e_{\mathbb{N}\times\mathbb{N}}: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}, (m, n) \mapsto m + \frac{1}{2}(m + n - 2)(m + n - 1).$$

The counting scheme corresponding to $e_{\mathbb{N}\times\mathbb{N}}$ is partially given in Figure 5.

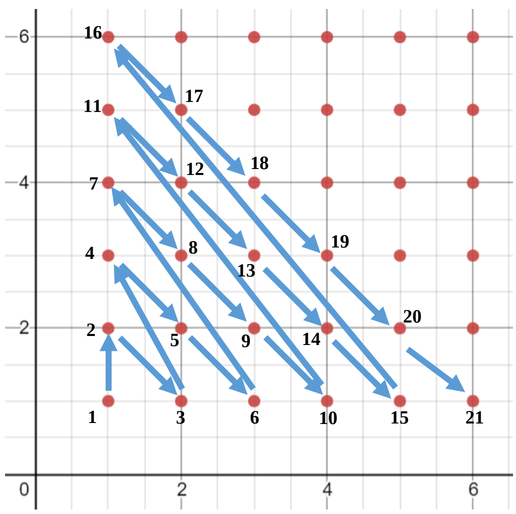

Figure 5. Counting scheme for $\mathbb{N} \times \mathbb{N}$

Using this above enumeration, we can also define the enumeration of $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ via the following composition of bijections:

$$e_{\mathbb{Z}\times(\mathbb{Z}-\{0\})} = e_{\mathbb{N}\times\mathbb{N}} \circ (e_{\mathbb{Z}} \times e_{\mathbb{Z}-\{0\}}),$$

where $e_{\mathbb{Z}} \times e_{\mathbb{Z}-\{0\}}: \mathbb{Z} \times (\mathbb{Z} - \{0\}) \longrightarrow \mathbb{N} \times \mathbb{N}, (p, q) \mapsto (e_{\mathbb{Z}}(p), e_{\mathbb{Z}-\{0\}}(q))$.

Check my thinking. Is this the only way to think about the enumeration of points in $\mathbb{Z} \times (\mathbb{Z} - \{0\})$? Interestingly we can label the horizontal axis and the vertical axis in a less conventional way; that is, mark out on the horizontal axis $0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$ and on the vertical axis $1, -1, 2, -2, 3, -3, 4, -4, \ldots$. Clearly, this manner of labelling ensures that none of the integers (respectively, non-zero integers) are missed out. This time, counting off all the elements in $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ can be enacted by a simpler and far more natural scheme (suppressing the counting indices) shown in Figure 6.

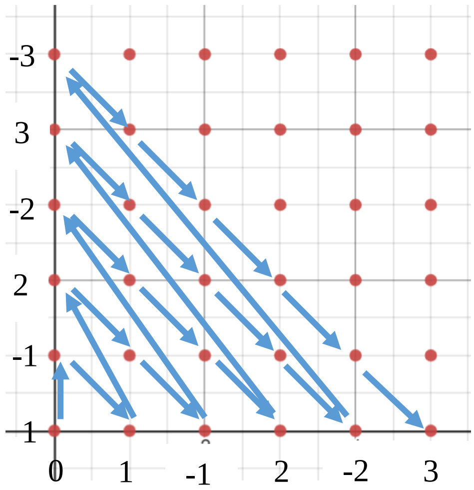

Figure 6. A natural way of enumerating $\mathbb{Z} \times (\mathbb{Z} - \{0\})$

The solution

Whichever bijection $e_{\mathbb{Z}\times(\mathbb{Z}-\{0\})}$ we define, all we care is to be able to list out (or enumerate) *every single* possibility of (p, q)-groundhogs day-by-day so as not to miss out any type of groundhogs. With whichever enumeration $e_{\mathbb{Z}\times(\mathbb{Z}-\{0\})}$ one has defined, we lay down the inspection schedule as follows. The original Groundhog Problem is now solved in two stages:

(1) enumerating the set of possible pairs (p, q)'s using the number, n, of nights that passed as the counting index allows us to target the corresponding (p, q)-groundhog (where $(p, q) = e^{-1}_{\mathbb{Z}\times(\mathbb{Z}-\{0\})}(n)$), and

