The Mathematician Educator
2020, Vol 1, No. 2, 75-82

Four Solutions of a Geometry Problem

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This article focuses on a challenging geometry problem that was originally posed to primary school students. Four solution approaches, ranging from elementary to advanced, are discussed. Reflections on these approaches and the problem solving processes are also shared.

Keywords: geometry, elementary mathematics, problem solving

A Primary School Geometry Problem?

One of the authors was informed of the following challenging geometry problem through a discussion group for parents whose children attend a certain primary school. The problem is restated as follows: *Figure 1 contains five equal sized squares arranged in the shape of a cross,* AB *and* CD *intersect at* O. *Find the measure of* ∠AOC. This problem is atypical as there are no numerical values and participants of the discussion group expressed frustration at the seemingly impossible problem that was foisted on their children.

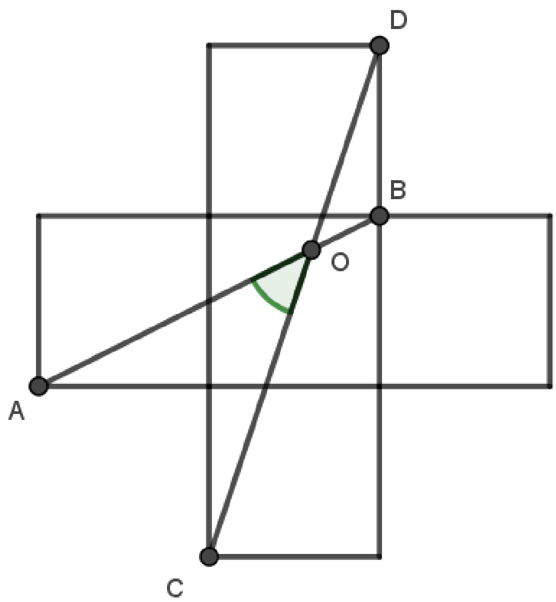

Figure 1. The problem

The problem was shared amongst the other authors, who independently attempted to solve the problem with varying degrees of success. Together, the authors came up with four different solutions to the problem. In the following sections, we present these solutions and consider implications both for the problem solver and the mathematics educator.

Solution 1: Trigonometry

In this approach, two triangles, ΔABE and ΔCFD, are identified in Figure 2. We have

$$\angle \mathrm{CDF} = \tan^{-1}\left(\frac{1}{3}\right) \text{ and } \angle \mathrm{ABE} = \tan^{-1}\left(\frac{1}{2}\right).$$

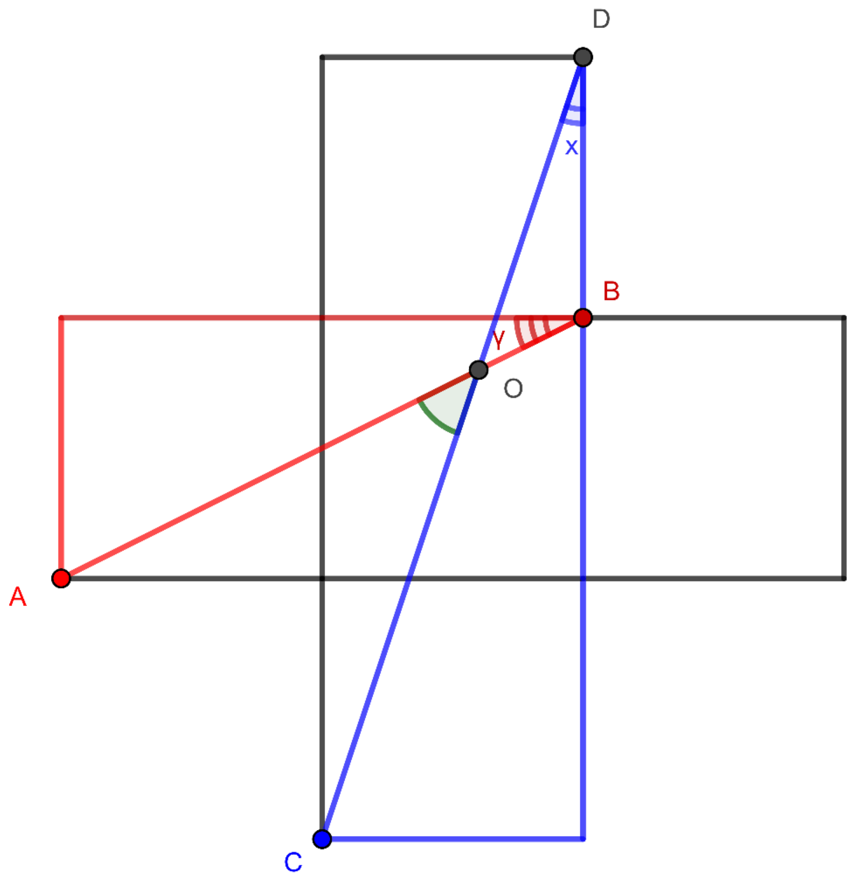

Figure 2. Trigonometric solution

Furthermore, ∠AOC = ∠DOB, and using the angle sum property of ΔDOB, we have

$$\begin{aligned} \angle \mathrm{AOC} &= 90° - \tan^{-1}\left(\frac{1}{3}\right) - \tan^{-1}\left(\frac{1}{2}\right) \\ &= 90° - \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}\right) \\ &= 90° - \tan^{-1}(1) = 45°. \end{aligned}$$

In the above, the arctangent addition formula was used. It can be argued that such an approach can be considered overly complicated as the original problem was supposed to be for primary school students, whereas trigonometry is usually taught in the upper levels at secondary schools. However, this was actually the first approach adopted by one of the authors, *Z*. The observation ∠AOC = ∠DOB was immediate, which led naturally to an attempt to identify the values of ∠CDF and ∠OBD. Since there were no numerical values specified in the problem, *Z* had to utilize the next best thing, the relative side lengths of the corresponding triangles to produce numerical quantities for calculation. Z next encountered the obstacle that neither $\tan^{-1}\left(\frac{1}{3}\right)$ nor $\tan^{-1}\left(\frac{1}{2}\right)$ evaluates to some well-known constant. But faced with the sum of two (inverse) trigonometric quantities, it was not too much of a stretch to believe that some trigonometric identity or formula could be used to resolve the problem. The above episode illustrates a natural and logical progression that might be undertaken by a problem solver who has learnt trigonometry, a branch of mathematics that was developed for the very purpose of studying the relationships between side lengths and angles of triangles.

It also illustrates the typical approach of a mathematician when trying to solve a problem. After understanding the problem à la Pólya (1957), she devises the most natural plan to solve the problem using whatever resources at her disposal. In the preface to his textbook, Nathanson (2000) wrote:

> In mathematics, when we want to prove a theorem, we may use any method. The rule is "no holds barred". It is OK to use complex variables, algebraic geometry, cohomology theory, and the kitchen sink to obtain a proof. But once a theorem is proved, once we know that it is true, particularly if it is a simply stated and easily understood fact …, then we may want to find another proof, one that uses only "elementary arguments" … Elementary proofs are not better than other proofs, nor are they necessarily easy. (p. ix)

Following Nathanson's suggestion, we next attempted to find an elementary geometry solution.

Solutions 2 and 3: Geometry

Re-examining Figure 1 again, we were once again confronted with the lack of data in the problem. We knew that the required answer was 45°. Actually, even if we had not completely solved the problem, it was reasonable to guess that the solution was either 30°, 45° or 60°. Prior experience informed us that we should introduce some auxiliary lines. Pólya (1957) had much to say about introducing auxiliary elements, including lines, unknowns or even theorems, to further one along the problem solving process. "In general, having recollected a formerly solved related problem and wishing to use it for our present one, we must often ask: Should we introduce some auxiliary element in order to make its use possible?" (p. 47). Here we were looking to introduce auxiliary lines in order to make the 45° evident.

Another of the authors, Y, introduced three auxiliary line segments as shown in Figure 3a. Since these segments were all diagonals of some 2 unit by 1 unit rectangle, a square (with O as one of its vertices) was formed. The diagram suggested that CD bisected this square and hence the required angle must be 45°. However, there was a problem with this line of reasoning, although the correct answer was obtained. Pólya (1957) remarked that "Geometry is the art of correct reasoning on incorrect figures." (p. 208). The previous line of reasoning would be the case of applying incorrect reasoning to correct figures. We cannot use the diagram to conclude that AB, CD and ST are concurrent!