(2) evaluating the movement function $\lceil n \rceil$ at the input n enables us to determine the exact location of the groundhog once the target is locked on. The movement function is defined as follows: for each $n \in \mathbb{N}$, define $\lceil n \rceil : \mathbb{N} \longrightarrow \mathbb{Z}$ by

$$\lceil n \rceil(m) = p + qm,$$

where $(p, q) = e^{-1}_{\mathbb{Z}\times(\mathbb{Z}-\{0\})}(n)$. Then the Hole No. to be inspected on the n<u>th</u> night is

$$H_n = \lceil n \rceil(n).$$

The above representation bears an uncanny resemblance to the Cantor's diagonalization argument which is a standard technique for (i) establishing the uncountability of certain sets, e.g., the set $\mathbb{R}$ of real numbers, and the power set $\mathcal{P}(\mathbb{N})$ of natural numbers (Cantor, 1891), and (ii) solving the Halting Problem (Turing, 1937). If you look through the lens of computability, those countable sets salient to the Groundhog Problem happen to be recursively enumerable, i.e., there exist algorithms (exactly those counting schedules) which enumerate the members of these sets. Every recursively enumerable set is countable, but there are countable sets which are not recursively enumerable. When we enact the inspection schedule by calculating $\lceil n \rceil(n)$, we are merely applying the algorithm encoded by the natural number n to n itself, and this is reminiscent of the standard fact that Turing machines can be encoded as natural numbers (Turing, 1937).

Conclusion

What has the Mathematician to say? The existence of a winning strategy, that is, one guarantees the eventual capture of the groundhog depends fundamentally on the countable infinitude of the groundhog types. The simplified Groundhog Problem with $p = 0$ and $q > 0$ relies on the enumeration of the set of natural numbers, $\mathbb{N}$, and that with just $p = 0$ can be regarded as the enumeration of the integers, $\mathbb{Z}$. Finally, the Groundhog Problem reduces to the enumeration of the set $\mathbb{Z} \times (\mathbb{Z} - \{0\})$, or equivalently the set of rational numbers, $\mathbb{Q}$.

The Groundhog Problem can be *extended* in many interesting ways. One extension is to consider a lattice set $\mathbb{Z} \times \mathbb{Z}$ of burrow holes, allowing two-dimensional movement for the groundhog. Another extension is to consider groundhogs of polynomial types, i.e., its motion in terms of *n* is determined by a polynomial $P(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k$, where $k \in \mathbb{N}$ and $a_k \neq 0$. Since the first extension relies on the countability of $\mathbb{Z}^4$ and the second that of $\bigcup_{k \in \mathbb{N}}^{\infty} \mathbb{Z}^k$, strategies for the eventual capture of the groundhog exist.

What has the Educator to say? The Groundhog Problem was given by the second author to the first during his visit to the National Institute of Education, Singapore, in 2016. This problem is a rare gem in that it presents itself as a genuine mathematics problem, and whose solution makes essential use of the concept of countability. By documenting an authentic problem solving trajectory of the first author who is a working mathematician (as he ploughed through the Groundhog Problem), this article makes explicit the fine-grained details of applying the Pólya Problem Solving framework (UP-DP-CP-CE) and various problem solving heuristics (working backwards, simplifying the problem, guess and check, making suppositions, drawing a diagram, acting it out), weaved together with problem solving control, resources and beliefs (Schoenfeld, 1992). It is hoped that the authors have given sufficient encouragement to mathematics students (especially, tertiary level) to put into practice whatever problem solving dispositions that have been illustrated.

What has the Mathematician Educator to say? A mathematics teacher who bases his or her mathematics pedagogy on rigorous mathematical disciplinarity and sound up-to-date educational research – termed as ‘Mathematician Educator’ – may now use the Groundhog Problem as a prototype for illustrating how genuine problem solving in a mathematics classroom can be realized and supported with non-trivial mathematical nuances. The concept of countability is abstract and can be difficult to understand for undergraduate first year students. The Mathematician Educator can make use of the Groundhog Problem as a humorous way to introduce the countability.

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