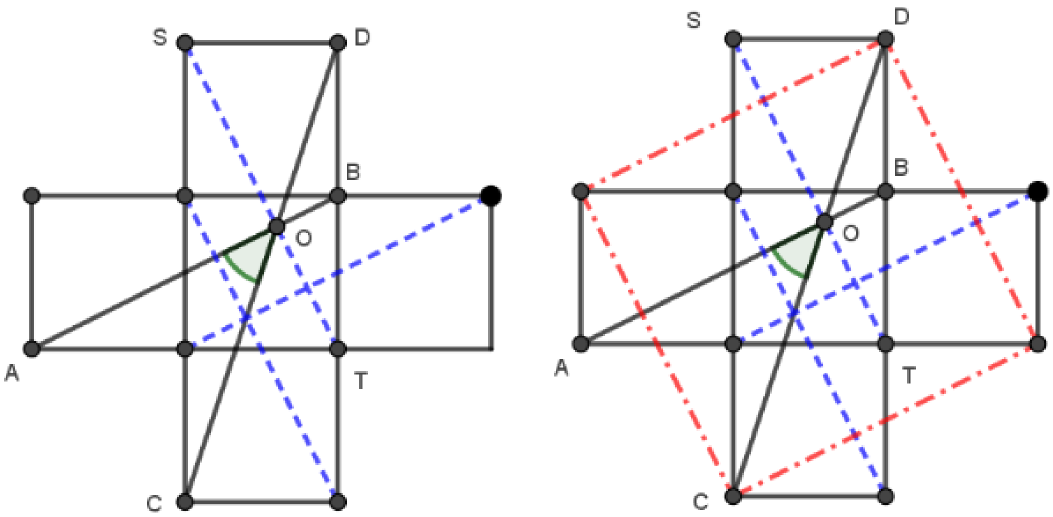

Figure 3. Adding auxiliary lines

To prove that CD bisected this square, *Y* introduced additional auxiliary line segments as shown in Figure 3b to form an outer square. Each side of the outer square was parallel to and equidistant from its corresponding side of the inner square. It was then possible to prove that CD, the diagonal of the outer square, coincided with the diagonal of the inner square. In retrospect, this solution is fairly complicated and possibly not suitable for primary school students who are mostly not exposed to the need for rigour in mathematical reasoning. However, it can serve as a suitable motivational context for secondary level students to learn about the rigours of proof. The teacher can present the incomplete "solution" based on Figure 3a and ask students to identify and rectify the gap in the argument.

A second attempt using geometry involved adding the auxiliary line segments CW and DW in Figure 4. The author who obtained the problem from his parent group, *X*, observed from this construction that CW was parallel to AB and thus $\angle AOC = \angle OCW$. Furthermore, ΔCWD was an isosceles right-angled triangle, which gave him the 45° that he was after. Among the authors, this solution was considered the most elegant.

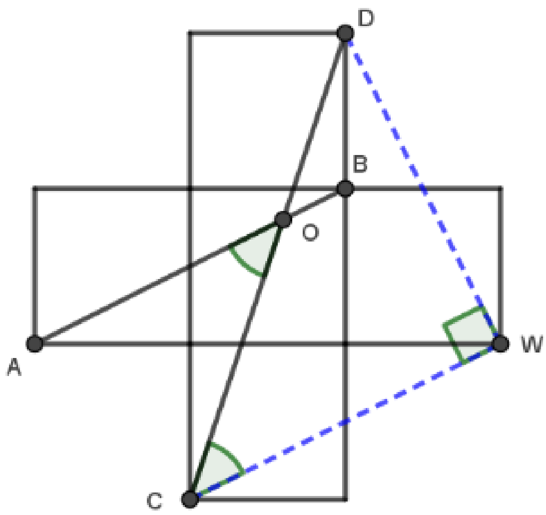

Figure 4. An elegant solution

Recall that the problem was supposedly meant for primary school students. Among the solutions we discussed so far, the one represented by Figure 4 is most accessible to primary school students, as it only utilizes properties of isosceles triangles and the fact that alternate angles on parallel lines are equal. However, this is a prime example of difficult problems whose solutions look deceptively easy. We cannot over emphasize how important it is for teachers to make a genuine attempt to solve a problem before giving it to the students. Without this prior effort to explore the terrain surrounding the problem, there is temptation for teachers to go straight to the solution technique (for example, showing the auxiliary lines one after the other directly). This can give students the impression that all that is required in the learning of mathematics is merely the acquisition of a "bag of such tricks". There is as such a wasted opportunity for students to struggle with the problem to learn the processes of productive exploration with respect to geometric problems.

In honesty, some of the authors who managed to solve the problem with trigonometry (Solution 1) or vectors (see Solution 4 below), struggled to come up with this elementary solution. An experienced teacher may be able to use Pólya's questioning strategies (1957, pp. 1 – 32) to lead a mature learner to the point of discovery. To illustrate how this problem can be exploited to teach the process of problem solving à la Pólya, we provide this description: the teacher presents the problem and gives students some time to attack it in their own ways. They will then experience frustration and hopefully the motivation to learn some ways forward. The teacher can prompt by directing students—since this is a problem requiring finding angles—to recall geometrical results that are about angles. It will invariably lead to the few that have to do with "parallel lines". This then becomes an opportunity for the teacher to ask students to provide the conditions on which they may then utilize these results—they will have to insert parallel lines into the diagram, hence the natural motivation for the use of auxiliary lines in problem solving! Another similar line of prompting comes from the inference that can be made from the fact that no specific value is given in the question for any of the angles (apart from the implicit value of 90° as the value of the interior angles of the squares). The teacher can then prompt students to recall angle properties of triangles and quadrilaterals where specific values of angles can be made—such as 60° in equilateral triangles and 45° in isosceles right-angled triangles. These two combined lines of inquiry—utilize parallel lines to see congruent angles, and look for special quadrilaterals or triangles—can be very helpful to guide (at least some) students towards success at solving the problem.

Solution 4: Vectors

Our fourth solution utilized more advanced mathematics, namely the concept of vectors in linear algebra. If we adopt the usual rectangular coordinates, we can represent AB by the vector $\begin{pmatrix}2\\1\end{pmatrix}$ and CD by the vector $\begin{pmatrix}1\\3\end{pmatrix}$. The angle ∠AOC between these two vectors satisfies

$$\cos(\angle AOC) = \frac{\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\3\end{pmatrix}}{\sqrt{2^2+1^2}\sqrt{1^2+3^2}} = \frac{2+3}{\sqrt{50}} = \frac{1}{\sqrt{2}}.$$

Hence, the required angle must be 45°. This solution is extremely elegant although one needs to be familiar with the concept of dot product of vectors.

Another advantage is that it is equally simple to use the same technique to solve the adapted problem of finding ∠EMC in Figure 5a. In this case, EG can be represented by the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

$$\cos(\angle \mathrm{EMC}) = \frac{\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\sqrt{2^2+1^2}\sqrt{1^2+3^2}} = \frac{-1}{\sqrt{50}}.$$

The difference here is that we require a calculator to evaluate ∠EMC which is approximately 98.13°. While it is possible to employ trigonometry, it is difficult to imagine an easy elementary method to evaluate this angle.

We can further generalize the problem to one where AB and CD are intersecting diagonals of two rectangles drawn on a grid array. Figure 5b shows an example with a 4 unit by 1 unit rectangle and a 1 unit by 5 unit rectangle. In this case,

$$\cos(\angle \mathrm{AOC}) = \frac{\begin{pmatrix} 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \end{pmatrix}}{\sqrt{4^2+1^2}\sqrt{1^2+5^2}} = \frac{9}{\sqrt{17}\sqrt{26}}.$$

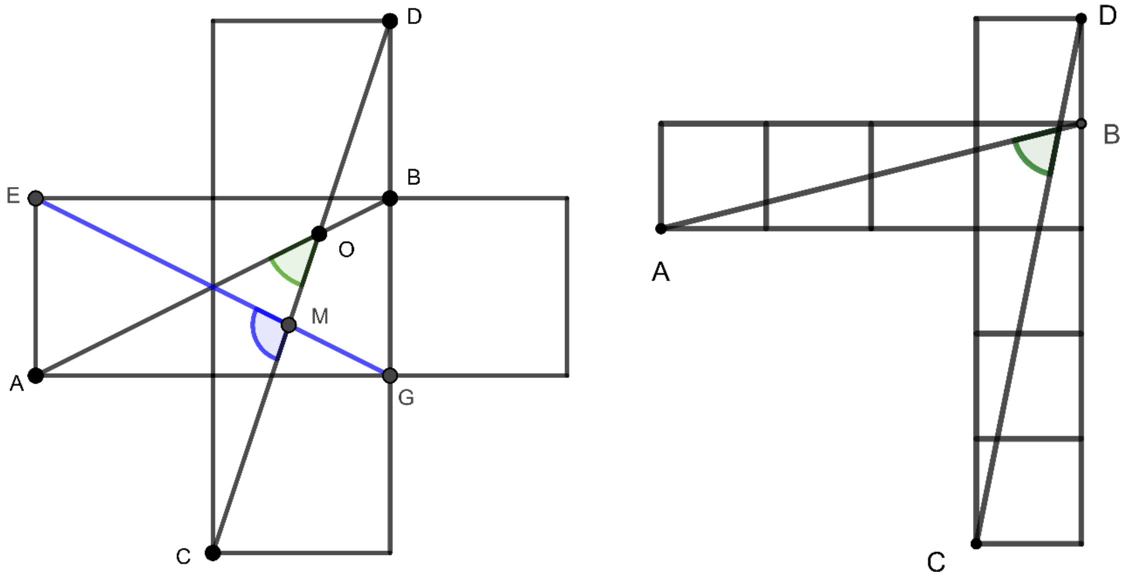

Figure 5. An adaptation and a generalization

Although we are still limited by the fact that the angle that we have found often cannot be easily evaluated, we can turn this limitation into an opportunity to create new problems. That is, we look for nonzero vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$, where a, b, c, d are integers, such that

$$\cos(\angle \mathrm{AOC}) = \frac{\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix}}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}} = \frac{1}{2}, \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{3}}{2}.$$

A computer search revealed a surprising phenomenon. There are many solutions when the right side of the above equation is $\frac{1}{\sqrt{2}}$, but no solutions can be found in the other two cases. A result in number theory (see for example Chapter 14 of Rosen, 2011) states that if an integer can be represented as a sum of two squares, then the prime 3 can only appear in its prime factorization as an even power. Hence it is impossible for the left side of the above expression to equal a rational multiple of $\sqrt{3}$. An alternate argument (Toh, 2020) can be used to show that $\cos(\angle AOC)$ cannot be equal to $\frac{1}{2}$. On the other hand, it can be proved that for every nonzero vector $\binom{a}{b}$, it is possible to write down a vector $\binom{c}{d}$ such that the two vectors intersect at 45°. Furthermore, there are essentially two ways to form such a vector and the resulting problem can always be solved using the approach described in Solution 3. One way is to set $\binom{c}{d} = \binom{a-b}{a+b}$. The required auxiliary line segments correspond to the vectors $\binom{a}{b}$ and $\binom{-b}{a}$, which are perpendicular to each other, have the same length and thus form an isosceles right-angled triangle with $\binom{a-b}{a+b}$. This illustrates the power of abstraction in mathematics. A single problem in geometry, when viewed through the lens of linear algebra, leads to a complete resolution of a whole class of analogous problems. We end by inviting the reader to solve two new problems illustrated in Figure 6.

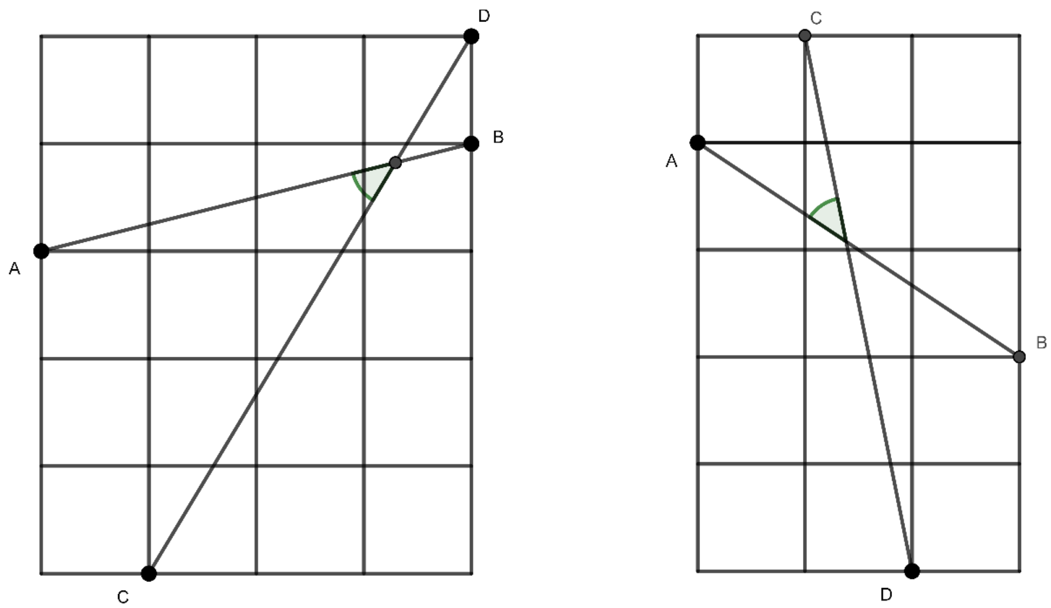

Figure 6. Two new problems

Conclusion

This article described four different solutions to an interesting geometry problem. We also shared our views from both the perspective of the problem solver and that of the mathematics educator. Although the problem was originally meant for primary school students, exposing secondary students to different approaches would serve to showcase mathematics as a connected enterprise and hopefully help to foster a deeper appreciation of mathematics.

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